Dingyong Bai; Yuming Chen

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GLOBAL CONTINUUM OF POSITIVE SOLUTIONS FOR DISCRETE p-LAPLACIAN EIGENVALUE PROBLEMS

Dingyong Bai, Guangzhou, Yuming Chen, Waterloo

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Abstract. We discuss the discrete p-Laplacian eigenvalue problem,

$$\begin{cases} \Delta(\varphi_p(\Delta u(k-1))) + \lambda a(k) g(u(k)) = 0, & k \in \{1, 2, \dots, T\}, \\ u(0) = u(T+1) = 0, \end{cases}$$

where $T > 1$ is a given positive integer and $\varphi_p(x) := |x|^{p-2}x$, $p > 1$. First, the existence of an unbounded continuum $\mathcal{C}$ of positive solutions emanating from $(\lambda, u) = (0, 0)$ is shown under suitable conditions on the nonlinearity. Then, under an additional condition, it is shown that the positive solution is unique for any $\lambda > 0$ and all solutions are ordered. Thus the continuum $\mathcal{C}$ is a monotone continuous curve globally defined for all $\lambda > 0$.

Keywords: discrete p-Laplacian eigenvalue problem; positive solution; continuum; Picone-type identity; lower and upper solutions method

MSC 2010: 39A12, 39A10, 34B09

1. Introduction

Let $\mathbb{Z}$ be the set of all integers. For a, $b \in \mathbb{Z}$ with $a < b$, define $[a, b]_{\mathbb{Z}} = \{a, a+1, a+2, \dots, b\}$. Let $T > 1$ be a given positive integer. We consider the following discrete p-Laplacian eigenvalue problem

$$\begin{cases} \Delta(\varphi_p(\Delta u(k-1))) + \lambda a(k) g(u(k)) = 0, & k \in [1, T]_{\mathbb{Z}}, \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.1}$$

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where $\varphi_p(x) := |x|^{p-2}x$, $p > 1$, $\Delta u(k) = u(k+1) - u(k)$, and λ is a nonnegative parameter. We assume that

(A1) $a\colon [1,T]_{\mathbb{Z}} \to (0,\infty)$ and $g\colon \mathbb{R}^+ = [0,\infty) \to (0,\infty)$ is continuous;

(A2) $\lim\limits_{u\to\infty} g(u)/\varphi_p(u) = 0$;

(A3) $g(u)/\varphi_p(u)$ is strictly decreasing on $(0,\infty)$.

Recently, solutions (especially positive ones) of discrete p-Laplacian boundary value problems have been widely studied (see, for example, [1], [4], [7], [11], [12], [14], [15], [17], and the references therein). However, there is no report on the global structure of solution sets of discrete p-Laplacian boundary value problems. The purpose of this paper is to study the global structure of the continuum $\mathcal{C}$ of positive solutions of (1.1).

Two results are established in this paper. The first one is that, under conditions (A1) and (A2), the continuum $\mathcal{C}$ emanates from $(\lambda, u) = (0,0)$ and can be extended to $\lambda = \infty$. This means that (1.1) has at least one positive solution for any $\lambda > 0$. The second result is that, under conditions (A1)–(A3), the positive solution of (1.1) is unique for any $\lambda > 0$ and all solutions are ordered. Therefore, the continuum $\mathcal{C}$ is a monotone continuous curve globally defined for all $\lambda > 0$. Our proofs are based on an existence theorem of a global continuum of solutions to an operator equation $T(\lambda, u) = u$, the lower and upper solutions method, and the Picone-type identity for discrete p-Laplacian operators due to Řehák [13].

This study is motivated by the results of Kim and Shi [8] and Bai and Xu [3]. In [8], the global continuum and three positive solutions of a differential p-Laplacian boundary value problem were studied. The results in [8] demonstrate the rich structure of the solution set of one-dimensional p-Laplacian eigenvalue problems. In [3], Bai and Xu established a result associated with lower and upper solutions for discrete φ-Laplacian boundary value problems and generalized the result on existence of three positive solutions of [8]. In [8], Kim and Shi proved the uniqueness of positive solutions by the generalized Picone identity for one-dimensional p-Laplacian operators due to Jaroš and Kusano [6], [9]. In this paper, the proof of uniqueness of positive solutions is based on the discrete Picone-type identity due to Řehák [13]. However, the discussion is much more complicated due to the discrete structure of difference p-Laplacian operators, which is demonstrated in the proof of Theorem 3.4.

The remaining part of this paper is organized as follows. In Section 2, we give some preliminary results. These results are crucial in the development of Section 3, where we present and prove the main results of this paper.

2. Preliminary results

Note that (1.1) is a special case of the φ-Laplacian boundary problem (3) with $\varphi(x) = \varphi_p(x)$ in Bai [2]. Some necessary results on solutions to (1.1) for the discussion in Section 3 are summarized below. We refer to [2] for the details.

Let $\mathbb{R}$ be the set of all real numbers and $E = \{u\colon [0, T+1]_{\mathbb{Z}} \to \mathbb{R}^{T+2}\}$ be equipped with the norm $\|u\| = \max\limits_{t\in[0,T+1]_{\mathbb{Z}}} |u(t)|$ for $u \in E$. Given $u, v \in E$, we say that $u \leqslant v$ if $u(k) \leqslant v(k)$ holds for all $k \in [0, T+1]_{\mathbb{Z}}$, and that $u \prec v$ if $u \leqslant v$ and $u(k) < v(k)$ for $k \in [1, T]_{\mathbb{Z}}$.

The function $\alpha \in E$ is called a lower solution of (1.1) if

$$\begin{cases} \Delta(\varphi_p(\Delta\alpha(k-1))) + \lambda a(k) g(\alpha(k)) \geqslant 0, & k \in [1, T]_{\mathbb{Z}}, \\ \alpha(0) \leqslant 0, \quad \alpha(T+1) \leqslant 0. \end{cases}$$

If the first inequality above is strict, then α is called a strict lower solution of (1.1). The upper solution and the strict upper solution of (1.1) can be defined similarly by reversing the above inequalities.

Lemma 2.1 ([3], [5]). *Assume that* (1.1) *has a lower solution α and an upper solution β such that $\alpha \leqslant \beta$. Then the problem* (1.1) *has at least one solution u which satisfies $\alpha \leqslant u \leqslant \beta$. Moreover, if α and β are strict lower solution and strict upper solution, respectively, then $\alpha \prec \beta$.*

A function u of integer variable is said to be concave on $[a, b]_{\mathbb{Z}}$ if $\Delta^2 u(k-1) \leqslant 0$ for all $k \in [a+1, b-1]_{\mathbb{Z}}$. Moreover, if $\Delta^2 u(k-1) < 0$ for all $k \in [a+1, b-1]_{\mathbb{Z}}$, then u is said to be strictly concave on $[a, b]_{\mathbb{Z}}$.

Let (A1) hold. If $u \in E$ is a solution of (1.1) for some $\lambda > 0$, then the following statements are true (see Bai [2] for the detail).

(i) u is strictly concave on $[0, T+1]_{\mathbb{Z}}$ and $u(k) > 0$ for all $k \in [1, T]_{\mathbb{Z}}$.

(ii) u satisfies

$$u(k) = \sum_{s=1}^{k} \varphi_p^{-1}\Big(C_1 + \lambda \sum_{l=s}^{T} a(l) g(u(l))\Big) = \sum_{s=k}^{T} \varphi_p^{-1}\Big(C_2 + \lambda \sum_{l=1}^{s} a(l) g(u(l))\Big) \tag{2.1}$$

for $k \in [0, T+1]_{\mathbb{Z}}$, where C_1 and C_2 satisfy

$$\sum_{k=1}^{T+1} \varphi_p^{-1}\Big(C_1 + \lambda \sum_{l=k}^{T} a(l) g(u(l))\Big) = 0 \tag{2.2}$$

and

$$\sum_{k=0}^{T} \varphi_p^{-1}\Big(C_2 + \lambda \sum_{l=1}^{k} a(l)g(u(l))\Big) = 0, \tag{2.3}$$

respectively.

(iii) Suppose $\|u\| = u(k^*)$ for some $k^* \in [1,T]_{\mathbb{Z}}$. Then

$$C_1 + \lambda \sum_{l=k^*}^{T} a(l)g(u(l)) \geqslant 0 \tag{2.4}$$

and

$$C_2 + \lambda \sum_{l=1}^{k^*} a(l)g(u(l)) \geqslant 0. \tag{2.5}$$

Let $K = \{u \in E\colon\ u$ is nonnegative and concave on $[0,T+1]_{\mathbb{Z}}\}$, a cone in E. With the help of (2.1), define $T\colon\ \mathbb{R}^+ \times K \to E$ by

$$T(\lambda,u)(k) = \sum_{s=1}^{k} \varphi_p^{-1}\Big(C_1 + \lambda \sum_{l=s}^{T} a(l)g(u(l))\Big), \quad k \in [0,T+1]_{\mathbb{Z}}.$$

Then $T\colon\ \mathbb{R}^+ \times K \to K$ is continuous. It is easy to see that, for $\lambda > 0$, u is a positive solution of (1.1) if and only if $u \in K \setminus \{0\}$ is a fixed point of $T(\lambda,\cdot)$.

Since $T(0,u) = 0$ and $T(\lambda,0) \neq 0$ for all $\lambda > 0$, applying a well-known result on the existence of a global continuum of solutions to $T(\lambda,u) = u$ in Zeidler [16], we have the following result.

Lemma 2.2. *Assume that* (A1) *holds. Then there exists an unbounded continuum $\mathcal{C}$ of positive solutions for* (1.1) *emanating from* $(0,0)$ *in* $\mathbb{R}^+ \times K$.

Finally, we present an identity satisfied by solutions of (1.1), which is a simplified version of the generalized Picone identity due to Řehák [13].

Lemma 2.3. *Let $u \in E$ and $v \in E$ be two positive solutions of* (1.1) *for some $\lambda > 0$. Then*

$$\begin{aligned} \Delta\Big\{ &\frac{u(k-1)}{\varphi_p(v(k-1))}[\varphi_p(v(k-1))\varphi_p(\Delta u(k-1)) - \varphi_p(u(k-1))\varphi_p(\Delta v(k-1))]\Big\} \\ &= -\lambda a(k)\Big[\frac{g(u(k))}{\varphi_p(u(k))} - \frac{g(v(k))}{\varphi_p(v(k))}\Big]|u(k)|^p + G(u,v)(k) \end{aligned} \tag{2.6}$$

for $k \in [2,T]_{\mathbb{Z}}$, *where*

$$
(2.7) \qquad G(u,v)(k) = |\Delta u(k-1)|^p - \frac{\varphi_p(\Delta v(k-1))}{\varphi_p(v(k))}|u(k)|^p + \frac{\varphi_p(\Delta v(k-1))}{\varphi_p(v(k-1))}|u(k-1)|^p \geqslant 0
$$

for $k \in [2,T]_{\mathbb{Z}}$ *and* $G(u,v)(k) = 0$ *if and only if*

$$
\Delta u(k-1) = u(k-1)\frac{\Delta v(k-1)}{v(k-1)}.
$$

3. Main results

Let $\mathcal{S}$ be the set of positive solutions of (1.1) in $\mathbb{R}^+ \times K$. Denote

$$
\underline{a} := \min_{k\in[1,T]_{\mathbb{Z}}} a(k) \qquad \text{and} \qquad \overline{a} := \max_{k\in[1,T]_{\mathbb{Z}}} a(k).
$$

Lemma 3.1. *Assume that* (A1) *holds. If there exists a sequence* $\{(\lambda_n,u_n)\}_{n=1}^{\infty} \in \mathcal{S}$ *such that* $\lambda_n \to \infty$ *as* $n \to \infty$, *then* $\|u_n\| \to \infty$ *as* $n \to \infty$.

Proof. By way of contradiction, suppose that there exists $M > 0$ such that $\|u_n\| \leqslant M$ for all n. Passing to a subsequence of $\{u_n\}_{n=1}^{\infty}$ if necessary, we can assume that $\{u_n\}_{n=1}^{\infty}$ is convergent. Since $g\colon \mathbb{R}^+ \to (0,\infty)$ is continuous, there exists $\delta > 0$ such that $g(u_n(k)) \geqslant \delta$ for $k \in [1,T]_{\mathbb{Z}}$. For each n, let $\|u_n\| = u_n(k_n)$ for some $k_n \in [1,T]_{\mathbb{Z}}$. We distinguish two cases to finish the proof.

Case 1. There is a subsequence of $\{k_n\}_{n=1}^{\infty}$, say itself, whose all terms are equal to 1. Then it follows from (2.1) and (2.5) that

$$
\begin{aligned}
u_n(T) &= \varphi_p^{-1}\Big(C_2^{(n)} + \lambda_n\sum_{l=1}^{T} a(l)g(u_n(l))\Big)\\
&= \varphi_p^{-1}\Big(C_2^{(n)} + \lambda_n a(1)g(u_n(1)) + \lambda_n\sum_{k=2}^{T} a(k)g(u_n(k))\Big)\\
&\geqslant \varphi_p^{-1}\Big(\lambda_n\sum_{k=2}^{T} a(k)g(u_n(k))\Big)\\
&\geqslant \varphi_p^{-1}(\lambda_n\underline{a}\delta(T-1)) \to \infty \quad \text{as } n\to\infty,
\end{aligned}
$$

a contradiction to $\|u_n\| \leqslant M$. Here $C_2^{(n)}$ satisfies (2.3) with λ and u being replaced by λ_n and u_n, respectively.

Case 2. There is no subsequence of $\{k_n\}_{n=1}^{\infty}$whose all terms are equal to 1. In this case, without loss of generality, we assume that $k_n > 1$. Similarly, with (2.1) and (2.4), we get

$$\begin{aligned}
u_n(1) &= \varphi_p^{-1}\Bigl(C_1^{(n)} + \lambda_n \sum_{l=1}^{T} a(l)g(u_n(l))\Bigr)\\
&= \varphi_p^{-1}\Bigl(C_1^{(n)} + \lambda_n \Bigl(\sum_{k=1}^{k_n-1} + \sum_{k=k_n}^{T}\Bigr) a(k)g(u_n(k))\Bigr)\\
&\geqslant \varphi_p^{-1}\Bigl(\lambda_n \sum_{k=1}^{k_n-1} a(k)g(u_n(k))\Bigr)\\
&\geqslant \varphi_p^{-1}(\lambda_n \underline{a}\delta(k_n-1))\\
&\geqslant \varphi_p^{-1}(\lambda_n\ \underline{a}\delta) \to \infty \quad \text{as } n \to \infty,
\end{aligned}$$

again a contradiction to $\|u_n\| \leqslant M$. Here $C_1^{(n)}$ satisfies (2.2) with λ and u being replaced by λ_n and u_n, respectively.

In summary, in either case, we have a contradiction. This completes the proof. $\square$

Lemma 3.2. *Assume that* (A1) *and* (A2) *hold. If there exists a sequence* $\{(\lambda_n, u_n)\}_{n=1}^{\infty} \in \mathcal{S}$ *such that* $\|u_n\| \to \infty$ *as* $n \to \infty$, *then* $\lambda_n \to \infty$ *as* $n \to \infty$.

Proof. By way of contradiction, assume that there exists $L > 0$ such that $\lambda_n \leqslant L$ for all n. Let $\varepsilon = 1/(2T^pL\overline{a})$. Then, by (A2), there exists $N_\varepsilon > 0$ such that $g(u) < \varepsilon\varphi_p(u)$ for all $u > N_\varepsilon$. Denote $M_\varepsilon = \max\limits_{0\leqslant u\leqslant N_\varepsilon} g(u)$, and $A_n = \{k \in [1,T]\colon\ u_n(k) \leqslant N_\varepsilon\}$ and $B_n = \{k \in [1,T]\colon\ u_n(k) > N_\varepsilon\}$ for each n. Moreover, for each n, assume $\|u_n\| = u_n(k_n)$ for some $k_n \in [1,T]_{\mathbb{Z}}$. Using (2.1), we have

$$\begin{aligned}
\|u_n\| = u_n(k_n) &= \sum_{s=1}^{k_n} \varphi_p^{-1}\Bigl(C_1^{(n)} + \lambda_n \sum_{l=s}^{T} a(l)g(u_n(l))\Bigr)\\
&\leqslant \sum_{s=1}^{T} \varphi_p^{-1}\Bigl(\lambda_n\Bigl(\sum_{l\in A_n} + \sum_{l\in B_n}\Bigr) a(l)g(u_n(l))\Bigr)\\
&\leqslant T\varphi_p^{-1}\Bigl(L\overline{a}\Bigl(TM_\varepsilon + \varepsilon \sum_{l\in B_n} \varphi_p(u_n(l))\Bigr)\Bigr)\\
&\leqslant T\varphi_p^{-1}(TL\overline{a}(M_\varepsilon + \varepsilon\varphi_p(\|u_n\|))),
\end{aligned}$$

where $C_1^{(n)}$ satisfies (2.2) with u being replaced by u_n. It follows that

$$\varphi_p(T^{-1}\|u_n\|) \leqslant TL\overline{a}(M_\varepsilon + \varepsilon\varphi_p(\|u_n\|)),$$

or

$$\frac{T^{1-p}}{2} = T^{1-p} - \varepsilon T L \overline{a} \leqslant \frac{T L \overline{a} M_\varepsilon}{\varphi_p(\|u_n\|)}.$$

Letting $n \to \infty$ yields $T^{1-p}/2 \leqslant 0$, a contradiction. This completes the proof. $\square$

The next theorem follows immediately from Lemmas 2.2, 3.1, and 3.2.

Theorem 3.3. *Assume that* (A1) *and* (A2) *hold. Then there exists an unbounded continuum $\mathcal{C}$ of positive solutions for* (1.1) *emanating from* $(0,0)$ *in* $\mathbb{R}^+ \times K$ *such that*

(i) *for each $\lambda > 0$, there exists a positive solution $u(\lambda)$ of* (1.1) *such that* $(\lambda, u(\lambda)) \in \mathcal{C}$*; and*
(ii) *for* $(\lambda, u(\lambda)) \in \mathcal{S}$, $\lambda \to \infty$ *if and only if* $\|u(\lambda)\| \to \infty$.

The following result shows that the continuum is indeed a monotone continuous curve globally defined for all $\lambda > 0$.

Theorem 3.4. *Assume that* (A1)–(A3) *hold. Then $\mathcal{S} = \mathcal{C}$ and $\mathcal{C}$ is the solution curve of positive solutions for* (1.1) *such that*

(i) *for each $\lambda > 0$, there exists a unique positive solution $u(\lambda)$ of* (1.1) *such that* $(\lambda, u(\lambda)) \in \mathcal{C}$;
(ii) $\lambda \to \infty$ *if and only if* $\|u(\lambda)\| \to \infty$*; and*
(iii) *for any* $0 < \lambda_a < \lambda_b$, $u(\lambda_a) \prec u(\lambda_b)$.

Proof. We only need to prove (i) and (iii) since (ii) follows from Theorem 3.3.

We first prove (i). By way of contradiction, assume that there exists $\lambda > 0$ such that (λ, u_1) and (λ, u_2) are two distinct positive solutions of (1.1). Without loss of generality, there exist $1 \leqslant k_1 \leqslant k_2 \leqslant T$ such that

$$u_1(k) > u_2(k) \quad \text{for } k \in [k_1, k_2]_{\mathbb{Z}} \tag{3.1}$$

and

$$u_1(k_1 - 1) \leqslant u_2(k_1 - 1), \quad u_1(k_2 + 1) \leqslant u_2(k_2 + 1). \tag{3.2}$$

Then

$$\Delta u_1(k_1 - 1) > \Delta u_2(k_1 - 1) \quad \text{and} \quad \Delta u_1(k_2) < \Delta u_2(k_2). \tag{3.3}$$

We reach a contradiction in each of three distinct cases listed below.

Case 1: $1 < k_1 \leqslant k_2 \leqslant T$. Summing (2.6) from k_1 to k_2 and with the help of (2.7), we obtain

$$\begin{aligned}(3.4)\qquad &\frac{u_1(k_2)}{\varphi_p(u_2(k_2))}[\varphi_p(u_2(k_2))\varphi_p(\Delta u_1(k_2)) - \varphi_p(u_1(k_2))\varphi_p(\Delta u_2(k_2))]\\ &\quad - \frac{u_1(k_1-1)}{\varphi_p(u_2(k_1-1))}[\varphi_p(u_2(k_1-1))\varphi_p(\Delta u_1(k_1-1))\\ &\quad - \varphi_p(u_1(k_1-1))\varphi_p(\Delta u_2(k_1-1))]\\ &\geqslant \sum_{k=k_1}^{k_2} \lambda a(k)\Big[\frac{g(u_2(k))}{\varphi_p(u_2(k))} - \frac{g(u_1(k))}{\varphi_p(u_1(k))}\Big]|u_1(k)|^p.\end{aligned}$$

First we show

$$(3.5)\qquad \varphi_p(u_2(k_2))\varphi_p(\Delta u_1(k_2)) - \varphi_p(u_1(k_2))\varphi_p(\Delta u_2(k_2)) \leqslant 0.$$

Obviously, (3.5) holds if either $k_2 = T$, or $0 \leqslant \Delta u_1(k_2) < \Delta u_2(k_2)$, or $\Delta u_1(k_2) < 0 \leqslant \Delta u_2(k_2)$. Since $\Delta u_1(k_2) < \Delta u_2(k_2)$, to finish the discussion, we only need to consider the case where $\Delta u_1(k_2) < \Delta u_2(k_2) < 0$ and $k_2 < T$. In fact,

$$\begin{aligned}(3.5) &\iff \frac{\varphi_p(u_1(k_2))}{\varphi_p(u_2(k_2))} \leqslant \frac{\varphi_p(\Delta u_1(k_2))}{\varphi_p(\Delta u_2(k_2))}\\ &\iff \Big[\frac{u_1(k_2)}{u_2(k_2)}\Big]^{p-1} \leqslant \Big[\frac{\Delta u_1(k_2)}{\Delta u_2(k_2)}\Big]^{p-1}\\ &\iff \frac{u_1(k_2)}{u_2(k_2)} \leqslant \frac{u_1(k_2)-u_1(k_2+1)}{u_2(k_2)-u_2(k_2+1)}\\ &\iff u_1(k_2)u_2(k_2+1) \geqslant u_2(k_2)u_1(k_2+1).\end{aligned}$$

The last inequality holds by (3.1) and (3.2). In summary, (3.5) holds.

Next we show

$$(3.6)\qquad \varphi_p(u_2(k_1-1))\varphi_p(\Delta u_1(k_1-1)) - \varphi_p(u_1(k_1-1))\varphi_p(\Delta u_2(k_1-1)) > 0.$$

This inequality holds if either $\Delta u_1(k_1-1) > \Delta u_2(k_1-1) \geqslant 0$ or $\Delta u_1(k_1-1) \geqslant 0 > \Delta u_2(k_1-1)$. To finish the discussion, it suffices to consider the case where $0 > \Delta u_1(k_1-1) > \Delta u_2(k_1-1)$. In this case,

$$\begin{aligned}(3.6) &\iff \frac{\varphi_p(u_1(k_1-1))}{\varphi_p(u_2(k_1-1))} > \frac{\varphi_p(\Delta u_1(k_1-1))}{\varphi_p(\Delta u_2(k_1-1))}\\ &\iff u_1(k_1-1)u_2(k_1) < u_2(k_1-1)u_1(k_1).\end{aligned}$$

Again, the last inequality holds because of (3.1) and (3.2). This proves (3.6).

It follows from (3.4)–(3.6) that

$$\sum_{k=k_1}^{k_2} \lambda a(k)\Big[\frac{g(u_2(k))}{\varphi_p(u_2(k))} - \frac{g(u_1(k))}{\varphi_p(u_1(k))}\Big]|u_1(k)|^p < 0.$$

However, by (3.1) and (A3),

$$\sum_{k=k_1}^{k_2} \lambda a(k)\Big[\frac{g(u_2(k))}{\varphi_p(u_2(k))} - \frac{g(u_1(k))}{\varphi_p(u_1(k))}\Big]|u_1(k)|^p > 0.$$

Hence, we reached a contradiction.

Case 2: $k_1 = 1 < k_2 \leqslant T$. This time, we sum (2.6) from 2 to k_2 and use (2.7) to get

$$\begin{aligned}\frac{u_1(k_2)}{\varphi_p(u_2(k_2))}&[\varphi_p(u_2(k_2))\varphi_p(\Delta u_1(k_2)) - \varphi_p(u_1(k_2))\varphi_p(\Delta u_2(k_2))]\\ &- \frac{u_1(1)}{\varphi_p(u_2(1))}[\varphi_p(u_2(1))\varphi_p(\Delta u_1(1)) - \varphi_p(u_1(1))\varphi_p(\Delta u_2(1))]\\ &\geqslant \sum_{k=2}^{k_2} \lambda a(k)\Big[\frac{g(u_2(k))}{\varphi_p(u_2(k))} - \frac{g(u_1(k))}{\varphi_p(u_1(k))}\Big]|u_1(k)|^p.\end{aligned}\tag{3.7}$$

We prove that

$$\varphi_p(u_2(1))\varphi_p(\Delta u_1(1)) - \varphi_p(u_1(1))\varphi_p(\Delta u_2(1)) > 0.\tag{3.8}$$

In fact, since

$$\begin{aligned}\varphi_p(\Delta u_1(1)) &= \varphi_p(u_1(1)) - \lambda a(1)g(u_1(1)),\\ \varphi_p(\Delta u_2(1)) &= \varphi_p(u_2(1)) - \lambda a(1)g(u_2(1)),\end{aligned}$$

it follows from assumption $u_1(1) > u_2(1)$ and (A3) that

$$\begin{aligned}&\frac{g(u_1(1))}{\varphi_p(u_1(1))} < \frac{g(u_2(1))}{\varphi_p(u_2(1))}\\ &\Longleftrightarrow \lambda a(1)g(u_1(1))\varphi_p(u_2(1)) < \lambda a(1)g(u_2(1))\varphi_p(u_1(1))\\ &\Longleftrightarrow [\varphi_p(u_1(1)) - \varphi_p(\Delta u_1(1))]\varphi_p(u_2(1)) < [\varphi_p(u_2(1)) - \varphi_p(\Delta u_2(1))]\varphi_p(u_1(1))\\ &\Longleftrightarrow (3.8).\end{aligned}$$

Combing (3.5) with (3.8) gives

$$\sum_{k=2}^{k_2} \lambda a(k)\Big[\frac{g(u_2(k))}{\varphi_p(u_2(k))} - \frac{g(u_1(k))}{\varphi_p(u_1(k))}\Big]|u_1(k)|^p < 0,$$

which, on the other hand, is larger than 0. Therefore, we have got a contradiction.

Case 3: $k_1 = k_2 = 1$. In this case, (3.8) still holds. First assume that either $0 \leqslant \Delta u_1(1) < \Delta u_2(1)$ or $\Delta u_1(1) < 0 \leqslant \Delta u_2(1)$. Then we have

$$\varphi_p(u_2(1))\varphi_p(\Delta u_1(1)) - \varphi_p(u_1(1))\varphi_p(\Delta u_2(1)) < 0,$$

contradicting with (3.8). Now assume that $\Delta u_1(1) < \Delta u_2(1) < 0$. It follows from $u_1(2) \leqslant u_2(2)$ and $u_2(1) < u_1(1)$ that $u_1(2)u_2(1) < u_1(1)u_2(2)$. Then

$$\begin{aligned} u_1(2)u_2(1) < u_1(1)u_2(2) &\Longrightarrow \frac{u_2(1)}{u_1(1)} > \frac{u_2(2) - u_2(1)}{u_1(2) - u_1(1)} \\ &\Longrightarrow \frac{\varphi_p(u_2(1))}{\varphi_p(u_1(1))} > \frac{\varphi_p(\Delta u_2(1))}{\varphi_p(\Delta u_1(1))} \\ &\Longrightarrow \varphi_p(u_2(1))\varphi_p(\Delta u_1(1)) - \varphi_p(u_1(1))\varphi_p(\Delta u_2(1)) < 0, \end{aligned}$$

contradicting (3.8) again. Anyway, there is a contradiction in this case.

In summary, we have a contradiction in any of the above three cases. Therefore, (1.1) has a unique positive solution $u(\lambda)$ for each $\lambda > 0$. This proves (i).

Next we prove (iii). Let v be the unique solution of the following boundary value problem

$$\begin{cases} \Delta(\varphi_p(\Delta u(k-1))) + a(k) = 0, & k \in [1, T]_{\mathbb{Z}}, \\ u(0) = u(T+1) = 0. \end{cases}$$

Then it follows from the remark after Lemma 2.2 and Lemma 2.3 of Bai [2] that $v(k) > 0$ for all $k \in [1, T]_{\mathbb{Z}}$. Define $g^*(u) = \max_{0 \leqslant s \leqslant u} g(s)$. Obviously, $g^*(s)$ is nondecreasing on $[0, \infty)$ and $\lim_{u \to \infty} g^*(u)/\varphi_p(u) = 0$ by (A2) (see, for example, [10]). So there exists $C_b > 0$ sufficiently large such that

$$\lambda_b \frac{g^*(\lambda_b C_b)}{\varphi_p(\lambda_b C_b)} < \frac{1}{\varphi_p(\|v\|)},$$

and $u(\lambda_a) \leqslant \beta := \lambda_b C_b v / \|v\|$. Then β is a strict upper solution of (1.1) at λ_b. In fact, for $k \in [1, T]_{\mathbb{Z}}$,

$$\begin{aligned} -\Delta(\varphi_p(\Delta\beta(k-1))) &= -\varphi_p\Big(\frac{\lambda_b C_b}{\|v\|}\Big)\Delta(\varphi_p(\Delta v(k-1))) \\ &= \varphi_p\Big(\frac{\lambda_b C_b}{\|v\|}\Big)a(k) \\ &> \lambda_b a(k) g^*(\lambda_b C_b) \geqslant \lambda_b a(k) g^*(\beta(k)) \geqslant \lambda_b a(k) g(\beta(k)). \end{aligned}$$

Clearly, $u(\lambda_a)$ is a strict lower solution of (1.1) at λ_b. By Lemma 2.1, there exists a positive solution u_b of (1.1) at $\lambda = \lambda_b$, such that $u(\lambda_a) \prec u_b \prec \beta$. The uniqueness of positive solution implies that $u_b = u(\lambda_b)$ and hence $u(\lambda_a) \prec u(\lambda_b)$. The proof is complete. $\square$

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Authors' addresses: *Dingyong Bai*, School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China, and Key Laboratory of Mathematics and Interdisciplinary Sciences of Guangdong Higher Education Institutes, Guangzhou University, Guangzhou 510006, China, e-mail: `baidy@gzhu.edu.cn`; *Yuming Chen*, Department of Mathematics, Wilfrid Laurier University, Waterloo, Ontario, N2L 3C5 Canada, e-mail: `ychen@wlu.ca`.