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EXISTENCE OF SOLUTIONS AND APPROXIMATE CONTROLLABILITY OF IMPULSIVE FRACTIONAL STOCHASTIC DIFFERENTIAL SYSTEMS WITH INFINITE DELAY AND POISSON JUMPS

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Abstract. The paper is motivated by the study of interesting models from economics and the natural sciences where the underlying randomness contains jumps. Stochastic differential equations with Poisson jumps have become very popular in modeling the phenomena arising in the field of financial mathematics, where the jump processes are widely used to describe the asset and commodity price dynamics. This paper addresses the issue of approximate controllability of impulsive fractional stochastic differential systems with infinite delay and Poisson jumps in Hilbert spaces under the assumption that the corresponding linear system is approximately controllable. The existence of mild solutions of the fractional dynamical system is proved by using the Banach contraction principle and Krasnoselskii's fixed-point theorem. More precisely, sufficient conditions for the controllability results are established by using fractional calculations, sectorial operator theory and stochastic analysis techniques. Finally, examples are provided to illustrate the applications of the main results.

Keywords: approximate controllability; fixed-point theorem; fractional stochastic differential system; Hilbert space, Poisson jumps

MSC 2010: 34K50, 93B05, 93E03

1. Introduction

Differential equations involving fractional derivatives in time, compared with those of integer order in time, are more realistic to describe many phenomena in nature (for

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instance, to describe the memory and hereditary properties of various materials and processes), the study of such equations has become an object of extensive research during recent years (see [11], [17]). Fractional differential equations have a wide range of applications in many physical phenomena such as seepage flow in porous media, fluid dynamics, traffic models, etc. The most important advantage of using fractional differential equations is their time delay property. This means that the next state of a system depends not only upon its current state but also upon all its historical states. This is probably the most relevant feature making this fractional tool useful from an applied standpoint and interesting from a mathematical standpoint, which led to the sustained study of the theory of fractional differential equations. Besides, noise or stochastic perturbation is unavoidable and omnipresent in nature as well as in man-made systems. Therefore, it is of great significance to import the stochastic effects into the investigation of fractional differential systems. Various evolutionary processes from fields as diverse as physics, population dynamics, aeronautics, economics and engineering are characterized by the fact that they undergo abrupt changes of state at certain moments of time between intervals of continuous evolution. Because the duration of these changes is often negligible compared to the total duration of the process, such changes can be reasonably well approximated as being instantaneous changes of state, or in the form of impulses (see [13]). These processes are suitably modeled by impulsive fractional stochastic differential equations. Moreover, the qualitative behavior such as the existence and controllability of fractional dynamical systems are current important issues explored by many researchers, for example see [7], [18], [20]. Tai et al. [26] addressed the controllability results of fractional-order impulsive neutral functional infinite delay integro-differential systems in Banach spaces by using Krasnoselskii's fixed-point theorem. In control theory, the main tool is to convert the controllability problem into a fixed-point problem with the assumption that the controllability operator has an induced inverse on a quotient space. To prove controllability, an assumption that the semigroup (or the resolvent operator) associated with the linear part is compact is often made. However, if the compactness condition holds on the bounded operator that maps the control function on the generated C_0-semigroup, then the controllability operator is also compact and its inverse does not exist if the state space is infinite-dimensional (see [28]). Sukavanam et al. [25] have proved some sufficient conditions for the approximate controllability of fractional order system in which the nonlinear term depends on both state and control variables. Balasubramaniam et al. [1] discussed the approximate controllability of impulsive fractional integro-differential systems with nonlocal conditions in Hilbert space by using Darbo-Sadovskii's fixed-point theorem.

In recent years there has been an accelerating interest in the development of stochastic models for describing the functions of intrinsic noise, due to the uncertainty of natural phenomena, and extrinsic noise, due to fluctuations in the environment. Thus, stochastic differential equations appear as a natural description of several observed phenomena of real world (see [6]). In infinite dimensions, the stochastic systems can be studied using Brownian motion with finite trace nuclear covariance operator. Cui et al. [4] proved the existence result for fractional neutral stochastic integro-differential equations with infinite delay by using Sadovskii's fixed-point theorem. The existence and uniqueness for a class of fractional stochastic delay differential equations has been established in [8]. Sakthivel et al. [22] addressed the issue of existence of mild solutions for a class of fractional stochastic differential equations with impulses in Hilbert spaces by using fractional calculations, fixed-point technique and stochastic analysis theory. In contrast, papers dealing with the approximate controllability of fractional order stochastic systems are scarce. Recently, the subject was addressed in Sakthivel et al. [23] without Poisson jumps.

The modelling of risky asset by stochastic processes with continuous paths, based on Brownian motion, suffers from several defects. First, the path continuity assumption does not seem reasonable in view of the possibility of sudden price variations (jumps) resulting of market crashes. A solution is to use stochastic processes with jumps, which will account for sudden variations of the asset prices. On the other hand, such jump models are generally based on the Poisson random measure. Many popular economic and financial models are described by stochastic differential equations with Poisson jumps (see [3], [29]). Taniguchi et al. [27] derived a new set of sufficient conditions for the existence of mild solutions of stochastic evolution equations with infinite delay driven by Poisson jump processes. Liu et al. [14] studied the existence and uniqueness of global mild solutions of jump-type stochastic fractional partial differential equations with fractional noise by using Green functions. Hausenblas et al. [10] studied the numerical approximation of parabolic stochastic partial differential equations driven by a Poisson random measure by using spectral methods, implicit Euler scheme and explicit Euler scheme. Very few authors studied the qualitative properties of stochastic differential equations driven by Poisson jumps (see [5], [19] and references therein). Sakthivel et al. [21] studied the complete controllability of stochastic evolution equations with jumps without assuming the compactness of the semigroup property. Long et al. [15] proved the sufficient condition for the approximate controllability of SPDE with infinite delays driven by Poisson jumps by using the Krasnoselskii-Schaefer fixed-point theorem. Here, we move from deterministic impulsive fractional differential equations to stochastic impulsive fractional differential equations with Poisson jumps for the study of existence of solutions and controllability properties. Motivated by few studies [7], [18], [23], [22], the existence of solutions and approximate controllability of the following impulsive fractional stochastic differential system with infinite delay and Poisson jumps remains an untreated topic in the literature:

$$
\begin{aligned}
\text{(1.1)}\qquad D_t^\alpha x(t) &= Ax(t) + Bu(t) + f(t,x_t) + g(t,x_t)\frac{\mathrm{d}W(t)}{\mathrm{d}t}\\
&\quad + \int_Z h(t,x_t,\eta)\widetilde{N}(\mathrm{d}t,\mathrm{d}\eta), \quad t\in J := [0,b],\ t\neq t_i,\\
\Delta x(t_i) &= I_i(x(t_i^-)), \quad i=1,2,\dots,m,\\
x(t) &= \varphi\in\mathcal{C}_v, \quad -\infty < t \leqslant 0, \quad t\in J_0 := (-\infty,0],
\end{aligned}
$$

where $0<\alpha<1$; D_t^α denotes the Caputo fractional derivative of order α. Here, $x(\cdot)$ takes values in the Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, $A\colon D(A)\subset H\to H$ is the infinitesimal generator of an α-resolvent family $S_\alpha(t)_{t>0}$. The control function $u(\cdot)$ takes values in $L_2^{\mathfrak{F}}(J,U)$, the space of admissible control functions, U is a Hilbert space, B is a bounded linear operator from U into H. Let K be another separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_K$ and the norm $\|\cdot\|_K$. Suppose $\{W(t)\}_{t\geqslant 0}$ is a given K-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q\geqslant 0$. We use the same notation $\|\cdot\|$ for the norm of $L(K,H)$, where $L(K,H)$ denotes the space of all bounded linear operators from K into H, simply $L(H)$ if $K=H$. The histories $x_t\colon(-\infty,0]\to\mathcal{C}_v$ defined by $x_t=\{x(t+\theta);\ \theta\in(-\infty,0]\}$ belong to the phase space $\mathcal{C}_v$, which is defined in Section 2. Let $\hat{q}=(\hat{q}(t))$, $t\in D_{\hat{q}}$, be a stationary $\mathfrak{F}_t$-Poisson point process with a characteristic measure $\hat{\lambda}$. Let $N(\mathrm{d}t,\mathrm{d}\eta)$ be the Poisson counting measure associated with $\hat{q}$. Thus, we have $N(t,Z)=\sum\limits_{s\in D_{\hat{q}},s\leqslant t} I_Z(\hat{q}(s))$ with a measurable set $Z\in\mathcal{B}(K-\{0\})$, which denotes the Borel σ-field of $K-\{0\}$. Let $\widetilde{N}(\mathrm{d}t,\mathrm{d}\eta)=N(\mathrm{d}t,\mathrm{d}\eta)-\mathrm{d}t\hat{\lambda}(\mathrm{d}\eta)$ be the compensated Poisson measure that is independent of Brownian motion. Let $p_2([0,b]\times Z;H)$ be the space of all predictable mappings $\widehat{\chi}\colon[0,b]\times Z\to H$ for which

$$\int_0^b\int_Z E\|\widehat{\chi}(t,\eta)\|_H^2\,\mathrm{d}t\hat{\lambda}(\mathrm{d}\eta)<\infty.$$

Then, we can define the H-valued stochastic integral $\int_0^b\int_Z\widehat{\chi}(t,\eta)\widetilde{N}(\mathrm{d}t,\mathrm{d}\eta)$, which is a centred square-integrable martingale. The functions $f\colon J\times\mathcal{C}_v\to H$, $g\colon J\times\mathcal{C}_v\to L_Q(K,H)$ and $h\colon J\times\mathcal{C}_v\times Z\to H$ are Borel measurable functions, where $L_Q(K,H)$ denotes the space of all Q-Hilbert-Schmidt operators from K into H. For $i=1,2,\dots,m$, $I_i\colon H\to H$ are appropriate functions. Here $0=t_0\leqslant t_1\leqslant\dots\leqslant t_m\leqslant t_{m+1}=b$, $\Delta x(t_i)=x(t_i^+)-x(t_i^-)$, $x(t_i^+)=\lim\limits_{\hat{h}\to 0}x(t_i+\hat{h})$ and $x(t_i^-)=\lim\limits_{\hat{h}\to 0}x(t_i-\hat{h})$ are respectively the right and left limits of $x(t)$ at $t=t_i$. The initial data $\varphi(t)$ is an $\mathfrak{F}_0$-adapted H-valued random variable independent of the Wiener process W.

The paper is organized as follows. In Section 2, we introduce the basic notations and assumptions which are necessary to formulate the main results. In Section 3, we derive the existence of mild solutions using fixed-point techniques. In Section 4, the approximate controllability result is investigated. Examples are provided in Section 5 to illustrate the desired theoretical results.

2. Preliminaries

For more details on the concepts presented in this section, the reader may refer to [16], [18], [22] and references therein. Throughout the paper, $(H, \|\cdot\|)$ and $(K, \|\cdot\|_K)$ denote real separable Hilbert spaces. Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space equipped with a normal filtration $\{\mathfrak{F}_t, t \in J\}$ satisfying the usual conditions (that is, right continuity and $\mathfrak{F}_0$ containing all P-null sets of $\mathfrak{F}$). An H-valued random variable is an $\mathfrak{F}$-measurable function $x(t)\colon\ \Omega \to H$ and the collection of random variables $S = \{x(t,\omega)\colon\ \Omega \to H|_{t\in J}\}$ is called a stochastic process. Generally, we just write $x(t)$ instead of $x(t,\omega)$ and $x(t)\colon\ J \to H$ in the space of S. Let $\{\zeta_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of K. Suppose that $\{W(t); t \geqslant 0\}$ is a K-valued Wiener process with finite trace nuclear covariance operator $Q \geqslant 0$, denote $\operatorname{Tr}(Q) = \sum\limits_{i=1}^{\infty}\lambda_i < \infty$, which satisfies $Q\zeta_i = \lambda_i\zeta_i$. So, actually, $W(t) = \sum\limits_{i=1}^{\infty}\sqrt{\lambda_i}\beta_i(t)\zeta_i$, where $\{\beta_i(t)\}_{i=1}^{\infty}$, are mutually independent one-dimensional standard Wiener processes. We assume that $\mathfrak{F}_t = \sigma\{W(s)\colon\ 0 \leqslant s \leqslant t\}$ is the σ-algebra generated by W and $\mathfrak{F}_b = \mathfrak{F}$. Let $\chi \in L(K,H)$ and define

$$\|\chi\|_Q^2 = \operatorname{Tr}(\chi Q \chi^*) = \sum_{i=1}^{\infty}\bigl\|\sqrt{\lambda_i}\chi\zeta_i\bigr\|^2.$$

If $\|\chi\|_Q < \infty$, then χ is called a Q-Hilbert-Schmidt operator. Let $L_Q(K,H)$ denote the space of all Q-Hilbert-Schmidt operators $\chi\colon\ K \to H$. The completion $L_Q(K,H)$ of $L(K,H)$ with respect to the topology induced by the norm $\|\cdot\|_Q$, where $\|\chi\|_Q^2 = \langle\chi,\chi\rangle$ is a Hilbert space with the above norm topology. The collection of all strongly measurable, square integrable H-valued random variables, denoted by $L_2(\Omega,\mathfrak{F},P;H) \equiv L_2(\Omega;H)$, is a Banach space equipped with the norm $\|x(\cdot)\|_{L_2} = (E\|x(\cdot;\omega)\|_H^2)^{1/2}$, where the expectation E is defined by $E(h_1) = \int_\Omega h_1(\omega)\,\mathrm{d}P$. Let $\widehat{J} = (-\infty, b]$ and let $C(\widehat{J}, L_2(\Omega;H))$ be the Banach space of all continuous maps from $\widehat{J}$ into $L_2(\Omega;H)$ satisfying the condition $\sup\limits_{t\in\widehat{J}} E\|x(t)\|^2 < \infty$. Now, we present the abstract phase space $\mathcal{C}_v$. Assume that $v\colon\ (-\infty,0] \to (0,\infty)$ is a continuous function satisfying $l = \int_{-\infty}^0 v(t)\,\mathrm{d}t < \infty$. The Banach space $(\mathcal{C}_v, \|\cdot\|_{\mathcal{C}_v})$ induced by the function v is defined as follows:

$$\mathcal{C}_v = \Big\{\varphi\colon\ (-\infty,0]\to H,\ \text{such that for any } a>0,\ E(|\varphi(\theta)|^2)^{1/2} \text{ is a bounded and measurable function on } [-a,0] \text{ with } \varphi(0)=0 \text{ and } \int_{-\infty}^{0} v(s)\sup_{s\leqslant\theta\leqslant 0} E(|\varphi(\theta)|^2)^{1/2}\,\mathrm{d}s<\infty\Big\}$$

and $\mathcal{C}_v$ is endowed with the norm $\|\varphi\|_{\mathcal{C}_v} = \int_{-\infty}^{0} v(s)\sup_{s\leqslant\theta\leqslant 0} E(|\varphi(\theta)|^2)^{1/2}\,\mathrm{d}s$, $\varphi\in\mathcal{C}_v$ (see [12], [19]). Let us consider the space

$$\mathcal{C}_b = \{x\colon\ (-\infty,b]\to H,\ \text{such that } x|_{J_i}\in C(J_i,H) \text{ and there exist } x(t_i^+) \text{ and } x(t_i^-) \text{ with } x(t_i)=x(t_i^-),\ x_0=\varphi\in\mathcal{C}_v,\ i=1,2,\dots m\},$$

where $x|_{J_i}$ is the restriction of x to $J_i=(t_i,t_{i+1}]$, $i=0,1,2,\dots,m$. Set $\|\cdot\|_{\mathcal{C}_b}$ to be a seminorm defined by

$$\|x\|_{\mathcal{C}_b} = \|\varphi\|_{\mathcal{C}_v} + \sup_{s\in[0,b]}(E|x(s)|^2)^{1/2},\quad x\in\mathcal{C}_b.$$

Lemma 2.1. *Assume that $x\in\mathcal{C}_b$. Then for $t\in J$, $x_t\in\mathcal{C}_v$. Moreover,*

$$l(E\|x(t)\|^2)^{1/2}\leqslant \|\varphi\|_{\mathcal{C}_v} + l\sup_{s\in[0,b]}(E|x(s)|^2)^{1/2}.$$

Definition 2.2 ([9])**.** A closed linear operator A is said to be sectorial if there are constants $w\in\mathbb{R}$, $\theta\in[\pi/2,\pi]$, $M>0$, such that the following two conditions are satisfied:

i) $\varrho(A)\subset\Sigma_{(\theta,w)}=\{\tilde\lambda\in\mathbb{C}\colon\ \tilde\lambda\neq w, |\arg(\tilde\lambda-w)|<\theta\}$,
ii) $\|R(\tilde\lambda,A)\|\leqslant M/|\tilde\lambda-w|$, $\tilde\lambda\in\Sigma_{(\theta,w)}$.

Definition 2.3 ([22])**.** Let A be a closed linear operator with the domain $D(A)$ defined in a Banach space H. Let $\varrho(A)$ be the resolvent set of A. We say that A is the generator of an α-resolvent family if there exist $w\geqslant 0$ and a strongly continuous function $S_\alpha\colon\ \mathbb{R}_+\to\mathcal{L}(H)$, where $\mathcal{L}(H)$ is a Banach space of all bounded linear operators from H into H and the corresponding norm is denoted by $\|\cdot\|$, such that $\{\tilde\lambda^\alpha\colon\ \operatorname{Re}\tilde\lambda>w\}\subset\varrho(A)$ and

$$(\tilde\lambda^\alpha I-A)^{-1}x=\int_0^\infty \mathrm{e}^{\tilde\lambda t}S_\alpha(t)x\,\mathrm{d}t,\quad \operatorname{Re}\tilde\lambda>w,\ x\in H,$$

where $S_\alpha(t)$ is called the α-resolvent family generated by A.

Definition 2.4 ([7]). Let A be a closed linear operator with the domain $D(A)$ defined in a Banach space H and $\alpha > 0$. We say that A is the generator of a solution operator if there exist $w \geqslant 0$ and a strongly continuous function $T_\alpha\colon \mathbb{R}_+ \to \mathcal{L}(H)$ such that $\{\tilde\lambda^\alpha\colon \operatorname{Re}\tilde\lambda > w\} \subset \varrho(A)$ and

$$\tilde\lambda^{\alpha-1}(\tilde\lambda^\alpha I - A)^{-1}x = \int_0^\infty \mathrm{e}^{\tilde\lambda t} T_\alpha(t)x\,\mathrm{d}t, \quad \operatorname{Re}\tilde\lambda > w,\ x \in H,$$

where $T_\alpha(t)$ is called the solution operator generated by A.

For more details on the α-resolvent family and solution operator, the reader may refer to [7], [9], [24] and references therein.

Definition 2.5 ([2]). The Caputo derivative of order α with the lower limit 0 for a function f can be written as

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}}\,\mathrm{d}s = I^{n-\alpha}f^n(t), \quad t > 0,\ 0 \leqslant n-1 < \alpha < n.$$

The Caputo derivative of a constant is equal to zero. The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given by

$$L\{D_t^\alpha f(t); \tilde\lambda\} = \tilde\lambda^\alpha \hat f(\tilde\lambda) - \sum_{k=0}^{n-1} \tilde\lambda^{\alpha-k-1} f^k(0), \quad n-1 \leqslant \alpha < n.$$

Lemma 2.6 ([7], [22]). *Let A be a sectorial operator. Then the unique solution of the linear fractional control problem*

$$\begin{aligned} D_t^\alpha x(t) &= Ax(t) + Bu(t), \quad t > t_0,\ t_0 \geqslant 0,\ 0 < \alpha < 1,\\ x(t) &= \varphi(t), \quad t \leqslant t_0, \end{aligned}$$

is given by

$$x(t) = T_\alpha(t-t_0)x(t_0) + \int_{t_0}^t S_\alpha(t-s)Bu(s)\,\mathrm{d}s,$$

where $T_\alpha(t) = (2\pi\mathrm{i})^{-1}\int_{\widehat{B}_r} \mathrm{e}^{\tilde\lambda t}\tilde\lambda^{\alpha-1}/(\tilde\lambda^\alpha - A)\,\mathrm{d}\tilde\lambda$, $S_\alpha(t) = (2\pi\mathrm{i})^{-1}\int_{\widehat{B}_r} \mathrm{e}^{\tilde\lambda t}/(\tilde\lambda^\alpha - A)\,\mathrm{d}\tilde\lambda$, *where $\widehat{B}_r$ denotes the Bromwich path, $S_\alpha(t)$ is the α-resolvent family, and $T_\alpha(t)$ is the solution operator generated by A.*

Let us now introduce the following operators. Define the operator $\Gamma_0^b\colon H \to H$ associated with the linear system of (1.1) (that is, in equation (1.1), $f = g = h = 0$) as

$$\Gamma_0^b = \int_0^b S_\alpha(b-s)BB^\star S_\alpha^\star(b-s)\,\mathrm{d}s, \quad R(\lambda, \Gamma_0^b) = (\lambda I + \Gamma_0^b)^{-1},$$

where $B^\star$ denotes the adjoint of B, $\|B\| = M_B$ and $S_\alpha(t)^\star$ is the adjoint of $S_\alpha(t)$.

Definition 2.7 (see [22], [26]). An $\mathfrak{F}_t$-adapted stochastic process $x\colon\ (-\infty,b]\to H$ is called a mild solution of the system (1.1) if $x_0=\varphi\in\mathcal{C}_v$ and the following conditions hold:

(i) $x(t)$ is $\mathcal{C}_v$-valued and the restriction of $x(\cdot)$ to the interval $(t_i,t_{i+1}]$, $i=1,2,\ldots,m$ is continuous,

(ii) for each $t\in J$, $x(t)$ satisfies the following integral equation:

$$
x(t)=\begin{cases}
\varphi(t), \qquad t\in(-\infty,0],\\
\displaystyle\int_0^t S_\alpha(t-s)Bu(s)\,\mathrm{d}s+\int_0^t S_\alpha(t-s)f(s,x_s)\,\mathrm{d}s\\
\displaystyle\quad+\int_0^t S_\alpha(t-s)g(s,x_s)\,\mathrm{d}W(s)\\
\displaystyle\quad+\int_0^t\int_Z S_\alpha(t-s)h(s,x_s,\eta)\widetilde{N}(\mathrm{d}s,\mathrm{d}\eta), \qquad t\in[0,t_1],\\
\displaystyle T_\alpha(t-t_i)(x(t_i^-)+I_i(x(t_i^-)))+\int_{t_i}^t S_\alpha(t-s)Bu(s)\,\mathrm{d}s\\
\displaystyle\quad+\int_{t_i}^t S_\alpha(t-s)f(s,x_s)\,\mathrm{d}s+\int_{t_i}^t S_\alpha(t-s)g(s,x_s)\,\mathrm{d}W(s)\\
\displaystyle\quad+\int_{t_i}^t\int_Z S_\alpha(t-s)h(s,x_s,\eta)\widetilde{N}(\mathrm{d}s,\mathrm{d}\eta), \quad t\in(t_i,t_{i+1}],\\
\quad i=1,2,\ldots,m,
\end{cases}
\tag{2.1}
$$

(iii) $\Delta x|_{t=t_i}=I_i(x(t_i^-))$, $i=1,2,\ldots,m$, the restriction of $x(\cdot)$ to the integral $[0,b)\setminus\{t_1,t_2,\ldots,t_m\}$ is continuous.

In general, let us denote $u(t)=B^\star S_\alpha^\star(b-t)R(\lambda,\Gamma_0^b)p(x(\cdot))$, where

$$
p(x(\cdot))=\begin{cases}
\displaystyle x_{t_1}-\int_0^{t_1} S_\alpha(t_1-s)f(s,x_s)\,\mathrm{d}s-\int_0^{t_1} S_\alpha(t_1-s)g(s,x_s)\,\mathrm{d}W(s)\\
\displaystyle\quad-\int_0^{t_1}\int_Z S_\alpha(t_1-s)h(s,x_s,\eta)\widetilde{N}(\mathrm{d}s,\mathrm{d}\eta), \qquad t\in[0,t_1],\\
\displaystyle x_{t_{i+1}}-T_\alpha(b-t_i)(x(t_i^-)+I_i(x(t_i^-)))-\int_{t_i}^b S_\alpha(b-s)f(s,x_s)\,\mathrm{d}s\\
\displaystyle\quad-\int_{t_i}^b S_\alpha(b-s)g(s,x_s)\,\mathrm{d}W(s)\\
\displaystyle\quad-\int_{t_i}^b\int_Z S_\alpha(b-s)h(s,x_s,\eta)\widetilde{N}(\mathrm{d}s,\mathrm{d}\eta),\\
\quad t\in(t_i,t_{i+1}],\ i=1,2,\ldots,m.
\end{cases}
\tag{2.2}
$$

Let $x(t;\varphi,u)$ denote the state value of system (1.1) at time t corresponding to the control $u\in L_2^{\mathfrak{F}}(J,U)$. In particular, the state of system (1.1) at $t=b, x(b;\varphi,u)$

is called the terminal state with control u. The set $\mathfrak{R}(b;\varphi,u) = \{x(b;\varphi,u)\colon u \in L_2^{\mathfrak{F}}(J,U)\}$ is called the reachable set of system (1.1).

Definition 2.8. System (1.1) is approximately controllable on J if $\overline{\mathfrak{R}(b;\varphi,u)} = L_2(\Omega,\mathfrak{F},H)$, where $\overline{\mathfrak{R}(b;\varphi,u)}$ is the closure of the reachable set.

To prove our main results, we need the following basic assumptions.

(H_1) If $\alpha \in (0,1)$ and $A \in \mathbb{A}^{\alpha}(\theta_0, w_0)$, then for any $x \in H$, $t > 0$, from Theorems 3.3 and 3.4 in [24], we have $\|T_\alpha(t)\| \leqslant M_T$ and $\|S_\alpha(t)\| \leqslant t^{\alpha-1}M_S$.

(H_2) The nonlinear functions f, g satisfy the Lipschitz condition and there exist positive constants M_f, M_g such that

$$\begin{aligned} E\|f(t,x)-f(t,y)\|_H^2 &\leqslant M_f\|x-y\|_{\mathcal{C}_v}^2, \\ E\|g(t,x)-g(t,y)\|_Q^2 &\leqslant M_g\|x-y\|_{\mathcal{C}_v}^2, \quad \text{for all } x,y \in \mathcal{C}_v,\ t \in J. \end{aligned}$$

(H_3) For each $i = 1,2,\ldots,m$, there exists $M_i > 0$ such that

$$E\|I_i(x)-I_i(y)\|_H^2 \leqslant M_i\|x-y\|_H^2 \quad \text{for all } x,y \in H.$$

(H_4) The nonlinear function h satisfies the Lipschitz condition and there exist positive constants M_h, L_h such that

$$\begin{aligned} &\int_Z E\|h(t,x,\eta)-h(t,y,\eta)\|_H^2\hat{\lambda}(\mathrm{d}\eta)\,\mathrm{d}t \leqslant M_h\|x-y\|_{\mathcal{C}_v}^2, \\ &\int_Z E\|h(t,x,\eta)-h(t,y,\eta)\|_H^4\hat{\lambda}(\mathrm{d}\eta)\,\mathrm{d}t \leqslant L_h\|x-y\|_{\mathcal{C}_v}^4, \quad \text{for all } x,y \in \mathcal{C}_v. \end{aligned}$$

Lemma 2.9 ([20])**.** *Let $\mathbb{H}$ be a Banach space. Let $\mathbb{E}$ be a bounded, closed, and convex subset of $\mathbb{H}$ and let Ψ_1, Ψ_2 be maps from $\mathbb{E}$ into $\mathbb{H}$ such that $\Psi_1 x + \Psi_2 x \in \mathbb{E}$, for every pair $x, y \in \mathbb{E}$. If Ψ_1 is contraction and Ψ_2 is compact and continuous, then the equation $\Psi_1 x + \Psi_2 x = x$ has a solution on $\mathbb{E}$.*

3. Existence of mild solutions

Taking into account the above notations, definitions and lemmas, we shall derive the existence of solutions for the nonlinear fractional stochastic system (1.1) by using the contraction mapping principle and Krasnoselskii's fixed-point theorem. The existence of solutions to system (1.1) is a natural premise to carry out the approximate controllability results.

Theorem 3.1. *If assumptions* (H_1)–(H_4) *are satisfied with* $A \in \mathbb{A}^\alpha(\theta_0, w_0)$, *then system* (1.1) *has a unique mild solution, provided that*

$$\begin{aligned}
(3.1)\qquad &\max_{1\leqslant i\leqslant m} 6\Big(1 + 5\frac{M_B^2M_S^4}{\lambda}\frac{(t_{i+1}-t_i)^{4\alpha-3}}{4\alpha-3}\Big)\\
&\times\bigg(M_T^2(1+M_i) + M_S^2 l\Big(\frac{(t_{i+1}-t_i)^{2\alpha}}{\alpha^2}M_f + \frac{(t_{i+1}-t_i)^{2\alpha-1}}{2\alpha-1}M_g\\
&\quad + \frac{(t_{i+1}-t_i)^{2\alpha}}{\alpha^2}[M_h + \sqrt{L_h}]\Big)\bigg) < 1\\
&\textit{with } \alpha \neq \frac{1}{2},\ \frac{3}{4}.
\end{aligned}$$

P r o o f. Define the operator $\Theta\colon \mathcal{C}_b \to \mathcal{C}_b$ by

$$(\Theta x)(t) = \begin{cases} \varphi(t), \qquad t \in (-\infty, 0],\\ \displaystyle\int_0^t S_\alpha(t-s)Bu(s)\,\mathrm{d}s + \int_0^t S_\alpha(t-s)f(s,x_s)\,\mathrm{d}s\\ \quad\displaystyle + \int_0^t S_\alpha(t-s)g(s,x_s)\,\mathrm{d}W(s)\\ \quad\displaystyle + \int_0^t\int_Z S_\alpha(t-s)h(s,x_s,\eta)\widetilde{N}(\mathrm{d}s,\mathrm{d}\eta), \qquad t\in[0,t_1],\\ \displaystyle T_\alpha(t-t_i)(x(t_i^-) + I_i(x(t_i^-))) + \int_{t_i}^t S_\alpha(t-s)Bu(s)\,\mathrm{d}s\\ \quad\displaystyle + \int_{t_i}^t S_\alpha(t-s)f(s,x_s)\,\mathrm{d}s + \int_{t_i}^t S_\alpha(t-s)g(s,x_s)\,\mathrm{d}W(s)\\ \quad\displaystyle + \int_{t_i}^t\int_Z S_\alpha(t-s)h(s,x_s,\eta)\widetilde{N}(\mathrm{d}s,\mathrm{d}\eta),\\ \quad t\in(t_i,t_{i+1}],\ i=1,2,\dots,m. \end{cases}$$

For $\varphi \in \mathcal{C}_v$, define

$$y(t) = \begin{cases} \varphi(t), & t\in(-\infty,0],\\ 0, & t\in J,\end{cases}$$

then $y_0 = \varphi$. Next we define the function

$$z(t) = \begin{cases} 0, & t\in(-\infty,0],\\ z(t), & t\in J,\end{cases}$$

for each $z \in C(J, \mathbb{R})$ with $z(0) = 0$. If $x(\cdot)$ satisfies (2.1), then $x(t) = y(t) + z(t)$ for $t \in J$, which implies $x_t = y_t + z_t$ for $t \in J$ and the function $z(\cdot)$ satisfies

$$z(t) = \begin{cases} \displaystyle\int_0^t S_\alpha(t-s)Bu(s)\,\mathrm{d}s + \int_0^t S_\alpha(t-s)f(s, y_s + z_s)\,\mathrm{d}s \\ \quad \displaystyle + \int_0^t S_\alpha(t-s)g(s, y_s + z_s)\,\mathrm{d}W(s) \\ \quad \displaystyle + \int_0^t\int_Z S_\alpha(t-s)h(s, y_s + z_s, \eta)\widetilde{N}(\mathrm{d}s, \mathrm{d}\eta), \qquad t \in [0, t_1], \\ \displaystyle T_\alpha(t-t_i)((y(t_i^-) + z(t_i^-)) + I_i(y(t_i^-) + z(t_i^-))) + \int_{t_i}^t S_\alpha(t-s)Bu(s)\,\mathrm{d}s \\ \quad \displaystyle + \int_{t_i}^t S_\alpha(t-s)f(s, y_s + z_s)\,\mathrm{d}s + \int_{t_i}^t S_\alpha(t-s)g(s, y_s + z_s)\,\mathrm{d}W(s) \\ \quad \displaystyle + \int_{t_i}^t\int_Z S_\alpha(t-s)h(s, y_s + z_s, \eta)\widetilde{N}(\mathrm{d}s, \mathrm{d}\eta), \\ \quad\quad t \in (t_i, t_{i+1}],\ i = 1, 2, \ldots, m. \end{cases}$$

Set $\mathcal{C}_b^0 = \{z \in \mathcal{C}_b,\ \text{such that } z_0 = 0\}$ and for any $z \in \mathcal{C}_b^0$, we have

$$\|z\|_{\mathcal{C}_b^0} = \|z_0\|_{\mathcal{C}_v} + \sup_{t\in J}(E\|z(t)\|^2)^{1/2} = \sup_{t\in J}(E\|z(t)\|^2)^{1/2},$$

thus $(\mathcal{C}_b^0, \|\cdot\|_{\mathcal{C}_b^0})$ is a Banach space. Define the operator $\Phi\colon \mathcal{C}_b^0 \to \mathcal{C}_b^0$ by

$$(\Phi z)(t) = \begin{cases} \displaystyle\int_0^t S_\alpha(t-s)Bu(s)\,\mathrm{d}s + \int_0^t S_\alpha(t-s)f(s, y_s + z_s)\,\mathrm{d}s \\ \quad \displaystyle + \int_0^t S_\alpha(t-s)g(s, y_s + z_s)\,\mathrm{d}W(s) \\ \quad \displaystyle + \int_0^t\int_Z S_\alpha(t-s)h(s, y_s + z_s, \eta)\widetilde{N}(\mathrm{d}s, \mathrm{d}\eta), \qquad t \in [0, t_1], \\ \displaystyle T_\alpha(t-t_i)(z(t_i^-) + I_i(z(t_i^-))) + \int_{t_i}^t S_\alpha(t-s)Bu(s)\,\mathrm{d}s \\ \quad \displaystyle + \int_{t_i}^t S_\alpha(t-s)f(s, y_s + z_s)\,\mathrm{d}s + \int_{t_i}^t S_\alpha(t-s)g(s, y_s + z_s)\,\mathrm{d}W(s) \\ \quad \displaystyle + \int_{t_i}^t\int_Z S_\alpha(t-s)h(s, y_s + z_s, \eta)\widetilde{N}(\mathrm{d}s, \mathrm{d}\eta), \\ \quad\quad t \in (t_i, t_{i+1}],\ i = 1, 2, \ldots, m. \end{cases}$$

In order to prove the existence results, it is enough to show that Φ has a unique fixed point. Let $z_1, z_2 \in \mathcal{C}_b^0$. Then for all $t \in [0, t_1]$, we have

$$
\begin{aligned}
&E\|(\Phi z_1)(t) - (\Phi z_2)(t)\|^2 \\
&\quad\leqslant 4E\left\|\int_0^t S_\alpha(t-s)(f(s, y_s + z_{1,s}) - f(s, y_s + z_{2,s}))\,\mathrm{d}s\right\|^2 \\
&\qquad + 4E\left\|\int_0^t S_\alpha(t-s)(g(s, y_s + z_{1,s}) - g(s, y_s + z_{2,s}))\,\mathrm{d}W(s)\right\|^2 \\
&\qquad + 4E\left\|\int_0^t\int_Z S_\alpha(t-s)(h(s, y_s + z_{1,s}, \eta) - h(s, y_s + z_{2,s}, \eta))\widetilde{N}(\mathrm{d}s, \mathrm{d}\eta)\right\|^2 \\
&\qquad + 4E\left\|\int_0^t S_\alpha(t-s)BB^\star S_\alpha^\star(t_1 - s)R(\lambda, \Gamma_0^{t_1})\right. \\
&\qquad \times \left\{\int_0^{t_1} S_\alpha(t_1 - s)(f(s, y_s + z_{1,s}) - f(s, y_s + z_{2,s}))\,\mathrm{d}s\right. \\
&\qquad + \int_0^{t_1} S_\alpha(t_1 - s)(g(s, y_s + z_{1,s}) - g(s, y_s + z_{2,s}))\,\mathrm{d}W(s) \\
&\qquad + \left.\left.\int_0^{t_1}\int_Z S_\alpha(t_1 - s)(h(s, y_s + z_{1,s}, \eta) - h(s, y_s + z_{2,s}, \eta))\widetilde{N}(\mathrm{d}s, \mathrm{d}\eta)\right\}\mathrm{d}s\right\|^2 \\
&\quad\leqslant 4M_S^2\int_0^t (t-s)^{\alpha-1}\,\mathrm{d}s\int_0^t (t-s)^{\alpha-1}[M_f\|z_{1,s} - z_{2,s}\|_{\mathcal{C}_v}^2]\,\mathrm{d}s \\
&\qquad + 4M_S^2\operatorname{Tr}(Q)\int_0^t (t-s)^{2(\alpha-1)}[M_g\|z_{1,s} - z_{2,s}\|_{\mathcal{C}_v}^2]\,\mathrm{d}s + 4M_S^2\int_0^t (t-s)^{\alpha-1}\,\mathrm{d}s \\
&\qquad \times \int_0^t\int_Z (t-s)^{\alpha-1}E\|h(s, y_s + z_{1,s}, \eta) - h(s, y_s + z_{2,s}, \eta)\|^2\hat{\lambda}(\mathrm{d}\eta)\,\mathrm{d}s \\
&\qquad + \int_0^t (t-s)^{\alpha-1}\left(\int_Z E\|h(s, y_s + z_{1,s}, \eta) - h(s, y_s + z_{2,s}, \eta)\|^4\hat{\lambda}(\mathrm{d}\eta)\right)^{1/2}\mathrm{d}s \\
&\qquad + 12\frac{M_B^2 M_S^4}{\lambda}\frac{t_1^{4\alpha-3}}{4\alpha-3} \\
&\qquad \times \left[M_S^2\int_0^{t_1}(t_1 - s)^{\alpha-1}\,\mathrm{d}s\int_0^{t_1}(t_1 - s)^{\alpha-1}[M_f\|z_{1,s} - z_{2,s}\|_{\mathcal{C}_v}^2]\,\mathrm{d}s\right. \\
&\qquad + M_S^2\operatorname{Tr}(Q)\int_0^{t_1}(t_1 - s)^{2(\alpha-1)}[M_g\|z_{1,s} - z_{2,s}\|_{\mathcal{C}_v}^2]\,\mathrm{d}s \\
&\qquad + M_S^2\int_0^{t_1}(t_1 - s)^{\alpha-1}\,\mathrm{d}s \\
&\qquad \times \int_0^{t_1}\int_Z (t_1 - s)^{\alpha-1}E\|h(s, y_s + z_{1,s}, \eta) - h(s, y_s + z_{2,s}, \eta)\|^2\hat{\lambda}(\mathrm{d}\eta)\,\mathrm{d}s \\
&\qquad + \left.\int_0^{t_1}(t_1 - s)^{\alpha-1}\left(\int_Z E\|h(s, y_s + z_{1,s}, \eta) - h(s, y_s + z_{2,s}, \eta)\|^4\hat{\lambda}(\mathrm{d}\eta)\right)^{1/2}\mathrm{d}s\right]
\end{aligned}
$$

$$\leqslant 4M_S^2 l\Big(1+3\frac{M_B^2M_S^4}{\lambda}\frac{t_1^{4\alpha-3}}{4\alpha-3}\Big)$$
$$\times\Big(\frac{t_1^{2\alpha}}{\alpha^2}M_f+\frac{t_1^{2\alpha-1}}{2\alpha-1}M_g+\frac{t_1^{2\alpha}}{\alpha^2}\big[M_h+\sqrt{L_h}\big]\Big)\|z_1-z_2\|^2_{\mathcal{C}_b^0}.$$

For $t\in(t_1,t_2]$, we have

$$E\|(\Phi z_1)(t)-(\Phi z_2)(t)\|^2\leqslant 6\Big(1+5\frac{M_B^2M_S^4}{\lambda}\frac{(t_2-t_1)^{4\alpha-3}}{4\alpha-3}\Big)$$
$$\times\Big(M_T^2(1+M_1)+M_S^2 l\Big(\frac{(t_2-t_1)^{2\alpha}}{\alpha^2}M_f+\frac{(t_2-t_1)^{2\alpha-1}}{2\alpha-1}M_g$$
$$+\frac{(t_2-t_1)^{2\alpha}}{\alpha^2}\big[M_h+\sqrt{L_h}\big]\Big)\Big)\|z_1-z_2\|^2_{\mathcal{C}_b^0}.$$

Similarly, when $t\in(t_i,t_{i+1}]$, $i=2,\dots,m$, we get

$$E\|(\Phi z_1)(t)-(\Phi z_2)(t)\|^2\leqslant 6\Big(1+5\frac{M_B^2M_S^4}{\lambda}\frac{(t_{i+1}-t_i)^{4\alpha-3}}{4\alpha-3}\Big)$$
$$\times\Big(M_T^2(1+M_i)+M_S^2 l\Big(\frac{(t_{i+1}-t_i)^{2\alpha}}{\alpha^2}M_f$$
$$+\frac{(t_{i+1}-t_i)^{2\alpha-1}}{2\alpha-1}M_g+\frac{(t_{i+1}-t_i)^{2\alpha}}{\alpha^2}\big[M_h+\sqrt{L_h}\big]\Big)\Big)\|z_1-z_2\|^2_{\mathcal{C}_b^0}.$$

Therefore, we conclude from (3.1) that Φ is a contraction mapping on $\mathcal{C}_b^0$. Then the mapping Φ has a unique fixed point $z(\cdot)\in\mathcal{C}_b^0$, which is the mild solution of (1.1). $\square$

Now, we prove another existence result of mild solutions for system (1.1), additionally we assume the following hypotheses:

(H_5) The nonlinear functions f,g are continuous and there exist continuous functions $\mu_1,\mu_2\colon J\to(0,\infty)$ such that

$$E\|f(t,x)\|_H^2\leqslant\mu_1(t)\|x\|^2_{\mathcal{C}_v},\quad E\|g(t,x)\|_Q^2\leqslant\mu_2(t)\|x\|^2_{\mathcal{C}_v},\quad\text{for all } x\in\mathcal{C}_v,\ t\in J,$$

where $\mu_1^\star=\sup\limits_{s\in[0,t]}\mu_1(t)$ and $\mu_2^\star=\sup\limits_{s\in[0,t]}\mu_2(t)$.

(H_6) The function $I_i\colon H\to H$, $i=1,2,\dots,m$, is continuous and there exists $\Lambda>0$ such that $\Lambda=\max\limits_{1\leqslant i\leqslant m,x\in\mathcal{C}_q}(E\|I_i(x)\|^2)$, where $\mathcal{C}_q=\{y\in\mathcal{C}_b^0,\ \|y\|^2_{\mathcal{C}_b^0}\leqslant q,\ q>0\}$.

(H_7) The nonlinear function h is continuous and there exist continuous functions $\mu_3,\mu_4\colon J\to(0,\infty)$ such that

$$\int_Z E\|h(t,x,\eta)\|^2\hat\lambda(\mathrm{d}\eta)\,\mathrm{d}t\leqslant\mu_3(t)\|x\|^2_{\mathcal{C}_v},$$
$$\int_Z E\|h(t,x,\eta)\|^4\hat\lambda(\mathrm{d}\eta)\,\mathrm{d}t\leqslant\mu_4(t)\|x\|^4_{\mathcal{C}_v},\quad\text{for every } x\in\mathcal{C}_v,$$

where $\mu_3^\star=\sup\limits_{s\in[0,t]}\mu_3(t)$ and $\mu_4^\star=\sup\limits_{s\in[0,t]}\mu_4(t)$.

(H_8) ([16], [23]) The linear system of (1.1) is approximately controllable on J, that is, equivalent to $\lambda R(\lambda,\Gamma_0^b)=\lambda(\lambda I+\Gamma_0^b)^{-1}\to 0$, as $\lambda\to 0$ in the strong operator topology.

The set $\mathcal{C}_q$ is clearly a bounded closed convex set in $\mathcal{C}_b^0$ for each q and for each $y\in\mathcal{C}_q$. From Lemma 2.1, we have

$$\begin{aligned}\|y_t+z_t\|^2_{\mathcal{C}_v}&\leqslant 2(\|y_t\|^2_{\mathcal{C}_v}+\|z_t\|^2_{\mathcal{C}_v})\leqslant 4\Big(l^2\sup_{t\in[0,t]}E\|y(t)\|^2_H+\|y_0\|^2_{\mathcal{C}_v}\Big)\\&\quad+4\Big(l^2\sup_{t\in[0,t]}E\|z(t)\|^2_H+\|z_0\|^2_{\mathcal{C}_v}\Big)\leqslant 4(l^2q+\|\varphi\|^2_{\mathcal{C}_v}).\end{aligned}$$

Theorem 3.2. *Assume that hypotheses* (H_1)–(H_7) *hold. Then the fractional stochastic control system* (1.1) *has at least one mild solution on* J, *provided that*

$$\begin{aligned}q&\geqslant 5\frac{M_B^2M_S^4}{\lambda}\frac{t_{i+1}^{4\alpha-3}}{4\alpha-3}\|x_{t_{i+1}}\|^2+5\Big(1+5\frac{M_B^2M_S^4}{\lambda}\frac{t_{i+1}^{4\alpha-3}}{4\alpha-3}\Big)M_T(q+\Lambda)\\&\quad+20M_S^2\Big(1+5\frac{M_B^2M_S^4}{\lambda}\frac{t_{i+1}^{4\alpha-3}}{4\alpha-3}\Big)\\&\quad\times\Big(\frac{t_{i+1}^{2\alpha}}{\alpha^2}\mu_1^\star+\frac{t_{i+1}^{2\alpha-1}}{2\alpha-1}\mu_2^\star+\frac{t_{i+1}^{2\alpha}}{\alpha^2}[\mu_3^\star+\sqrt{\mu_4^\star}]\Big)(\|\varphi\|^2_{\mathcal{C}_v}+l^2q)\end{aligned}$$

and

$$\begin{aligned}&\max_{1\leqslant i\leqslant m}\Bigg(4\Big(1+5\frac{M_B^2M_S^4}{\lambda}\frac{t_{i+1}^{4\alpha-3}}{4\alpha-3}\Big)\Big(M_T^2(1+M_i)+M_S^2l\frac{t_{i+1}^{2\alpha}}{\alpha^2}M_f\Big)\\&\quad+5\frac{M_B^2M_S^4}{\lambda}\frac{t_{i+1}^{4\alpha-3}}{4\alpha-3}M_S^2l\Big(\frac{t_{i+1}^{2\alpha-1}}{2\alpha-1}M_g+\frac{t_{i+1}^{2\alpha}}{\alpha^2}[M_h+\sqrt{L_h}]\Big)\Bigg)<1\\&\quad\textit{with } \alpha\neq\frac12,\ \frac34.\end{aligned}\tag{3.2}$$

P r o o f. Let $\Psi_1\colon\mathcal{C}_q\to\mathcal{C}_q$ and $\Psi_2\colon\mathcal{C}_q\to\mathcal{C}_q$ be defined as

$$(\Psi_1z)(t)=\begin{cases}\displaystyle\int_0^t S_\alpha(t-s)Bu(s)\,\mathrm{d}s+\int_0^t S_\alpha(t-s)f(s,y_s+z_s)\,\mathrm{d}s, & t\in[0,t_1],\\ \displaystyle T_\alpha(t-t_i)(z(t_i^-)+I_i(z(t_i^-)))+\int_{t_i}^t S_\alpha(t-s)Bu(s)\,\mathrm{d}s & \\ \displaystyle\quad+\int_{t_i}^t S_\alpha(t-s)f(s,y_s+z_s)\,\mathrm{d}s, & t\in(t_i,t_{i+1}],\ i=1,2,\dots,m,\end{cases}$$

$$
\begin{aligned}
(\Psi_2 z)(t) = &\int_0^t S_\alpha(t-s) g(s, y_s + z_s)\,\mathrm{d}W(s)\\
&+ \int_0^t \int_Z S_\alpha(t-s) h(s, y_s+z_s, \eta)\widetilde{N}(\mathrm{d}s, \mathrm{d}\eta),\ \ t \in J.
\end{aligned}
$$

In order to prove the existence of solutions to (1.1), it is enough to show that $\Psi_1 + \Psi_2$ has a fixed point on $\mathcal{C}_q$, which is then a solution of system (1.1). We prove that $\Psi_1 z + \Psi_2 z \in \mathcal{C}_q$ for $z \in \mathcal{C}_q$. For $t \in [0, t_1]$, we have

$$
\begin{aligned}
&E\|(\Psi_1 z)(t) + (\Psi_2 z)(t)\|^2\\
&\quad \leqslant 4E\bigg\|\int_0^t S_\alpha(t-s)BB^\star S_\alpha^\star(t_1-s)R(\lambda, \Gamma_0^{t_1})\\
&\qquad \times \bigg\{x_{t_1} - \int_0^{t_1} S_\alpha(t_1-s) f(s, y_s+z_s)\,\mathrm{d}s - \int_0^{t_1} S_\alpha(t_1-s) g(s, y_s+z_s)\,\mathrm{d}W(s)\\
&\qquad - \int_0^{t_1}\int_Z S_\alpha(t_1-s) h(s, y_s+z_s, \eta)\widetilde{N}(\mathrm{d}s, \mathrm{d}\eta)\bigg\}\,\mathrm{d}s\bigg\|^2\\
&\qquad + 4E\bigg\|\int_0^t S_\alpha(t-s) f(s, y_s+z_s)\,\mathrm{d}s\bigg\|^2 + 4E\bigg\|\int_0^t S_\alpha(t-s) g(s, y_s+z_s)\,\mathrm{d}W(s)\bigg\|^2\\
&\qquad + 4E\bigg\|\int_0^t\int_Z S_\alpha(t-s) h(s, y_s+z_s, \eta)\widetilde{N}(\mathrm{d}s, \mathrm{d}\eta)\bigg\|^2\\
&\quad \leqslant 4\frac{M_B^2 M_S^4}{\lambda}\frac{t_1^{4\alpha-3}}{4\alpha-3}\|x_{t_1}\|^2 + 16M_S^2\Big(1 + 4\frac{M_B^2 M_S^4}{\lambda}\frac{t_1^{4\alpha-3}}{4\alpha-3}\Big)\\
&\qquad \times \Big(\frac{t_1^{2\alpha}}{\alpha^2}\mu_1^\star + \frac{t_1^{2\alpha-1}}{2\alpha-1}\mu_2^\star + \frac{t_1^{2\alpha}}{\alpha^2}[\mu_3^\star + \sqrt{\mu_4^\star}]\Big)(\|\varphi\|_{\mathcal{C}_v}^2 + l^2 q) \leqslant q.
\end{aligned}
$$

For $t \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots, m$, we have

$$
\begin{aligned}
&E\|(\Psi_1 z)(t) + (\Psi_2 z)(t)\|^2 \leqslant 5E\|T_\alpha(t-t_i)(z(t_i^-) + I_i(z(t_i^-)))\|^2\\
&\quad + 5E\bigg\|\int_0^t S_\alpha(t-s)BB^\star S_\alpha^\star(t_{i+1}-s)R(\lambda, \Gamma_0^b)\\
&\quad \times \bigg\{x_{t_{i+1}} - T_\alpha(t_{i+1}-t_i)(z(t_i^-) + I_i(z(t_i^-))) - \int_0^{t_{i+1}} S_\alpha(t_{i+1}-s) f(s, y_s+z_s)\,\mathrm{d}s\\
&\quad - \int_0^{t_{i+1}} S_\alpha(t_{i+1}-s) g(s, y_s+z_s)\,\mathrm{d}W(s)\\
&\quad - \int_0^{t_{i+1}}\int_Z S_\alpha(t_{i+1}-s) h(s, y_s+z_s, \eta)\widetilde{N}(\mathrm{d}s, \mathrm{d}\eta)\bigg\}\,\mathrm{d}s\bigg\|^2\\
&\quad + 5E\bigg\|\int_0^t S_\alpha(t-s) f(s, y_s+z_s)\,\mathrm{d}s\bigg\|^2 + 5E\bigg\|\int_0^t S_\alpha(t-s) g(s, y_s+z_s)\,\mathrm{d}W(s)\bigg\|^2\\
&\quad + 5E\bigg\|\int_0^t\int_Z S_\alpha(t-s) h(s, y_s+z_s, \eta)\widetilde{N}(\mathrm{d}s, \mathrm{d}\eta)\bigg\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant 5\frac{M_B^2M_S^4}{\lambda}\frac{t_{i+1}^{4\alpha-3}}{4\alpha-3}\|x_{t_{i+1}}\|^2+5\Big(1+5\frac{M_B^2M_S^4}{\lambda}\frac{t_{i+1}^{4\alpha-3}}{4\alpha-3}\Big)M_T(q+\Lambda)\\
&\quad+20M_S^2\Big(1+5\frac{M_B^2M_S^4}{\lambda}\frac{t_{i+1}^{4\alpha-3}}{4\alpha-3}\Big)\\
&\quad\times\Big(\frac{t_{i+1}^{2\alpha}}{\alpha^2}\mu_1^\star+\frac{t_{i+1}^{2\alpha-1}}{2\alpha-1}\mu_2^\star+\frac{t_{i+1}^{2\alpha}}{\alpha^2}[\mu_3^\star+\sqrt{\mu_4^\star}]\Big)(\|\varphi\|_{\mathcal{C}_v}^2+l^2q)\leqslant q.
\end{aligned}
$$

Next, we prove that Ψ_1 is a contraction mapping. For any $z_1, z_2 \in \mathcal{C}_q$ and $t \in [0, t_1]$, we have

$$
\begin{aligned}
&E\|(\Psi_1z_1)(t)-(\Psi_1z_2)(t)\|^2\\
&\quad\leqslant 2E\bigg\|\int_0^t S_\alpha(t-s)(f(s,y_s+z_{1,s})-f(s,y_s+z_{2,s}))\,\mathrm{d}s\bigg\|^2\\
&\qquad+2E\bigg\|\int_0^t S_\alpha(t-s)BB^\star S_\alpha^\star(t_1-s)R(\lambda,\Gamma_0^{t_1})\\
&\qquad\times\bigg\{\int_0^{t_1} S_\alpha(t_1-s)(f(s,y_s+z_{1,s})-f(s,y_s+z_{2,s}))\,\mathrm{d}s\\
&\qquad+\int_0^{t_1} S_\alpha(t_1-s)(g(s,y_s+z_{1,s})-g(s,y_s+z_{2,s}))\,\mathrm{d}W(s)\\
&\qquad+\int_0^{t_1}\int_Z S_\alpha(t_1-s)(h(s,y_s+z_{1,s},\eta)-h(s,y_s+z_{2,s},\eta))\widetilde{N}(\mathrm{d}s,\mathrm{d}\eta)\bigg\}\,\mathrm{d}s\bigg\|^2\\
&\quad\leqslant\bigg(\Big(1+3\frac{M_B^2M_S^4}{\lambda}\frac{t_1^{4\alpha-3}}{4\alpha-3}\Big)2M_S^2l\frac{t_1^{2\alpha}}{\alpha^2}M_f+6\frac{M_B^2M_S^4}{\lambda}\frac{t_1^{4\alpha-3}}{4\alpha-3}M_S^2l\\
&\qquad\times\Big(\frac{t_1^{2\alpha-1}}{2\alpha-1}M_g+\frac{t_1^{2\alpha}}{\alpha^2}[M_h+\sqrt{L_h}]\Big)\bigg)\|z_1-z_2\|_{\mathcal{C}_b^0}^2.
\end{aligned}
$$

For any $z_1, z_2 \in \mathcal{C}_q$ and $t \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots, m$, we have

$$
\begin{aligned}
E\|(\Psi_1z_1)(t)-(\Psi_1z_2)(t)\|^2&\leqslant 4\|T_\alpha(t-t_i)\|^2E\|z_1(t_i^-)-z_2(t_i^-)\|^2\\
&\quad+4\|T_\alpha(t-t_i)\|^2E\|I_i(z_1(t_i^-))-I_i(z_2(t_i^-))\|^2\\
&\quad+4E\bigg\|\int_0^t S_\alpha(t-s)(f(s,y_s+z_{1,s})-f(s,y_s+z_{2,s}))\,\mathrm{d}s\bigg\|^2\\
&\quad+4E\bigg\|\int_0^t S_\alpha(t-s)BB^\star S_\alpha^\star(t_{i+1}-s)R(\lambda,\Gamma_0^b)\\
&\quad\times\Big\{T_\alpha(t_{i+1}-t_i)(z_1(t_i^-)-z_2(t_i^-))+T_\alpha(t_{i+1}-t_i)(I_i(z_1(t_i^-))-I_i(z_2(t_i^-)))
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^{t_{i+1}} S_\alpha(t_{i+1}-s)(f(s,y_s+z_{1,s})-f(s,y_s+z_{2,s}))\,\mathrm{d}s\\
&+\int_0^{t_{i+1}} S_\alpha(t_{i+1}-s)(g(s,y_s+z_{1,s})-g(s,y_s+z_{2,s}))\,\mathrm{d}W(s)\\
&+\int_0^{t_{i+1}}\int_Z S_\alpha(t_{i+1}-s)(h(s,y_s+z_{1,s},\eta)-h(s,y_s+z_{2,s},\eta))\widetilde{N}(\mathrm{d}s,\mathrm{d}\eta)\bigg\}\,\mathrm{d}s\bigg\|^2\\
\leqslant{}&\bigg(4\Big(1+5\frac{M_B^2M_S^4}{\lambda}\frac{t_{i+1}^{4\alpha-3}}{4\alpha-3}\Big)\Big(M_T^2(1+M_i)+M_S^2l\frac{t_{i+1}^{2\alpha}}{\alpha^2}M_f\Big)\\
&+5\frac{M_B^2M_S^4}{\lambda}\frac{t_{i+1}^{4\alpha-3}}{4\alpha-3}M_S^2l\Big(\frac{t_{i+1}^{2\alpha-1}}{2\alpha-1}M_g+\frac{t_{i+1}^{2\alpha}}{\alpha^2}[M_h+\sqrt{L_h}]\Big)\bigg)\|z_1-z_2\|^2_{\mathcal{C}_b^0}.
\end{aligned}
$$

By inequality (3.2), it follows that Ψ_1 is a contraction mapping for $t\in J$. Now, we prove that Ψ_2 is continuous and compact. First we show that Ψ_2 is continuous. Let $\{z^n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{C}_q$ with $\lim z^n\to z\in\mathcal{C}_q$. Since the functions g and h are continuous, for all $\varepsilon>0$, there exists N such that for $n>N$, we have $E\|g(s,y_s+z_s^n)-g(s,y_s+z_s)\|^2<\varepsilon$ and $E\|h(s,y_s+z_s^n,\eta)-h(s,y_s+z_s,\eta)\|^2<\varepsilon$. Now, for all $t\in J$, we obtain

$$
\begin{aligned}
&E\|(\Psi_2z^n)(t)-(\Psi_2z)(t)\|^2\\
&\leqslant 2E\bigg\|\int_0^t S_\alpha(t-s)(g(s,y_s+z_s^n)-g(s,y_s+z_s))\,\mathrm{d}W(s)\bigg\|^2\\
&\quad+2E\bigg\|\int_0^t\int_Z S_\alpha(t-s)(h(s,y_s+z_s^n,\eta)-h(s,y_s+z_s,\eta))\widetilde{N}(\mathrm{d}s,\mathrm{d}\eta)\bigg\|^2<\varepsilon.
\end{aligned}
$$

Now, we prove that $\Psi_2(\mathcal{C}_q)$ is equicontinuous. The functions $\{(\Psi_2z);z\in\mathcal{C}_q\}$ are equicontinuous at $t=0$. For any $z\in\mathcal{C}_q$ and $0<t_1<t_2\leqslant b$, we have

$$
\begin{aligned}
&E\|(\Psi_2z)(t_2)-(\Psi_2z)(t_1)\|^2\\
&\quad\leqslant 4E\bigg\|\int_0^{t_1}[S_\alpha(t_2-s)-S_\alpha(t_1-s)]g(s,y_s+z_s)\,\mathrm{d}W(s)\bigg\|^2\\
&\qquad+4E\bigg\|\int_{t_1}^{t_2} S_\alpha(t_2-s)g(s,y_s+z_s)\,\mathrm{d}W(s)\bigg\|^2\\
&\qquad+4E\bigg\|\int_0^{t_1}\int_Z[S_\alpha(t_2-s)-S_\alpha(t_1-s)]h(s,y_s+z_s,\eta)\widetilde{N}(\mathrm{d}s,\mathrm{d}\eta)\bigg\|^2\\
&\qquad+4E\bigg\|\int_{t_1}^{t_2}\int_Z S_\alpha(t_2-s)h(s,y_s+z_s,\eta)\widetilde{N}(\mathrm{d}s,\mathrm{d}\eta)\bigg\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant 16\,\mathrm{Tr}(Q)\int_0^{t_1}\|S_\alpha(t_2-s)-S_\alpha(t_1-s)\|^2\mu_2^\star(\|\varphi\|^2_{\mathcal{C}_v}+l^2q)\,\mathrm{d}s\\
&\quad+16\,\mathrm{Tr}(Q)M_S^2\frac{(t_2-t_1)^{2\alpha-1}}{2\alpha-1}\mu_2^\star(\|\varphi\|^2_{\mathcal{C}_v}+l^2q)\\
&\quad+16\int_0^{t_1}\|S_\alpha(t_2-s)-S_\alpha(t_1-s)\|^2(\mu_3^\star+\sqrt{\mu_4^\star})(\|\varphi\|^2_{\mathcal{C}_v}+l^2q)\,\mathrm{d}s\\
&\quad+16M_S^2\frac{(t_2-t_1)^{2\alpha}}{\alpha^2}(\mu_3^\star+\sqrt{\mu_4^\star})(\|\varphi\|^2_{\mathcal{C}_v}+l^2q).
\end{aligned}
$$

The above inequality tends to 0 by the continuity of the function $t\to\|S_\alpha(t)\|$ as $t_1\to t_2$. Therefore, the right-hand side of the above inequality tends to 0 as $t_1\to t_2$. This implies that $\{(\Psi_2 z); z\in\mathcal{C}_q\}$ is a family of equicontinuous functions. Finally, we prove the compactness of Ψ_2. To prove this, we first prove that the set $\{(\Psi_2 z); z\in\mathcal{C}_q\}$ is relatively compact in H. Subsequently, we show that $\{(\Psi_2 z); z\in\mathcal{C}_q\}$ is uniformly bounded. We have that

$$
E\|(\Psi_2 z)(t)\|^2\leqslant 8\Big(\mathrm{Tr}(Q)M_S^2\frac{b^{2\alpha-1}}{2\alpha-1}\mu_2^\star+M_S^2\frac{b^{2\alpha}}{\alpha^2}(\mu_3^\star+\sqrt{\mu_4^\star})\Big)(\|\varphi\|^2_{\mathcal{C}_v}+l^2q)<\infty.
$$

Therefore, the set $\{(\Psi_2 z); z\in\mathcal{C}_q\}$ is uniformly bounded. Hence, in view of Arzelà-Ascoli theorem, Ψ_2 is compact. Thus, the Krasnoselskii fixed-point theorem allows us to conclude that the system (1.1) has at least one mild solution on J. $\square$

4. Approximate controllability

Theorem 4.1. *Assume that the hypotheses* (H_1)–(H_8) *hold, the functions f,g, and h are uniformly bounded in H, $L(K,H)$, and H, respectively. Then the fractional stochastic control system* (1.1) *is approximately controllable on J.*

P r o o f. Let $x^\lambda(\cdot)$ be a fixed point of $\Psi_1+\Psi_2$. By using the stochastic Fubini theorem, any fixed point of $\Psi_1+\Psi_2$ is a mild solution of (1.1), if the control $u^\lambda(t)$ satisfies

$$
x^\lambda(b)=x_b-\lambda R(\lambda,\Gamma_0^b)p(x^\lambda(\cdot)), \tag{4.1}
$$

where

$$p(x^\lambda(\cdot)) = \begin{cases} x_{t_1} - \int_0^{t_1} S_\alpha(t_1 - s) f(s, x_s^\lambda)\,\mathrm{d}s - \int_0^{t_1} S_\alpha(t_1 - s) g(s, x_s^\lambda)\,\mathrm{d}W(s) \\ \quad - \int_0^{t_1}\int_Z S_\alpha(t_1 - s) h(s, x_s^\lambda, \eta)\widetilde{N}(\mathrm{d}s, \mathrm{d}\eta), \qquad t \in [0, t_1], \\ x_{t_{i+1}} - T_\alpha(b - t_i)(x^\lambda(t_i^-) + I_i(x^\lambda(t_i^-))) - \int_{t_i}^b S_\alpha(b - s) f(s, x_s^\lambda)\,\mathrm{d}s \\ \quad - \int_{t_i}^b S_\alpha(b - s) g(s, x_s^\lambda)\,\mathrm{d}W(s) \\ \quad - \int_{t_i}^b\int_Z S_\alpha(b - s) h(s, x_s^\lambda, \eta)\widetilde{N}(\mathrm{d}s, \mathrm{d}\eta), \\ \quad t \in (t_i, t_{i+1}],\ i = 1, 2, \ldots, m. \end{cases}$$

Moreover, by assumption, f, g, and h are uniformly bounded on J. Then there is a subsequence, still denoted by $\{f(s, x_s^\lambda)\}$, $\{g(s, x_s^\lambda)\}$ and $\{h(s, x_s^\lambda, \eta)\}$, which converges weakly to, say, $f(s), g(s)$ and $h(s, \eta)$ in $H, L(K, H)$, and H, respectively. Denote

$$\widehat{w} = \begin{cases} x_{t_1} - \int_0^{t_1} S_\alpha(t_1 - s) f(s)\,\mathrm{d}s - \int_0^{t_1} S_\alpha(t_1 - s) g(s)\,\mathrm{d}W(s) \\ \quad - \int_0^{t_1}\int_Z S_\alpha(t_1 - s) h(s, \eta)\widetilde{N}(\mathrm{d}s, \mathrm{d}\eta), \qquad t \in [0, t_1], \\ x_{t_{i+1}} - T_\alpha(b - t_i)(x(t_i^-) + I_i(x(t_i^-))) - \int_{t_i}^b S_\alpha(b - s) f(s)\,\mathrm{d}s \\ \quad - \int_{t_i}^b S_\alpha(b - s) g(s)\,\mathrm{d}W(s) - \int_{t_i}^b\int_Z S_\alpha(b - s) h(s, \eta)\widetilde{N}(\mathrm{d}s, \mathrm{d}\eta), \\ \quad t \in (t_i, t_{i+1}],\ i = 1, 2, \ldots, m. \end{cases}$$

It follows that for $t \in [0, t_1]$ and $t \in (t_i, t_{i+1}]$, we have

$$\begin{aligned} (4.2) \qquad E\|p(x^\lambda) - \widehat{w}\|^2 = E\bigg\| &\int_0^b S_\alpha(b - s)[f(s, x_s^\lambda) - f(s)]\,\mathrm{d}s \\ &+ \int_0^b S_\alpha(b - s)[g(s, x_s^\lambda) - g(s)]\,\mathrm{d}W(s) \\ &+ \int_0^b\int_Z S_\alpha(b - s)[h(s, x_s^\lambda, \eta) - h(s, \eta)]\widetilde{N}(\mathrm{d}s, \mathrm{d}\eta) \\ &+ T_\alpha(b - t_i)\big((x^\lambda(t_i^-) - x(t_i^-)) + I_i(x^\lambda(t_i^-) - x(t_i^-))\big)\bigg\|^2. \end{aligned}$$

By using the infinite-dimensional version of the Arzelà-Ascoli theorem, one can show that the operator $\hat{l}(\cdot) \to \int_0^\cdot S_\alpha(\cdot - s)\hat{l}(s)\,\mathrm{d}s\colon\ L_2(J; H) \to C(J; H)$ is compact. Hence, for all $t \in J$, we obtain that $E\|p(x^\lambda) - \widehat{w}\|^2 \to 0$ as $\lambda \to 0^+$. Moreover, from (4.1) we get

$$E\|x^\lambda(b) - x_b\|^2 \leqslant E\|\lambda R(\lambda, \Gamma_0^b)(\widehat{w})\|^2 + E\|\lambda R(\lambda, \Gamma_0^b)\|^2 E\|p(x^\lambda) - \widehat{w}\|^2.$$

It follows from hypothesis (H_8) and estimation of (4.2) that $E\|x^\lambda(b) - x_b\|^2 \to 0$ as $\lambda \to 0^+$. This proves the approximate controllability of the system (1.1). $\square$

5. Example

In this section, we consider some applications for our theoretical results. Let $H = L_2([0, \pi])$ and define the operator $A\colon H \to H$ by $Ay = y''$ with domain $D(A) = \{y \in H\colon y'' \in H,\ y(0) = y(\pi) = 0\}$. Then A generates an analytic semigroup $\{T(t),\ t \geqslant 0\}$ in H, given by $T(t)y = \sum\limits_{n=1}^{\infty} \mathrm{e}^{-n^2 t}\langle y, e_n\rangle e_n$, $y \in H$, where $e_n(x) = (2/\pi)^{1/2}\sin nx$, $n = 1, 2, \ldots$, is the orthogonal set of eigenvectors of A. From these expressions, it follows that $(T(t))_{t\geqslant 0}$ in H is uniformly bounded and compact semigroup, so $R(\tilde{\lambda}, A) = (\tilde{\lambda} - A)^{-1}$ is a compact operator for all $\tilde{\lambda} \in \varrho(A)$, that is, $A \in \mathbb{A}^\alpha(\theta_0, w_0)$. It follows from [24] that the α-resolvent operator $S_\alpha(t)$ and the solution operator $T_\alpha(t)$ satisfy the hypothesis (H_1). Define an infinite-dimensional space U by

$$U = \Big\{u\colon\ u = \sum_{n=2}^{\infty} u_n e_n \text{ with } \sum_{n=2}^{\infty} u_n^2 < \infty\Big\}.$$

The norm in U is defined by $\|u\|_U = \Big(\sum\limits_{n=2}^{\infty} u_n^2\Big)^{1/2}$.

Let B be the bounded linear operator from U to H defined by

$$Bu = 2u_2 e_1 + \sum_{n=2}^{\infty} u_n e_n \quad \text{for } u = \sum_{n=2}^{\infty} u_n e_n \in U.$$

Let $\{\hat{q}(t), t \in J\}$ be the Poisson point process (independent of the Brownian motion) taking values in the space $\mathbb{K} = [0, \infty)$ with a σ-finite intensity measure $\hat{\lambda}(\mathrm{d}\eta)$. Let us denote by $N(\mathrm{d}s, \mathrm{d}\eta)$ be the Poisson counting measure, which is induced by $\hat{q}(\cdot)$, and the compensating martingale measure by $\widetilde{N}(\mathrm{d}s, \mathrm{d}\eta) = N(\mathrm{d}s, \mathrm{d}\eta) - \hat{\lambda}(\mathrm{d}\eta)\,\mathrm{d}s$.

E x a m p l e 5.1. Consider the impulsive fractional stochastic partial differential equation with infinite delay and Poisson jumps in the following form:

$$
\begin{aligned}
(5.1)\qquad D_t^\alpha y(t,x) &= \frac{\partial^2}{\partial x^2} y(t,x) + \nu(t,x) + \int_{-\infty}^t a_1(t,x,s-t)a_2(y(s,x))\,\mathrm{d}s \\
&\quad + \Big(\int_{-\infty}^t \mathrm{e}^{4(s-t)} y(s,x)\,\mathrm{d}s\Big)\frac{\mathrm{d}\beta(t)}{\mathrm{d}t} \\
&\quad + \int_Z \eta\Big(\int_{-\infty}^t a_3(s-t)y(s,x)\,\mathrm{d}s\Big)\widetilde{N}(\mathrm{d}t,\mathrm{d}\eta), \\
&\quad 0 \leqslant x \leqslant \pi, \quad t \in J := [0,b],\ t \neq t_i, \\
&\quad y(t,0) = y(t,\pi) = 0, \\
&\quad y(t,x) = \varphi(t,x), \quad t \in (-\infty,0],\ x \in [0,\pi], \\
&\quad \Delta y(t_i)(x) = \int_{-\infty}^{t_i} d_i(t_i-s)y(s,x)\,\mathrm{d}s, \quad x \in [0,\pi],
\end{aligned}
$$

where $\beta(t)$ denotes a standard one-dimensional Wiener process in $H = L_2([0,\pi])$ defined on a stochastic space $(\Omega,\mathfrak{F},P)$, D_t^α is the Caputo fractional derivative of order $0<\alpha<1$, $0<t_1<t_2<\ldots<t_n<b$ are prefixed numbers, and $\varphi \in \mathcal{C}_v$. Define the bounded linear operator $B\colon U \to H$ by $Bu(t)(x) = \nu(t,x), 0 \leqslant x \leqslant \pi$, $u \in U$. Now, we present a special phase space $\mathcal{C}_v$. Let $v(s) = \mathrm{e}^{2s}$, $s<0$. Then $l = \int_{-\infty}^0 v(s)\,\mathrm{d}s = 1/2$. Let $\|\varphi\|_{\mathcal{C}_v} = \int_{-\infty}^0 v(s) \sup\limits_{s\leqslant\theta\leqslant 0} (E|\varphi(\theta)|^2)^{1/2}\,\mathrm{d}s$ (see [12]). For $(t,\varphi) \in J \times \mathcal{C}_v$, where $\varphi(\theta)(x) = \varphi(\theta,x)$, $(\theta,x) \in (-\infty,0]\times[0,\pi]$, and define the functions $f\colon J\times\mathcal{C}_v \to H$, $g\colon J\times\mathcal{C}_v \to L_Q(H)$ and $h\colon J\times\mathcal{C}_v\to H$ for the infinite delay as follows:

$$
\begin{aligned}
f(t,\varphi)(x) &= \int_{-\infty}^0 a_1(t,x,\theta)a_2(\varphi(\theta)(x))\,\mathrm{d}\theta, \\
g(t,\varphi)(x) &= \int_{-\infty}^0 \mathrm{e}^{4\theta}(\varphi(\theta)(x))\,\mathrm{d}\theta, \\
h(t,\varphi)(x) &= \int_{-\infty}^0 a_3(\theta)(\varphi(\theta)(x))\,\mathrm{d}\theta, \\
I_i(\varphi)(x) &= \int_{-\infty}^0 d_i(-\theta)(\varphi(\theta)(x))\,\mathrm{d}\theta.
\end{aligned}
$$

Moreover, we assume that

a) the function $a_1(t,x,\theta) \geqslant 0$ is continuous in $J\times[0,\pi]\times(-\infty,0]$ and

$$\int_{-\infty}^0 a_1(t,x,\theta)\,\mathrm{d}\theta = p_1(t,x) < \infty,$$

b) the function $a_2(\cdot)$ is continuous, $0 \leqslant a_2(z(\theta,x)) \leqslant \Phi(\int_{-\infty}^0 \mathrm{e}^{2s}\|z(s,\cdot)\|_{L_2}\,\mathrm{d}s)$ for $(\theta,x) \in (-\infty,0]\times[0,\pi]$, where $\Phi(\cdot)\colon [0,\infty)\to(0,\infty)$ is continuous and nondecreasing.

Thus under the hypotheses as above, we have

$$\begin{aligned}
E\|f(t,\varphi)\|_{L_2} &= \biggl[\int_0^\pi \biggl\{\int_{-\infty}^0 a_1(t,x,\theta)a_2(\varphi(\theta)(x))\,\mathrm{d}\theta\biggr\}^2\,\mathrm{d}x\biggr]^{1/2}\\
&\leqslant \biggl[\int_0^\pi \biggl\{\int_{-\infty}^0 a_1(t,x,\theta)\Phi\biggl(\int_{-\infty}^0 \mathrm{e}^{2s}\|\varphi(s)(\cdot)\|_{L_2}\,\mathrm{d}s\biggr)\,\mathrm{d}\theta\biggr\}^2\,\mathrm{d}x\biggr]^{1/2}\\
&\leqslant \biggl[\int_0^\pi \biggl\{\int_{-\infty}^0 a_1(t,x,\theta)\Phi\biggl(\int_{-\infty}^0 \mathrm{e}^{2s}\sup\|\varphi(s)(\cdot)\|_{L_2}\,\mathrm{d}s\biggr)\,\mathrm{d}\theta\biggr\}^2\,\mathrm{d}x\biggr]^{1/2}\\
&\leqslant \biggl[\int_0^\pi \biggl(\int_{-\infty}^0 a_1(t,x,\theta)\,\mathrm{d}\theta\biggr)^2\,\mathrm{d}x\biggr]^{1/2}\Phi(\|\varphi\|_{\mathcal{C}_v})\\
&\leqslant \biggl(\int_0^\pi p_1^2(t,x)\,\mathrm{d}x\biggr)^{1/2}\Phi(\|\varphi\|_{\mathcal{C}_v})\\
&\leqslant \mathcal{P}(t)\Phi(\|\varphi\|_{\mathcal{C}_v}),\quad \text{where } \mathcal{P}(t) = \biggl(\int_0^\pi p_1^2(t,x)\,\mathrm{d}x\biggr)^{1/2}.\\
E\|g(t,\varphi)\|_{L_2} &= \biggl[\int_0^\pi \biggl\{\int_{-\infty}^0 \mathrm{e}^{4\theta}\varphi(\theta)(x)\,\mathrm{d}\theta\biggr\}^2\,\mathrm{d}x\biggr]^{1/2}\\
&\leqslant \biggl\{\int_{-\infty}^0 \mathrm{e}^{4\theta}\,\mathrm{d}\theta\biggr\}^{1/2}\biggl[\int_0^\pi\int_{-\infty}^0 \mathrm{e}^{4\theta}\varphi^2(\theta)(x)\,\mathrm{d}\theta\,\mathrm{d}x\biggr]^{1/2}\\
&\leqslant \frac{1}{2}\biggl\{\int_{-\infty}^0 \mathrm{e}^{4\theta}\int_0^\pi \varphi^2(\theta)(x)\,\mathrm{d}x\,\mathrm{d}\theta\biggr\}^{1/2}\\
&\leqslant \frac{1}{2}\biggl\{\int_{-\infty}^0 \mathrm{e}^{4\theta}\|\varphi(\theta)\|_{L_2}^2\,\mathrm{d}\theta\biggr\}^{1/2}\\
&\leqslant \frac{1}{2}\biggl\{\int_{-\infty}^0 \mathrm{e}^{4\theta}\Bigl[\sup_{\theta\leqslant s\leqslant 0}\|\varphi(s)\|_{L_2}\Bigr]^2\,\mathrm{d}\theta\biggr\}^{1/2}\\
&\leqslant \frac{1}{2}\int_{-\infty}^0 \mathrm{e}^{2\theta}\sup_{\theta\leqslant s\leqslant 0}\|\varphi(s)\|_{L_2}\,\mathrm{d}\theta\\
&\leqslant \frac{1}{2}\|\varphi\|_{\mathcal{C}_v}.
\end{aligned}$$

Therefore, the nonlinear functions f, g satisfy the hypothesis (H_5). Similarly, the functions I_i, h also satisfy the hypotheses (H_6), (H_7), assuming that $\int_Z \eta^2\hat{\lambda}(\mathrm{d}\eta) < \infty$, $\int_Z \eta^4\hat{\lambda}(\mathrm{d}\eta) < \infty$. Following the same argument as in [16], [23], we can prove that (H_8) is valid and that the corresponding linear system (5.1) is approximately controllable on J. Then, we can rewrite the system (5.1) in the abstract form of (1.1). All conditions stated in Theorems 3.2 and 4.1 are satisfied, therefore the system (5.1) is approximately controllable on J.

Example 5.2. Consider the fractional stochastic partial differential equation with infinite delay and Poisson jumps in the following form:

$$
\begin{aligned}
(5.2)\qquad D_t^{\alpha} y(t,x) &= \frac{\partial^2}{\partial x^2} y(t,x) + Bu(t,x) + \left(\int_{-\infty}^{0} \hat{a}(s)\sin y(t+s,x)\,\mathrm{d}s\right)\frac{\mathrm{d}\beta(t)}{\mathrm{d}t} \\
&\quad + \int_Z \eta\left(\int_{-\infty}^{0} \check{a}\mathrm{e}^{\xi s} y(t+s,x)\,\mathrm{d}s\right)\widetilde{N}(\mathrm{d}t,\mathrm{d}\eta), \\
&\quad 0 \leqslant x \leqslant \pi,\ t \in J := [0,b], \\
&\quad y(t,0) = y(t,\pi) = 0, \\
&\quad y(t,x) = \varphi(t,x), \quad t \in (-\infty,0],\ x \in [0,\pi],
\end{aligned}
$$

where D_t^{α} is the Caputo fractional derivative of order $0 < \alpha < 1$. Define the operators $g\colon J \times \mathcal{C}_v \to L_Q(H)$ and $h\colon J \times \mathcal{C}_v \to H$ for the infinite delay as follows:

$$
\begin{aligned}
g(t,\varphi)(x) &= \int_{-\infty}^{0} \hat{a}(\theta)\sin(\varphi(\theta)(x))\,\mathrm{d}\theta, \\
h(t,\varphi)(x) &= \int_{-\infty}^{0} \check{a}\mathrm{e}^{\xi\theta}\varphi(\theta)(x)\,\mathrm{d}\theta,
\end{aligned}
$$

where $\check{a} \geqslant 0, \xi > 0$ and $\int_{-\infty}^{0} \|\hat{a}(\theta)\|^2\,\mathrm{d}\theta < \infty$. Moreover,

$$
\begin{aligned}
E\|g(t,\varphi_1) - g(t,\varphi_2)\|^2 &\leqslant \int_{-\infty}^{0} \|\hat{a}(\theta)\|^2\,\mathrm{d}\theta E\|\varphi_1 - \varphi_2\|_{\mathcal{C}_v}^2, \\
\text{and}\quad E\|g(t,\varphi_1)\|^2 &\leqslant \int_{-\infty}^{0} \|\hat{a}(\theta)\|^2\,\mathrm{d}\theta \quad \text{for any } \varphi_1,\varphi_2 \in \mathcal{C}_v.
\end{aligned}
$$

Therefore, the nonlinear functions g, h satisfy the hypotheses (H_2), (H_4). Then, the system (5.2) can be written in the abstract form (1.1), by setting $f = 0$ and without the impulsive term. All the conditions stated in Theorems 3.1 and 4.1 have been satisfied for the system (5.2) and so, the system (5.2) is approximately controllable.

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