SECOND ORDER QUASILINEAR FUNCTIONAL EVOLUTION EQUATIONS

László Simon, Budapest

(Received September 30, 2013)

Abstract. We consider second order quasilinear evolution equations where also the main part contains functional dependence on the unknown function. First, existence of solutions in $(0, T)$ is proved and examples satisfying the assumptions of the existence theorem are formulated. Then a uniqueness theorem is proved. Finally, existence and some qualitative properties of the solutions in $(0, \infty)$ (boundedness and stabilization as $t \to \infty$) are shown.

Keywords: functional evolution equation; second order quasilinear equation; monotone operator

MSC 2010: 35R10

1. Introduction

It is well known the importance of functional differential equations for applications, thus the theory of functional differential equations in one variable case and the theory of partial functional equations (evolution equations of first order) have been intensively studied for several years (see, e.g., [6], [7], [14]). In the field of partial functional equations the theory of monotone type operators can be applied, too (see, e.g., [9], [10], [11], [13]). It turned out that by using this theory not only first order functional evolution equations (parabolic functional equations) but also second order functional evolution equations can be dealt with, by using arguments similar to those, which were applied to second order evolution equations (without nonlocal terms), see, e.g., [4], [15] and [16].

There are several papers on second order semilinear functional equations (semilinear hyperbolic functional equations, see, e.g., [3], [5]).

This work was supported by the Hungarian National Foundation for Scientific Research under grant OTKA K 81403.

The aim of this paper is to consider some second order evolution equations of the form

$$u''(t) + [N(t, u'(t); u', u)](t) + Q[u(t)] + [M(t, u'(t); u', u)](t) = f(t) \tag{1.1}$$

with the initial condition

$$u(0) = u_0, \quad u'(0) = u_1, \tag{1.2}$$

by using the theory of monotone type operators and ideas of [4], [15], and [16]. Here $1 < p < \infty$,

$$N\colon\ L^p(0,T;V) \times L^p(0,T;V_1) \times L^p(0,T;V_1) \to L^q(0,T;V^\star),$$
$$M\colon\ L^p(0,T;V) \times L^p(0,T;V) \times L^p(0,T;V) \to L^q(0,T;V_1^\star),$$

are demicontinuous and bounded nonlinear operators, $Q\colon\ V \to V^\star$ is a linear and continuous operator, V, V_1 are reflexive Banach spaces such that $V \subset V_1$, the imbedding is compact, finally, $V \subset H \subset V^\star$ is an evolution triple.

Conditions will be formulated which imply the existence of solutions for $t \in (0,T)$ and for $t \in (0,\infty)$. Further, the boundedness of u', u for $t \in (0,\infty)$ and the stabilization of u as $t \to \infty$ will be shown. Several applications will be considered.

In a previous paper a similar equation, basically a particular case of the above equation was considered, here we shall use analogous arguments.

2. Existence of solutions for $t \in (0,T)$

Let V, V_1 be reflexive Banach spaces such that $V \subset V_1$ and the imbedding is compact, $V \subset H \subset V^\star$ an evolution triple (see, e.g., [15], [16]). Denote by $L^p(0,T;V)$ ($1 < p < \infty$) the Banach space of measurable functions $u\colon\ (0,T) \to V$ with the norm

$$\|u\|^p_{L^p(0,T;V)} = \int_0^T \|u(t)\|^p_V \,\mathrm{d}t.$$

The dual space of $L^p(0,T;V)$ is $L^q(0,T;V^\star)$ where $1/p + 1/q = 1$ and $V^\star$ is the dual space of V (see, e.g., [15], [16]). The duality between $V^\star$ and V will be denoted by $\langle \cdot, \cdot \rangle$ and between $L^q(0,T;V^\star)$ and $L^p(0,T;V)$ by $[\cdot, \cdot]$.

(i) Assumptions on N:

$$N\colon\ L^p(0,T;V) \times L^p(0,T;V_1) \times L^p(0,T;V_1) \to L^q(0,T;V^\star)$$

is demicontinuous (i.e., it maps strongly convergent sequences into weakly convergent sequences), and there is a constant c_1 such that

$$\|[N(t,z;v,w)](t)\|_{V^\star} \leqslant c_1 \|z\|_V^{p-1} \tag{2.1}$$

for all $t \in (0,T)$, $z \in V$, $v, w \in L^p(0,T;V_1)$. Further, for arbitrary fixed $t \in (0,T)$, $z \in V$,

$$(v,w) \mapsto N(t,z;v,w) \ \text{ is continuous} \tag{2.2}$$

(as an operator $L^p(0,T;V_1) \times L^p(0,T;V_1) \to L^q(0,T;V^\star)$). Finally, there exist constants $\sigma^\star \geqslant 0$ and $c_2 > 0$ such that $\sigma^\star < p-1$ and

$$\begin{aligned} &\langle [N(t,z_1;v,w)](t) - [N(t,z_2;v,w)](t), z_1(t) - z_2(t)\rangle \\ &\qquad \geqslant \frac{c_2}{[1 + \|v\|_{L^p(0,T;V)} + \|w\|_{L^p(0,T;V)}]^{\sigma^\star}} \|z_1(t) - z_2(t)\|_V^p \end{aligned} \tag{2.3}$$

for all $t \in (0,T)$, $z_1, z_2 \in V$, $v, w \in L^p(0,T;V)$.

(ii) Assumptions on Q:

$$\begin{gathered} Q\colon V \to V^\star \text{ is a linear and continuous operator satisfying} \\ \langle Qz_1, z_2\rangle = \langle Qz_2, z_1\rangle, \quad \langle Qz, z\rangle \geqslant 0, \quad z_1, z_2, z \in V. \end{gathered}$$

(iii) Assumptions on M:

$$M\colon L^p(0,T;V) \times L^p(0,T;V) \times L^p(0,T;V) \to L^q(0,T;V_1^\star)$$

is (nonlinear) bounded, demicontinuous and with some nonnegative constants c_3 and $\sigma < p - \sigma^\star - 1$

$$\langle [M(t,z;v,w)](t), z(t)\rangle \geqslant -c_3 \big[1 + \|v\|_{L^p(0,T;V)} + \|w\|_{L^p(0,T;V)}\big]^{\sigma+1}. \tag{2.4}$$

Theorem 2.1. *Assume* (i)–(iii). *Then for any $f \in L^q(0,T;V^\star)$, $u_0 \in V$ and $u_1 \in H$ there exists $u \in C([0,T];V)$ such that $u' \in L^p(0,T;V)$, $u'' \in L^q(0,T;V^\star)$ and u satisfies* (1.1), (1.2).

For the definition of the generalized derivatives u', u'' see, e.g., [15], page 417.

P r o o f. For simplicity, first consider the case $u_0 = 0$, $u_1 = 0$. Define an operator $S\colon L^p(0,T;V) \to L^p(0,T;V)$ by

$$(Sv)(t) = \int_0^t v(s)\,\mathrm{d}s.$$

Then S is a linear and continuous operator and u is a solution of (1.1), (1.2) with $u_0 = 0$, $u_1 = 0$ if and only if $v = u' \in L^p(0,T;V)$ satisfies

$$\begin{aligned} &v'(t) + [N(t,v(t);v,Sv)](t) + Q[Sv(t)] \\ &\qquad + [M(t,v(t);v,Sv)](t) = f(t), \quad t \in (0,T), \quad v(0) = 0. \end{aligned} \tag{2.5}$$

Consider the operator $A\colon L^p(0,T;V) \to L^q(0,T;V^\star)$ defined by

$$[A(v)](t) = [N(t,v(t);v,Sv)](t) + Q[Sv(t)] + [M(t,v(t);v,Sv)](t).$$

By (i)–(iii), it is not difficult to show that A is bounded and demicontinuous.

Further, by (i)

$$\begin{aligned} \int_0^T \langle [N(t,v(t);v,Sv)](t), v(t)\rangle \,\mathrm{d}t &\geqslant \frac{c_2}{[1 + \|v\|_{L^p(0,T;V)} + \|Sv\|_{L^p(0,T;V)}]^{\sigma^\star}} \\ &\quad\times \int_0^T \|v(t)\|_V^p \,\mathrm{d}t - \|N(t,0;v,Sv)\|_{L^q(0,T;V^\star)}\|v\|_{L^p(0,T;V)} \\ &\geqslant \frac{c_2}{[1 + \|v\|_{L^p(0,T;V)} + \|Sv\|_{L^p(0,T;V)}]^{\sigma^\star}} \int_0^T \|v(t)\|_V^p \,\mathrm{d}t \\ &\geqslant \mathrm{const}\|v\|_{L^p(0,T;V)}^{p-\sigma^\star} \end{aligned} \tag{2.6}$$

because

$$\|Sv\|_{L^p(0,T;V)} \leqslant \mathrm{const}\|v\|_{L^p(0,T;V)}.$$

Thus by (iii)

$$\begin{aligned} &\int_0^T \langle [A(v)](t), v(t)\rangle \,\mathrm{d}t \\ &\quad \geqslant \mathrm{const}\|v\|_{L^p(0,T;V)}^{p-\sigma^\star} - \mathrm{const}[1 + \|v\|_{L^p(0,T;V)} + \|Sv\|_{L^p(0,T;V)}]^{\sigma+1} \\ &\quad \geqslant \mathrm{const}\|v\|_{L^p(0,T;V)}^{p-\sigma^\star} - \mathrm{const}[1 + \|v\|_{L^p(0,T;V)}]^{\sigma+1}, \end{aligned} \tag{2.7}$$

which implies that

$$\lim_{\|v\|_{L^p(0,T;V)} \to \infty} \int_0^T \langle [A(v)](t), v(t)\rangle \,\mathrm{d}t = \infty$$

since $\sigma < p - \sigma^\star - 1$, thus A is coercive.

Now we show that A belongs to $(S)_+$ with respect to

$$D(L) = \{v \in L^p(0,T;V)\colon\ v' \in L^q(0,T;V^\star),\ v(0) = 0\},$$

which means:

$$v_j \in D(L), \quad (v_j) \to v \quad \text{weakly in } L^p(0,T;V), \tag{2.8}$$

$$(v_j') \to v' \quad \text{weakly in } L^q(0,T;V^\star), \tag{2.9}$$

$$\limsup [A(v_j), v_j - v] \leqslant 0 \tag{2.10}$$

imply

$$(v_j) \to v \quad \text{strongly in } L^p(0,T;V). \tag{2.11}$$

To prove that (2.8)–(2.10) imply (2.11), observe that

$$[QS(v_j), v_j - v] = [QS(v_j - v), v_j - v] + [QS(v), v_j - v],$$

where the first term on the right hand side is nonnegative (see, e.g., [15], [16]) and the second term tends to 0 by (2.9), thus

$$\liminf [QS(v_j), v_j - v] \geqslant 0. \tag{2.12}$$

Further, since V is compactly imbedded in V_1, by the compact imbedding theorem (see [8]), (2.8), (2.9) imply that

$$(v_j) \to v \quad \text{in } L^p(0,T;V_1) \tag{2.13}$$

for a subsequence (again denoted by (v_j), for simplicity), hence

$$\int_0^T \langle [M(t, v_j(t); v_j, Sv_j)](t), v_j(t) - v(t)\rangle \, \mathrm{d}t \to 0 \tag{2.14}$$

because the first term in (2.14) is bounded in $L^q(0,T;V_1^*)$ since M is a bounded operator. (2.10), (2.12), (2.14) imply that (for a subsequence)

$$\limsup \int_0^T \langle [N(t, v_j(t); v_j, Sv_j)](t), v_j(t) - v(t)\rangle \, \mathrm{d}t \leqslant 0. \tag{2.15}$$

Observe that

$$\begin{aligned} &\langle [N(t, v_j(t); v_j, Sv_j)](t), v_j(t) - v(t)\rangle \\ &\quad = \langle [N(t, v_j(t); v_j, Sv_j)](t) - [N(t, v(t); v_j, Sv_j)](t), v_j(t) - v(t)\rangle \\ &\quad\quad + \langle [N(t, v(t); v_j, Sv_j)](t) - [N(t, v(t); v, Sv)](t), v_j(t) - v(t)\rangle \\ &\quad\quad + \langle [N(t, v(t); v, Sv)](t), v_j(t) - v(t)\rangle. \end{aligned} \tag{2.16}$$

By (2.2), (2.13), (2.8) the second and third terms on the right hand side of (2.16) converge to 0. Thus (2.8), (2.15), (2.16) and (2.3) imply (2.11).

So we have shown that A is bounded, demicontinuous, coercive and belongs to $(S)_+$. Consequently (see, e.g., [2]), there is a solution of (2.5), and thus there is a solution of (1.1), (1.2) in the case $u_0 = 0$, $u_1 = 0$. The case $u_0 \in V$, $u_1 \in H$ can be easily reduced to the case $u_0 = 0$, $u_1 = 0$. (See, e.g., [11].) □

3. Examples

Example 3.1. Let V be a closed linear subspace of the Sobolev space $W^{1,p}(\Omega)$ containing $C_0^\infty(\Omega)$, $V_1 = L^p(\Omega)$ where Ω is a bounded domain with sufficiently smooth boundary, see [1]. The following examples satisfy the assumptions of Theorem 2.1:

$$\begin{aligned}&\langle [N(t,z(t);v,w)](t), y(t)\rangle\\ &\quad= \sum_{i=1}^n \int_\Omega b(t,x;[H(v)](t,x),[\widetilde{H}(v)](t,x))(D_i z)|Dz|^{p-2} D_i y\,\mathrm{d}x\\ &\qquad+ \int_\Omega b_0(t,x;[H_0(v)](t,x),[\widetilde{H}_0(v)](t,x))z|z|^{p-2} y\,\mathrm{d}x\end{aligned}$$

where b, b_0 are Carathéodory functions satisfying with some positive constants

$$\frac{c_2}{1+|\theta_1|^{\sigma^\star}+|\theta_1|^{\sigma^\star}} \leqslant b, \quad b_0(t,x;\theta_1,\theta_2) \leqslant c_3,$$

$$H,\widetilde{H},H_0,\widetilde{H}_0\colon\ L^p(Q_T)\to L^p(Q_T) \text{ are continuous linear operators.}$$

Further,

$$\langle Qu, v\rangle = \int_\Omega \Bigg[\sum_{k,l=1}^n a_{kl} D_k u D_l v + d_0 uv\Bigg]\,\mathrm{d}x$$

where a_{kl}, $d_0 \in L^\infty(\Omega)$, $a_{kl} = a_{lk}$, $\sum\limits_{k,l=1}^n a_{kl}(x)\xi_k\xi_l \geqslant 0$, $d_0 \geqslant 0$.

$$\begin{aligned}M(t,z;v,w) &= g_0\big(t,z(t,x),[F_1(v,Dv)](t,x),[F_2(w,Dw)](t,x)\big)\\ &\quad+ g\big(t,z(t,x),[F_3(v,Dv)](t,x),[F_4(w,Dw)](t,x)\big)\end{aligned}$$

where

$$g_0, g\colon\ (0,T)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$$

are Carathéodory functions satisfying

$$g_0(t,\theta_1,\theta_2,\theta_3)\theta_1 \geqslant 0, \quad |g_0(t,\theta_1,\theta_2,\theta_3)| \leqslant \mathrm{const}|\theta_1|^{p-1},$$

$$|g(t,\theta_1,\theta_2,\theta_3)| \leqslant \mathrm{const}[1+|\theta_1|+|\theta_2|+|\theta_3|]^\sigma; \quad F_j\colon\ [L^p(Q_T)]^{n+1}\to L^p(Q_T)$$

are continuous linear operators.

R e m a r k 3.1. In this case the solution of (1.1) can be viewed as a weak solution of the partial functional differential equation

$$\begin{aligned}D_t^2 u - \sum_{i=1}^{n} D_i\{b(t,x;[H(u')](t,x),[\widetilde{H}(u)](t,x))|Du'|^{p-2}D_i u'\}\\ + b_0(t,x;[H_0(u')](t,x),[\widetilde{H}_0(u)](t,x))|u'|^{p-2}u' - \sum_{j,k=1}^{n} D_l(a_{kl}D_k u) + d_0 u\\ + g_0(t,u',[F_1(u',Du')](t,x),[F_2(u,Du)](t,x))\\ + g(t,u',[F_3(u',Du')](t,x),[F_4(u,Du)](t,x)) = f(t,x),\end{aligned}$$

in the case $V = W_0^{1,p}(\Omega)$ with homogeneous Dirichlet boundary condition and in the case $V = W^{1,p}(\Omega)$ with homogeneous Neumann type boundary condition.

E x a m p l e 3.2. Similarly, let V be a closed linear subspace of the Sobolev space $W^{m,p}(\Omega)$ $(m \geqslant 1)$, $V_1 = W^{m-1,p}(\Omega)$, $H = L^2(\Omega)$. Then Q may be an analogous $2m$ order linear symmetric elliptic differential operator, N and M may be higher order nonlinear partial functional differential operators.

4. Uniqueness of the solution

Theorem 4.1. *Assume that the conditions of Theorem* 2.1 *are fulfilled so that N does not depend on v and w, M does not depend on w and M is Lipschitz in the following sense:*

$$\int_0^t \|M(\tau,v_1(\tau);v_1) - M(\tau,v_2(\tau);v_2)\|_H^2 \,\mathrm{d}\tau \leqslant \text{const} \int_0^t \|v_1(\tau) - v_2(\tau)\|_H^2 \,\mathrm{d}\tau$$

for arbitrary $v_1, v_2 \in L^p(0,T;V)$, $t \in (0,T)$.

Then for arbitrary solutions u_1, u_2 of (1.1) *with $f = f_1$ and $f = f_2$ we have*

$$\begin{aligned}(4.1)\qquad &\|u_1'(t) - u_2'(t)\|_H^2 + c_2\|u_1' - u_2'\|_{L^p(0,t;V)}\\ &\leqslant \text{const}\, \mathrm{e}^t\big[\|u_1'(0) - u_2'(0)\|_H^2 + \|f_1 - f_2\|_{L^q(0,t;V^\star)}^q\big].\end{aligned}$$

R e m a r k 4.1. The inequality (4.1) implies the uniqueness of the solution of (1.1), (1.2).

Further, from (4.1) one obtains estimates for $u_1 - u_2$ where u_j is the solution of (1.1), (1.2) with $u_0 = u_{0j}$, $u_1 = u_{1j}$, $f = f_j$, $j = 1,2$, since $u_j(t) = u_{0j} + \int_0^t u_j'(s)\,\mathrm{d}s$.

Proof of Theorem 4.1. Define v_j by $v_j(t) = \int_0^t u_j'(s)\,\mathrm{d}s = Su_j'$. Then v_j satisfies (2.5) with $f = f_j$. Apply the difference of these equations to $v_1 - v_2$. By using the arguments in the proof of Theorem 2.1, we obtain

$$\begin{aligned}(4.2)\qquad &\frac{1}{2}\|v_1(t) - v_2(t)\|_H^2 - \frac{1}{2}\|v_1(0) - v_2(0)\|_H^2 \\ &\quad + \int_0^t \langle [A(v_1)](\tau) - [A(v_2)](\tau), v_1(\tau) - v_2(\tau)\rangle \,\mathrm{d}\tau \\ &= \int_0^t \langle f_1(\tau) - f_2(\tau), v_1(\tau) - v_2(\tau)\rangle \,\mathrm{d}\tau \\ &\leqslant \left[\int_0^t \|f_1(\tau) - f_2(\tau)\|_{V^\star}^q \,\mathrm{d}\tau\right]^{1/q} \left[\int_0^t \|v_1(\tau) - v_2(\tau)\|_V^p \,\mathrm{d}\tau\right]^{1/p} \\ &\leqslant \varepsilon \|v_1 - v_2\|_{L^p(0,t;V)}^p + C(\varepsilon)\|f_1 - f_2\|_{L^q(0,t;V^\star)}^q\end{aligned}$$

for arbitrary $\varepsilon > 0$.

The assumptions of our theorem and the Cauchy-Schwarz inequality imply

$$\begin{aligned}(4.3)\qquad &\int_0^t \langle [A(v_1)](\tau) - [A(v_2)](\tau), v_1(\tau) - v_2(\tau)\rangle \,\mathrm{d}\tau \\ &\geqslant c_2\|v_1 - v_2\|_{L^p(0,t;V)}^p + \langle [(QS)(v_1)](t) \\ &\quad - [(QS)(v_2)](t), v_1(t) - v_2(t)\rangle - \mathrm{const}\int_0^t \|v_1(\tau) - v_2(\tau)\|_H^2 \,\mathrm{d}\tau.\end{aligned}$$

Since the second term on the right hand side of (4.3) is nonnegative (see the proof of Theorem 2.1), choosing $\varepsilon > 0$ sufficiently small, we obtain from (4.2), (4.3)

$$\begin{aligned}&\|v_1(t) - v_2(t)\|_H^2 + c_2\|v_1 - v_2\|_{L^p(0,t;V)}^p \\ &\quad \leqslant \|v_1(0) - v_2(0)\|_H^2 + c_3\|f_1 - f_2\|_{L^q(0,t;V^\star)}^q \\ &\qquad + \mathrm{const}\int_0^t \|v_1(\tau) - v_2(\tau)\|_H^2 \,\mathrm{d}\tau\end{aligned}$$

with some positive constants, which implies by Gronwall's lemma

$$\begin{aligned}(4.4)\qquad &\|v_1(t) - v_2(t)\|_H^2 + c_2\|v_1 - v_2\|_{L^p(0,t;V)}^p \\ &\qquad \leqslant \mathrm{const}\;\mathrm{e}^t\big[\|v_1(0) - v_2(0)\|_H^2 + \|f_1 - f_2\|_{L^q(0,t;V^\star)}^q\big].\end{aligned}$$

Since $v_j = u_j'$, from (4.4) one obtains (4.1). □

Now we formulate an existence theorem for $t \in (0, \infty)$ which can be obtained from Theorem 2.1, by using a diagonal process and the Volterra property of N and M (see, e.g. [11], [12]). Denote by $L^p_{\mathrm{loc}}(0, \infty; V)$ the set of functions $u\colon (0, \infty) \to V$ such that for each fixed finite $T > 0$, for the restriction of u to $(0, T)$, $u|_{(0,T)} \in L^p(0, T; V)$.

Theorem 5.1. *Assume that* $Q\colon V \to V^\star$ *satisfies* (ii). *Let*

$$N\colon L^p_{\mathrm{loc}}(0, T; V) \times L^p_{\mathrm{loc}}(0, T; V_1) \times L^p_{\mathrm{loc}}(0, T; V_1) \to L^q_{\mathrm{loc}}(0, T; V^\star),$$
$$M\colon L^p_{\mathrm{loc}}(0, T; V) \times L^p_{\mathrm{loc}}(0, T; V) \times L^p_{\mathrm{loc}}(0, T; V) \to L^q_{\mathrm{loc}}(0, T; V_1^\star)$$

be operators of Volterra type (i.e., for all $t > 0$ their restrictions to $v, w \in L^p(0, t; V_1)$ and $L^p(0, t; V)$, respectively, depend only on $v|_{(0,t)}$, $w|_{(0,t)}$) and assume that for each finite $T > 0$ their restrictions to $(0, T)$ satisfy (i) *and* (iii).

Then for arbitrary $f \in L^q_{\mathrm{loc}}(0, \infty; V^\star)$, $u_0 \in V$, $u_1 \in H$ there exists u such that $u \in C([0, \infty); V)$, $u' \in L^p_{\mathrm{loc}}(0, \infty; V)$, $u'' \in L^q_{\mathrm{loc}}(0, \infty; V^\star)$; further, u satisfies (1.1) *for $t \in (0, \infty)$ and the initial condition* (1.2).

Now we formulate a theorem on the boundedness of the solutions of (1.1), (1.2) for $t \in (0, \infty)$.

Theorem 5.2. *Let the assumptions of Theorem* 5.1 *be satisfied (with the same constants for all $T > 0$) such that instead of* (2.4)

$$\langle [M(t, z; v, w)](t), z(t)\rangle \geqslant -\mathrm{const}[1 + \|v\|_{L^p(0,t;V)}]^{\sigma+1}$$

holds and $f \in L^q(0, \infty; V^\star)$.

Then for a solution u of (1.1), (1.2), *in* $(0, \infty)$

$$\|u'(t)\|_H \quad \text{is bounded for } t \in (0, \infty), \tag{5.1}$$
$$u' \in L^p(0, \infty; V) \quad \text{and} \quad \langle Q[u(t)], u(t)\rangle \quad \text{is bounded for } t \in (0, \infty).$$

Further, if

$$\langle Qz, z\rangle \geqslant \hat{c}\|z\|_V^2 \quad \text{for } z \in V \tag{5.2}$$

with some constant $\hat{c} > 0$ then

$$u \in L^\infty(0, \infty; V). \tag{5.3}$$

P r o o f. Applying both sides of (1.1) to u' we obtain

$$\begin{aligned}(5.4)\qquad &\langle u''(t), u'(t)\rangle + \langle [N(t, u'(t); u', u)](t), u'(t)\rangle + \langle Q[u(t)], u'(t)\rangle \\ &+ \langle [M(t, u'(t); u', u)](t), u'(t)\rangle = \langle f(t), u'(t)\rangle.\end{aligned}$$

Integrating over $(0, t)$, we find by (2.6), (2.4) and the equality

$$\int_0^t \langle Q[u(\tau)], u'(\tau)\rangle \,\mathrm{d}t = \frac{1}{2}\langle Q[u(t)], u(t)\rangle - \frac{1}{2}\langle Q[u(0)], u(0)\rangle$$

(see, e.g., [11], [15], [16]) that

$$\begin{aligned}(5.5)\qquad &\frac{1}{2}\|u'(t)\|_H^2 - \frac{1}{2}\|u'(0)\|_H^2 + \mathrm{const}\|u'\|^{p-\sigma^\star}_{L^p(0,T;V)} + \frac{1}{2}\langle Q[u(t)], u(t)\rangle \\ &- \frac{1}{2}\langle Q[u(0)], u(0)\rangle - \mathrm{const}[1 + \|u'\|_{L^p(0,T;V)}]^{\sigma+1} \\ &\leqslant \varepsilon\|u'\|^{p-\sigma^\star}_{L^p(0,T;V)} + C(\varepsilon)\|f\|^{q^\star}_{L^q(0,T;V^\star)}\end{aligned}$$

with some positive constants for arbitrary $\varepsilon > 0$, where $q^\star$ is defined by

$$\frac{1}{p-\sigma^\star} + \frac{1}{q^\star} = 1, \quad \text{i.e., } q^\star = \frac{p-\sigma^\star}{p-\sigma^\star-1}.$$

Choosing $\varepsilon > 0$ sufficiently small, from (5.5) we obtain (5.1), since $p - \sigma^\star > \sigma + 1$.

Finally, (5.2) and (5.5) imply (5.3). □

Now we prove a theorem on the stabilization of the solution as $t \to \infty$.

Theorem 5.3. *Assume that the assumptions of Theorem 5.2 are satisfied so that for all $t > 0$*

$$(5.6)\quad \langle [M(t, z; v, w)](t), z(t)\rangle \geqslant \tilde c\|z(t)\|_H^2 - \mathrm{const}[1 + \|v\|_{L^p(0,t;V)}]^{\sigma+1}, \quad t \in (0,\infty)$$

with some constant $\tilde c > 0$, and there exists $a > 0$ such that

$$(5.7)\qquad \|[M(t, z; v, w)](t)\|_{V_1^\star} \leqslant \mathrm{const}\big[\|z\|_V + \|v\|^{p-1}_{L^p(t-a,t;V)}\big].$$

Further, let there exist $f_\infty \in V^\star$ such that $(f - f_\infty) \in L^2(0, T; H)$.

Then for a solution u of (1.1), (1.2) in $(0,\infty)$ we have

$$(5.8)\qquad \|u'(t)\|_H \leqslant \mathrm{const}\ \mathrm{e}^{-\tilde c t}, \quad t \in (0,\infty).$$

Further, there exists $w_0 \in V$ such that

$$\|u(t) - w_0\|_H \leqslant \text{const } \mathrm{e}^{-\tilde{c}t}, \quad t \in (0, \infty), \tag{5.9}$$

and w_0 satisfies the equation

$$Qw_0 = f_\infty. \tag{5.10}$$

P r o o f. Since $Q\colon V \to V^\star$ is linear, continuous, uniformly monotone, there exists a unique solution $u_\infty \in V$ of

$$Qu_\infty = f_\infty \tag{5.11}$$

(see, e.g., [15], [16]). By (5.3) and (5.11) we obtain for $w = u - u_\infty$

$$\begin{aligned}\langle w''(t), w'(t)\rangle &+ \langle [N(t, u'(t); u', u)](t), u'(t)\rangle + \langle Q[w(t)], w'(t)\rangle \\ &+ \langle [M(t, u'(t); u', u)](t), u'(t)\rangle = \langle f(t) - f_\infty, u'(t)\rangle\end{aligned}$$

since $w'(t) = u'(t)$, $w''(t) = u''(t)$; hence, integrating over $(0, T)$ with respect to t, we find by (5.6)

$$\begin{aligned}\frac{1}{2}\|w'(T)\|_H^2 - \frac{1}{2}\|w'(0)\|_H^2 &+ \text{const}\|w'\|_{L^p(0,T;V)}^{p-\sigma^\star} \\ &+ \frac{1}{2}\langle Q[w(T)], w(T)\rangle - \frac{1}{2}\langle Q[w(0)], w(0)\rangle \\ &+ \tilde{c}\int_0^T \|w'\|_H^2 \,\mathrm{d}t - \text{const}[1 + \|w'\|_{L^p(0,T;V)}]^{\sigma+1} \\ \leqslant \varepsilon &\int_0^T \|w'(t)\|_H^2 \,\mathrm{d}t + C(\varepsilon)\|f - f_\infty\|_{L^2(0,T;H)}^2\end{aligned} \tag{5.12}$$

with some positive constants. Choosing $\varepsilon > 0$ sufficiently small, we obtain from inequality (5.12)

$$\|u'(T)\|_H^2 + 2\tilde{c}\int_0^T \|u'(T)\|_H^2 \,\mathrm{d}t \leqslant \text{const}, \quad \text{for all } T > 0 \tag{5.13}$$

since $\|u'\|_{L^p(0,T;V)}$ is bounded for all $T > 0$. The constant $\tilde{c}$ is positive, thus by Gronwall's lemma

$$\|u'(T)\|_H^2 \leqslant c^\star \mathrm{e}^{-\tilde{c}T}, \quad T > 0$$

with some constant $c^\star$, i.e. we have (5.8).

The inequality (5.8) implies (5.9) since for any T_1, T_2 ($T_2 > T_1$)

$$\begin{aligned}\|u(T_2) - u(T_1)\|_H^2 &= (u(T_2), u(T_2) - u(T_1))_H - (u(T_1), u(T_2) - u(T_1))_H \\ &= \int_{T_1}^{T_2} \langle u'(t), u(T_2) - u(T_1)\rangle \,\mathrm{d}t = \int_{T_1}^{T_2} (u'(t), u(T_2) - u(T_1))_H \,\mathrm{d}t \\ &\leqslant \|u(T_2) - u(T_1)\|_H \int_{T_1}^{T_2} \|u'(t)\|_H \,\mathrm{d}t,\end{aligned}$$

hence

$$\|u(T_2) - u(T_1)\|_H \leqslant \int_{T_1}^{T_2} \|u'(t)\|_H \,\mathrm{d}t, \tag{5.14}$$

which implies

$$\|u(T_2) - u(T_1)\|_H \to 0 \quad \text{as } T_1, T_2 \to \infty.$$

Consequently, there exists $w_0 \in H$ such that

$$\|u(T) - w_0\|_H \to 0 \quad \text{as } T \to \infty \tag{5.15}$$

and by (5.14)

$$\|u(T) - w_0\|_H \leqslant \int_T^\infty \|u'(t)\| \,\mathrm{d}t \leqslant \text{const } \mathrm{e}^{-\tilde{c}T},$$

i.e., we have (5.9). Since $u \in L^\infty(0, \infty; V)$,

$$u(T_k) \to w_0^\star \quad \text{weakly in } V,\ w_0^\star \in V \tag{5.16}$$

for some sequence (T_k), $\lim(T_k) = \infty$. Clearly, (5.16) implies

$$u(T_k) \to w_0^\star \quad \text{weakly in } H,$$

thus by (5.15) $w_0 = w_0^\star \in V$ and (5.16) holds for an arbitrary sequence (T_k) converging to ∞.

Finally, we show (5.10). Consider an arbitrary fixed $v \in V$ and

$$\chi_T(t) = \chi(t - T) \quad \text{where } \chi \in C_0^\infty,\ \operatorname{supp}\chi \subset (0, 1),\ \int_0^1 \chi(t) \,\mathrm{d}t = 1.$$

Applying (1.1) to $v \in V$, multiplying by $\chi_T(t)$ and integrating over $(0, \infty)$ with respect to t, we obtain

$$\begin{aligned}&\int_0^\infty \langle u''(t), v\rangle \chi_T(t) \,\mathrm{d}t + \int_0^\infty \langle [N(t, u'(t); u', u)](t), v\rangle \chi_T(t) \,\mathrm{d}t \\ &\qquad + \int_0^\infty \langle Q[u(t)], v\rangle \chi_T(t) \,\mathrm{d}t + \int_0^\infty \langle [M(t, u'(t); u', u)](t), v\rangle \chi_T(t) \,\mathrm{d}t \\ &\quad = \int_0^\infty \langle f(t)(t), v\rangle \chi_T(t) \,\mathrm{d}t.\end{aligned} \tag{5.17}$$

Let (T_k) be an arbitrary sequence converging to ∞. For the first term on the left hand side of (5.17) (with $T = T_k$) we have by (5.8)

$$\int_0^\infty \langle u''(t), v\rangle \chi_{T_k}(t)\,\mathrm{d}t = -\int_0^\infty \langle u'(t), v\rangle \chi'_{T_k}(t)\,\mathrm{d}t \to 0 \quad \text{as } k\to\infty, \tag{5.18}$$

further by (ii), (5.16) and Lebesgue's dominated convergence theorem

$$\begin{aligned}\int_0^\infty \langle Q[u(t)], v\rangle \chi_{T_K}(t)\,\mathrm{d}t &= \int_0^\infty \langle Qv, u(t)\rangle \chi_{T_K}(t)\,\mathrm{d}t \\ &= \int_0^1 \langle Qv, u(T_k+\tau)\rangle \chi(\tau)\,\mathrm{d}\tau \to \int_0^1 \langle Qv, w_0\rangle \chi(\tau)\,\mathrm{d}\tau = \langle Qv, w_0\rangle = \langle Qw_0, v\rangle \\ &\quad \text{as } k\to\infty.\end{aligned} \tag{5.19}$$

For the second term on the left hand side of (5.17) we have by (5.1)

$$\begin{aligned}&\left|\int_0^\infty \langle [N(t,u'(t);u',u)](t), v\rangle \chi_{T_k}(t)\,\mathrm{d}t\right| \\ &\quad = \left|\int_0^1 \langle [N(T_k+\tau, u'(t_k+\tau);u',u)](T_k+\tau), v\rangle \chi(\tau)\,\mathrm{d}\tau\right| \\ &\quad \leqslant \text{const} \int_0^1 \|u'(T_k+\tau)\|_V^{p-1}\,\mathrm{d}\tau\cdot\|v\|_V \to 0 \quad \text{as } k\to\infty.\end{aligned} \tag{5.20}$$

For the fourth term on the left hand side of (5.17) we have by (5.7), (5.1)

$$\begin{aligned}&\left|\int_0^\infty \langle [M(t,u'(t);u',u)](t), v\rangle \chi_{T_k}(t)\,\mathrm{d}t\right| \\ &= \left|\int_0^1 \langle [M(T_k+\tau, u'(t_k+\tau);u',u)](T_k+\tau), v\rangle \chi(\tau)\,\mathrm{d}\tau\right| \\ &\leqslant \text{const}\|v\|_{V_1} \int_0^1 \|[M(T_k+\tau, u'(T_k+\tau);u',u)](T_k+\tau)\|_{V_1^\star}\,\mathrm{d}\tau \\ &\leqslant \text{const}\|v\|_{V_1} \int_0^1 \left[\|u'(T_k+\tau)\|_V + \|u'\|^{p-1}_{L^p(T_k+\tau-a;T_k+\tau;V)}\right]\mathrm{d}\tau \to 0 \\ &\quad \text{as } k\to\infty.\end{aligned} \tag{5.21}$$

Finally, for the right hand side of (5.17) we obtain, by using $(f-f_\infty)\in L^2(0,\infty;H)$ and the Cauchy-Schwarz inequality

$$\begin{aligned}\int_0^\infty \langle f(t)(t), v\rangle \chi_{T_k}(t)\,\mathrm{d}t &= \int_0^\infty (f(t)(t), v)\chi_{T_k}(t)\,\mathrm{d}t \\ &= \int_0^1 \langle f(T_k+\tau), v\rangle \chi(\tau)\,\mathrm{d}\tau \to \int_0^1 \langle f_\infty, v\rangle \chi(\tau)\,\mathrm{d}\tau = \langle f_\infty, v\rangle.\end{aligned} \tag{5.22}$$

From (5.17)–(5.22) one obtains (5.10) as $k\to\infty$. $\square$

R e m a r k 5.1. By using examples in Section 3 it is not difficult to formulate examples satisfying the assumptions of Theorems 5.1–5.3.

References

[1] *R. A. Adams*: Sobolev Spaces. Pure and Applied Mathematics 65. A Series of Monographs and Textbooks, Academic Press, New York, 1975.

[2] *J. Berkovits, V. Mustonen*: Topological degree for perturbations of linear maximal monotone mappings and applications to a class of parabolic problems. Rend. Mat. Appl., VII. Ser. *12* (1992), 597–621.

[3] *W. Czernous*: Global solutions of semilinear first order partial functional differential equations with mixed conditions. Funct. Differ. Equ. *18* (2011), 135–154.

[4] *H. Gajewski, K. Gröger, K. Zacharias*: Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen. Mathematische Lehrbücher und Monographien II. Abteilung 38, Akademie, Berlin, 1974. (In German.)

[5] *C. Giorgi, J. E. Muñoz Rivera, V. Pata*: Global attractors for a semilinear hyperbolic equation in viscoelasticity. J. Math. Anal. Appl. *260* (2001), 83–99.

[6] *J. Hale*: Theory of Functional Differential Equations. Applied Mathematical Sciences 3, Springer, New York, 1977.

[7] *F. Hartung, J. Turi*: Stability in a class of functional-differential equations with state-dependent delays. Qualitative Problems for Differential Equations and Control Theory (C. Corduneanu, ed.). World Scientific, Singapore, 1995, pp. 15–31.

[8] *J. L. Lions*: Quelques méthodes de résolution des problèmes aux limites non linéaires. Études mathématiques, Dunod; Gauthier-Villars, Paris, 1969. (In French.)

[9] *L. Simon*: Nonlinear functional parabolic equations. Integral Methods in Science and Engineering: Computational Methods 2. Selected papers based on the presentations at the 10th International Conference on Integral Methods in Science and Engineering, Santander, Spain, 2008 (C. Constanda et al., eds.). Birkhäuser, Boston, 2010, pp. 321–326.

[10] *L. Simon*: On nonlinear functional parabolic equations with state-dependent delays of Volterra type. Int. J. Qual. Theory Differ. Equ. Appl. *4* (2010), 88–103.

[11] *L. Simon*: Application of monotone type operators to parabolic and functional parabolic PDE's. Handbook of Differential Equations: Evolutionary Equations IV (C. M. Dafermos et al., eds.). Elsevier/North-Holland, Amsterdam, 2008, pp. 267–321.

[12] *L. Simon*: On nonlinear hyperbolic functional differential equations. Math. Nachr. *217* (2000), 175–186.

[13] *L. Simon, W. Jäger*: On non-uniformly parabolic functional differential equations. Stud. Sci. Math. Hung. *45* (2008), 285–300.

[14] *J. Wu*: Theory and Applications of Partial Functional-Differential Equations. Applied Mathematical Sciences 119, Springer, New York, 1996.

[15] *E. Zeidler*: Nonlinear Functional Analysis and Its Applications. II/A: Linear Monotone Operators. Springer, New York, 1990.

[16] *E. Zeidler*: Nonlinear Functional Analysis and Its Applications. II/B: Nonlinear Monotone Operators. Springer, New York, 1990.

Author's address: *László Simon*, Institute of Mathematics, Loránd Eötvös University, Pázmány Péter sétány 1/c, H-1117 Budapest, Hungary, e-mail: `simonl@cs.elte.hu`.