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GENERALIZED SYNCHRONIZATION IN A SYSTEM OF SEVERAL NON-AUTONOMOUS OSCILLATORS COUPLED BY A MEDIUM

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An abstract theory on general synchronization of a system of several oscillators coupled by a medium is given. By generalized synchronization we mean the existence of an invariant manifold that allows a reduction in dimension. The case of a concrete system modeling the dynamics of a chemical solution on two containers connected to a third container is studied from the basics to arbitrary perturbations. Conditions under which synchronization occurs are given. Our theoretical results are complemented with a numerical study.

Keywords: coupled oscillators, synchronization, invariant manifolds

Classification: 34D06, 34D35, 34C15

1. INTRODUCTION

We will introduce a general framework of generalized synchronization of periodic oscillators coupled by a medium. Consider a very general coupling framework

$$\begin{cases} \dot{x}_1 = f_1(x_1, y, t) \\ \vdots \\ \dot{x}_m = f_m(x_m, y, t) \\ \dot{y} = g(x_1, \dots, x_m, y, t) \end{cases} \tag{1}$$

of the oscillators $x_i \in \mathbb{R}^n$, $i = 1, \dots, m$ through a medium $y \in \mathbb{R}^p$. Throughout this paper is assumed the system is $T-$periodic in t, i.e, for all $t \in \mathbb{R}$ we have $f_i(x_i, y, t+T) = f_i(x_i, y, t)$ for $i = 1, \dots, m$ and $g(x_1, \dots, x_m, y, t+T) = g(x_1, \dots, x_m, y, t)$. This coupling scheme is rather natural, just imagine a group of cells immersed in a common medium, each cell interacts chemically with all the other cells trough the medium. We can find this type of coupling in several situations (see for example [4] and [7]).

By generalized synchronization we mean the existence of an invariant time dependent manifold $\mathcal{A}_t$, that can be seen as graph over certain subspaces and that attracts the orbits in the future, i. e., given a metric d and a solution $Z(t) = (x_1, x_2, \dots, x_n, y)^T$ of system (1),

$$d\left(Z(t), \mathcal{A}_t\right) \to 0$$

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as $t \to +\infty$. In this case we will call $\mathcal{A}_t$ the synchronization manifold. If we assure the existence of $\mathcal{A}_t$, with certain characteristics, this can allow us, for example, to know the asymptotic state of some oscillators from the state of the others. Depending on the type of synchronization we have, the particular geometry of the synchronization manifold will change. For example, the existence of an attracting manifold of the type

$$\mathcal{A}_t = \{x_1 = x_2 = \ldots = x_n\}$$

is a very special case of synchronization, that we call identical synchronization. In this case, we can predict the asymptotic behavior of all the oscillators from the knowledge of any one of them and the respective synchronization manifold will be of diagonal type.

The existence of invariant synchronization manifolds will be, in general, obtained from a general theory introduced by Russell Smith in [8]. Similar ideas were used by Martins and Margheri in [5] in order to identify generalized synchronization of coupled oscillators. The novelty of our case is that the coupling is done by the medium, the nature of this coupling will introduce some new aspects to the geometrical structure of the synchronization manifold that will, somehow, differ from the one given in [5].

All these ideas will be applied to a system modelling two containers with some chemical solution. These containers are connected trough a semi-permeable membrane to another container. We assume that the concentration of the chemical in the three containers is measured by the variables x_1, x_2 and y respectively. Then the evolution of these concentrations is described by the linear system of differential equations

$$\begin{cases} \dot{x}_1 = k\,(y - x_1) \\ \dot{x}_2 = k\,(y - x_2) \\ \dot{y} = k\,(x_1 - y) + k\,(x_2 - y)\,. \end{cases} \tag{2}$$

In the second part of this paper we study the perturbations of this system, giving conditions under which there occurs generalized synchronization. The option to study this particular example is rather arbitrary. Our goal is to give an example of the kind of results derived from the so called *Theorem of Generalized Synchronization*, introduced in section 2, whose proof is refered to section A. Analogous results could be obtained to a larger number of oscillators with a similar coupling. With our choice we just want to increase the intuition level of the results presented here.

2. GENERALIZED SYNCHRONIZATION

The system (1) may be written in a more condensed form, defining $X = (x_1, \ldots, x_m)^T$ and $F(X, y, t) = (f_1(x_1, y, t), \ldots, f_m(x_m, y, t))^T$. Hence, the system (1) can be rewritten as

$$\begin{cases} \dot{X} = F(X, y, t) \\ \dot{y} = g(X, y, t). \end{cases} \tag{3}$$

Throughout this paper is assumed that it is valid the assumption of existence and uniqueness of solutions of system (1) and that all its solutions are defined in $\mathbb{R}$. In the same condensed form we denote by $(X(t), y(t)) = (X(t; X_0, y_0, t_0), y(t; X_0, y_0, t_0))$ the solution of (1) so that $(X(t_0), y(t_0)) = (X_0, y_0)$.

In a more geometric flavor, for a system like the one presented in (1), we pretend to identify a k-dimensional invariant submanifold $\mathcal{A}_t \subset \mathbb{R}^{mn+p}$, with $k < nm + p$, that can be seen as a graph over a k-dimensional subspace that attracts the solutions of system (1). In the literature, the typical case exposed is the identical synchronization. In this case, the dimension of the synchronization manifold is equal to the dimension of a single oscillator, whose behavior rigidly catches the overall behavior of the system.

Definition 2.1. If $\mathcal{A}_t$ is a k-dimensional invariant submanifold of $\mathbb{R}^{nm+p}$, for each $t \in \mathbb{R}$, that is a graph over a k-dimensional subspace, and attracts all the (bounded) orbits in the future, i. e., given a metric d, for every (bounded) solution $Z(t) = (x_1, x_2, \ldots, x_n, y)^T$ of system (1),

$$d\,(Z(t), \mathcal{A}_t) \to 0$$

as $t \to +\infty$, then we call $\mathcal{A}_t$ a *synchronization manifold* and we say that there is *(bounded) generalized synchronization.*

The existence of a candidate for synchronization manifold will be obtained directly from considerations of symmetry, from Lyapunov function, or using a general result introduced by Russel Smith in [8]. In our framework, the hereafter called *Russel Smith's condition* is equivalent to the existence of a symmetric matrix $P \in M_{(nm+p)\times(nm+p)}(\mathbb{R})$, with precisely k negative eigenvalues, and positive constants ε, λ, so that for any $\left(\begin{smallmatrix} X \\ y \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} Q \\ w \end{smallmatrix}\right) \in \mathbb{R}^{nm+p}$ and $t \in \mathbb{R}$ we have

$$\begin{pmatrix} X-Q \\ y-w \end{pmatrix}^T P \left[\begin{pmatrix} F(X,y,t) - F(Q,w,t) \\ g(X,y,t) - g(Q,w,t) \end{pmatrix} + \lambda \begin{pmatrix} X-Q \\ y-w \end{pmatrix} \right] \leq -\varepsilon \left\| \begin{pmatrix} X-Q \\ y-w \end{pmatrix} \right\|^2. \tag{4}$$

Let V be the quadratic form associated with matrix P. The Russel Smith's condition is equivalent to say that, for any pair of solutions $\left(\begin{smallmatrix} X \\ y \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} Q \\ w \end{smallmatrix}\right)$ of system (1), the map $e^{2\lambda t} V\left(\begin{smallmatrix} X-Q \\ y-w \end{smallmatrix}\right)$ is strictly decreasing. Indeed is valid the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\{ e^{2\lambda t} V \begin{pmatrix} X-Q \\ y-w \end{pmatrix} \right\} \leq -2e^{2\lambda t} \varepsilon \left\| \begin{pmatrix} X-Q \\ y-w \end{pmatrix} \right\|^2.$$

This inequality shows that the Russell Smith's condition may be seen as a dissipative condition over the set of solutions of system (1). Intimate related with it are the *amenable points.*

Definition 2.2. A point $\begin{pmatrix} X_0, & y_0, & t_0 \end{pmatrix}^T \in \mathbb{R}^{nm+p} \times \mathbb{R}$ is an *amenable point* if it gives rise to a solution $\left(\begin{smallmatrix} X(t) \\ y(t) \end{smallmatrix}\right)$ so that

$$\int_{-\infty}^{t_0} e^{2\lambda t} \left\| \begin{pmatrix} X(t; X_0, y_0, t_0) \\ y(t; X_0, y_0, t_0) \end{pmatrix} \right\|^2 \mathrm{d}t < +\infty.$$

Solutions for which the initial conditions are amenable points are called amenable solutions and its locus amenable orbits. Obviously the bounded solutions are amenable. In sequence of this definition we define the set

$$\mathcal{A}_{t_0} = \left\{ \begin{pmatrix} X_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^{nm+p} \colon \begin{pmatrix} X_0 \\ y_0 \\ t_0 \end{pmatrix} \text{ is an amenable point} \right\}.$$

For the matrix P defined in (4), consider the respective eigenvalues, counted with multiplicities,

$$\lambda_1^-, \dots, \lambda_k^-, \lambda_{k+1}^+, \dots, \lambda_{nm+p}^+,$$

with $\lambda_i^- < 0$ and $\lambda_j^+ > 0$ for all $i = 1, \dots, k$ and $j = k+1, \dots, nm+p$. By symmetry of matrix P we know that there are $nm+p$ linearly independent eigenvectors respectively

$$v_1^-, \dots, v_k^-, v_{k+1}^+, \dots, v_{nm+p}^+.$$

We can then define the k-dimensional subspace $\mathcal{V}_- \subset \mathbb{R}^{nm}$ associated to the negative eigenvalues of P

$$\mathcal{V}_- = \operatorname{span}\{v_1^-, \dots, v_k^-\}.$$

From the previous points we are in conditions to state the main theorem of this paper, adapted from [5] to the case where the coupling is produced through the medium. The proof of this theorem will be given in appendix A. A different proof can also be obtained following the ideas in [8].

Theorem 2.3. (Theorem of Generalized Synchronization) Suppose that the system (1) satisfies the Russell Smith's condition and there is at least one amenable point. Then there is bounded generalized synchronization with $\mathcal{A}_t$ as a synchronization manifold. Moreover, for each $t \in \mathbb{R}$, $\mathcal{A}_t$ can be seen as a graph over $\mathcal{V}_-$.

There are two important points about this theorem. First, we notice that the dimension of the synchronization manifold is equal to the dimension of the subspace $\mathcal{V}_-$, which is given by the number of the negative eigenvalues of matrix P. On the other hand, for the same system, we can use Russel Smith's condition with different values for λ and this give us $\mathcal{A}_t$'s of different dimensions. So, in general, we can look for different levels of synchronization in the same system, as we will have the opportunity to see in the next sections.

3. A LINEAR EXAMPLE

In this section, we will consider the case of a linear system (1) that can be solved by direct methods. In the next sections we will study a nonlinear perturbation of this system.

We consider a system of three containers, say containers 1, 2 and 3. All the containers have the same capacity, say 1 liter. The containers 1 and 2 are connected to the container 3 through a semi-permeable membrane. There is a chemical solution diluted in the containers and we assume that the concentration of the chemical in the three containers is measured by the variables x_1, x_2 and y respectively. Then the evolution of these concentrations is described by the linear system of differential equations (2) where k is a constant that depends on the permeability of the membrane. We can also write this equation in the condensed form

$$\begin{pmatrix} \dot{X} \\ \dot{y} \end{pmatrix} = kA \begin{pmatrix} X \\ y \end{pmatrix}$$

where

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

The matrix A has eigenvalues -3, -1, and 0 with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

So the structure of the phase portrait of this system is very clear: there is a one-dimensional stable central manifold, the subspace generated by $(1,1,1)$, and there is an invariant two-dimensional stable subspace spanned by the vectors $(1,-1,0)$ and $(1,1,1)$.

This system is linear. Being so it can be easily integrated. However we will study it from the point of view of the tools introduced in last section. Doing so, it will give us an useful intuition when we introduce nonlinear perturbations in section 4 and it also illustrates the ideas given in section 2.

A question that naturally arises is: how can we find the matrix P in the Russel Smith's condition? To answer this question we need to go back a little bit and recall the notion of the Lyapunov's Equation. The following theorem is adapted from the Corollary (4.4.7) of Theorem (4.4.6), page 270 of [3], where is presented in full generality.

Theorem 3.1. Given a square matrix D, for every square matrix C the Lyapunov's equation

$$D^T P + PD = C$$

has an unique solution P if and only if $\sigma(D) \cap \overline{\sigma(-D)} = \emptyset$, where $\sigma(D)$ is the spectrum of matrix D.

Actually, when C is a positive definite matrix there is a relation between the eigenvalues of D and P. This is a consequence of another result from matrix analysis known as *general inertia theorem* (see page 105 of [3]). If all the eigenvalues of matrix D have non-zero real part and C is a positive definite matrix we may guarantee that the matrix P, solution of the above Lyapunov's equation, has the same inertia of the matrix D, meaning that the number of eigenvalues with positive real part is equal in both matrix D and P. In the case that the matrix C is negative definite, as will be our case, it is not difficult to show that the number of eigenvalues of matrix D with negative real part are exactly the same as the number of eigenvalues of the matrix P with positive real part.

This theorem, for the case where D is negative definite, guarantees the existence of a Lyapunov function for the linear system

$$\dot{x} = Dx.$$

Just choose any negative definite C and obtain a positive definite P by the results above. Then $V(x) = x^T P x$ is a Lyapunov function because the Lie derivative of V along the solutions verifies

$$\dot{V}(x) = x^T (D^T P + PD)x = x^T C x < 0.$$

In our case, in order to guarantee that the matrix D in the Lyapunov's equation does not have eigenvalues with null real part, we introduce a positive parameter λ and we replace the matrix kA by the perturbed matrix $kA+\lambda I$ in the Lyapunov's equation. Assuming that λ is chosen in such a way that $\sigma(kA+\lambda I)\cap\overline{\sigma(-kA-\lambda I)}=\emptyset$, the matrix P that comes up as the solution of Lyapunov's Equation

$$(kA+\lambda I)^T P + P(kA+\lambda I) = -I, \tag{5}$$

is a symmetric matrix, with a number of eigenvalues with positive real part equal to the number of eigenvalues of matrix $kA+\lambda I$ with negative real part. Both points are easy to prove. The last assumption comes directly from the previous observation while the symmetry of matrix P is a direct consequence of the uniqueness and the fact that both P and P^T are solutions of the Lyapunov's equation.

The matrix $kA+\lambda I$ has eigenvalues $-3k+\lambda$, $-k+\lambda$, and λ. We would like to study the dynamic in the situation where one or two of its eigenvalues are positive, but in order to P be well defined we must choose λ so that $\sigma(kA+\lambda I)\cap\overline{\sigma(-kA-\lambda I)}=\emptyset$. So we have two different qualitative scenarios, if $\lambda\in(0,k)\backslash\{\frac{k}{2}\}$ then $kA+\lambda I$ has one positive and two negative eigenvalues, if $\lambda\in(k,3k)\backslash\{\frac{3}{2}k,2k\}$ then $kA+\lambda I$ has two positive eigenvalues and one negative eigenvalue. In both cases we can compute the solution P of equation (5) for each λ, and we obtain the not so much friendly matrix

$$P=\begin{pmatrix} -\frac{k^2-3k\lambda+\lambda^2}{2\lambda(\lambda-3k)(\lambda-k)} & -\frac{k^2}{2\lambda(\lambda-3k)(\lambda-k)} & \frac{k}{2\lambda(\lambda-3k)} \\ -\frac{k^2}{2\lambda(\lambda-3k)(\lambda-k)} & -\frac{k^2-3k\lambda+\lambda^2}{2\lambda(\lambda-3k)(\lambda-k)} & \frac{k}{2\lambda(\lambda-3k)} \\ \frac{k}{2\lambda(\lambda-3k)} & \frac{k}{2\lambda(\lambda-3k)} & \frac{k-\lambda}{2\lambda(\lambda-3k)} \end{pmatrix}. \tag{6}$$

Although from the matrix theory we can only guarantee existence and uniqueness of a solution of Lyapunov equation for $\lambda\in(0,k)\backslash\{\frac{k}{2}\}$ and $\lambda\in(k,3k)\backslash\{\frac{3}{2}k,2k\}$, we see that the matrix P above is defined and is also solution for $\lambda\in(0,k)$ and $\lambda\in(k,3k)$, so from now on we consider this solution for λ in those intervals. We can even compute the eigenvalues of P. They are

$$-\frac{1}{2(\lambda-3k)},-\frac{1}{2(\lambda-k)},-\frac{1}{2\lambda},$$

with corresponding eigenvectors

$$\begin{pmatrix}1\\1\\-2\end{pmatrix},\begin{pmatrix}1\\-1\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

In any case, if we make

$$\begin{pmatrix}F(X,y,t)\\g(X,y,t)\end{pmatrix}=kA$$

in (4), then we obtain

$$\begin{pmatrix}X-Q\\y-w\end{pmatrix}^T P(kA+\lambda I)\begin{pmatrix}X-Q\\y-w\end{pmatrix}=-\frac{1}{2}\left\|\begin{pmatrix}X-Q\\y-w\end{pmatrix}\right\|^2$$

so Russel Smith's condition is satisfied with $\varepsilon = 1/2$. Then Theorem 2.3 with $\lambda \in (0, k)$ says that there is an invariant one-dimensional synchronization manifold, that we know to be the central subspace spanned by $(1,1,1)^T$ and if $\lambda \in (k, 3k)$ we get a two-dimensional synchronization manifold that we know to be the subspace spanned by $(1,1,1)^T$ and $(1,-1,0)^T$.

4. GENERAL CONDITIONS FOR SYNCHRONIZATION OF A NONLINEAR PERTURBATION

In this section, we will study a perturbation of the linear system of section 3 and we will see what Russel Smith's theory tell us about it. Consider the nonlinear non-autonomous time-periodic perturbation of system (2)

$$\begin{cases} \dot{x}_1 = k\,(y - x_1) + f_1(x_1, t) \\ \dot{x}_2 = k\,(y - x_2) + f_2(x_2, t) \\ \dot{y} = k\,(x_1 - y) + k\,(x_2 - y) + h(y, t). \end{cases} \tag{7}$$

The functions f_1, f_2, and h are assumed sufficiently regular, in such a way that there is existence and uniqueness of solutions of (7) and that all the solutions are defined in $\mathbb{R}$. Moreover they are T-periodic in t for some $T > 0$.

Our idea will be to see when this perturbed system still satisfies Russel Smith's condition with the matrix P computed in the last section, that is a matrix that suits the linear part. The general problem in its matricial form is written by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y} \end{pmatrix} = k \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} + \begin{pmatrix} f_1(x_1, t) \\ f_2(x_2, t) \\ h(y, t) \end{pmatrix}.$$

Given two solutions of the last system

$$\begin{pmatrix} X \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} \text{ and } \begin{pmatrix} Q \\ w \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ w \end{pmatrix},$$

the inequality in the Russel Smith's condition is thus given by

$$\begin{pmatrix} X-Q \\ y-w \end{pmatrix}^T P \left[\begin{pmatrix} f_1(x_1,t) - f_1(q_1,t) \\ f_2(x_2,t) - f_2(q_2,t) \\ h(y,t) - h(w,t) \end{pmatrix} + (kA + \lambda I) \begin{pmatrix} X-Q \\ y-w \end{pmatrix} \right] \le -\varepsilon \left\| \begin{pmatrix} X-Q \\ y-w \end{pmatrix} \right\|^2.$$

Considering the bilinear form associated to the matrix $P(kA + \lambda I)$, and assuming that P is the solution of the Lyapunov equation $(kA + \lambda I)^T P + P(kA + \lambda I) = -I$, we can rewrite the last inequality by

$$\left(\frac{1}{2} - \varepsilon\right) \left\| \begin{pmatrix} X-Q \\ y-w \end{pmatrix} \right\|^2 - \begin{pmatrix} X-Q \\ y-w \end{pmatrix}^T P \begin{pmatrix} f_1(x_1,t) - f_1(q_1,t) \\ f_2(x_2,t) - f_2(q_2,t) \\ h(y,t) - h(w,t) \end{pmatrix} \ge 0.$$

On the other hand, if $x_1 \neq q_1$, $x_2 \neq q_2$, and $y \neq w$, defining $\alpha = \alpha(x_1, q_2, t) = \frac{f_1(x_1,t) - f_1(q_1,t)}{x_1 - q_1}$, $\beta = \beta(x_2, q_2, t) = \frac{f_2(x_2,t) - f_2(q_2,t)}{x_2 - q_2}$, and $\gamma = \gamma(y, w, t) = \frac{h(y,t) - h(w,t)}{y - w}$, we can write the second term in the left hand side of the last inequality as

$$\begin{pmatrix} X-Q \\ y-w \end{pmatrix}^T P \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} X-Q \\ y-w \end{pmatrix}.$$

So the inequality in Russel Smith's condition is equivalent to

$$\begin{pmatrix} X-Q \\ y-w \end{pmatrix}^T \left[\left(\frac{1}{2}-\varepsilon\right) I - P \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \right] \begin{pmatrix} X-Q \\ y-w \end{pmatrix} \geq 0. \tag{8}$$

Consider the symmetric matrix Ω of the associate quadratic form

$$\Omega = \frac{1}{2}\left(\left[\left(\frac{1}{2}-\varepsilon\right) I - P \left(\begin{smallmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{smallmatrix}\right) \right]^T + \left(\frac{1}{2}-\varepsilon\right) I - P \left(\begin{smallmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{smallmatrix}\right) \right). \tag{9}$$

This matrix, written in an explicit form, is given by

$$\Omega = \begin{pmatrix} \frac{1}{2}-\varepsilon+\frac{\alpha(k^2-3k\lambda+\lambda^2)}{2\lambda(\lambda-3k)(\lambda-k)} & \frac{(\alpha+\beta)k^2}{4\lambda(\lambda-3k)(\lambda-k)} & \frac{(\alpha+\gamma)k}{4\lambda(3k-\lambda)} \\ \frac{(\alpha+\beta)k^2}{4\lambda(\lambda-3k)(\lambda-k)} & \frac{1}{2}-\varepsilon+\frac{\beta(k^2-3k\lambda+\lambda^2)}{2\lambda(\lambda-3k)(\lambda-k)} & \frac{(\beta+\gamma)k}{4\lambda(3k-\lambda)} \\ \frac{(\alpha+\gamma)k}{4\lambda(3k-\lambda)} & \frac{(\beta+\gamma)k}{4\lambda(3k-\lambda)} & \frac{1}{2}-\varepsilon+\frac{\gamma(k-\lambda)}{2\lambda(3k-\lambda)} \end{pmatrix}.$$

So all the discussion about the inequality in Russel Smith's condition is therefore equivalent to study under which circumstances the quadratic form defined by matrix Ω is positive definite. The result of the last observations can be resumed in the following theorem that is a direct consequence of Theorem 2.3.

Theorem 4.1. Suppose that there is $\lambda \in (0,k) \cup (k,3k)$ and ε for which Ω is positive definite for all x_1, x_2, y, q_1, q_2, w, with $x_1 \neq x_2$, $q_1 \neq q_2$, $y \neq w$, where P is the solution of the Lyapunov equation $(kA+\lambda I)^T P + P(kA+\lambda I) = -I$ given by (6). Then there is bounded generalized synchronization for system (7). If $\lambda \in (0,k)$, the synchronization manifold $\mathcal{A}_t$ is one-dimensional and can be seen as a graph over the subspace spanned by $(1,1,1)^T$, if $\lambda \in (k,3k)$, the synchronization manifold $\mathcal{A}_t$ is two-dimensional and can be seen as a graph over the subspace spanned by $(1,-1,0)^T$ and $(1,1,1)^T$.

We can try to see in what conditions Ω is positive definite. This can be done in two ways, computing the eigenvalues and see if they are all positive or studying the minors. Since the former is unmanageable, we use the second. Consider the coeficients a, b, c, and d given as function of parameters λ and k

$$\begin{aligned} a =& \frac{(k^2-3k\lambda+\lambda^2)}{2\lambda(\lambda-3k)(\lambda-k)} = \frac{1}{12}\left(\frac{2}{\lambda}+\frac{1}{-3k+\lambda}+\frac{3}{-k+\lambda}\right), \\ b =& \frac{k^2}{4\lambda(\lambda-3k)(\lambda-k)}, \\ c =& \frac{k}{4\lambda(3k-\lambda)}, \\ d =& \frac{\lambda-k}{2\lambda(\lambda-3k)}. \end{aligned} \tag{10}$$

Then the matrix Ω is thus given by

$$\Omega = \begin{pmatrix} \frac{1}{2}-\varepsilon+\alpha a & (\alpha+\beta)b & (\alpha+\gamma)c \\ (\alpha+\beta)b & \frac{1}{2}-\varepsilon+\beta a & (\beta+\gamma)c \\ (\alpha+\gamma)c & (\beta+\gamma)c & \frac{1}{2}-\varepsilon+\gamma d \end{pmatrix},$$

where the three minors of matrix Ω are respectively

$$\begin{aligned} m_1(\alpha) &= \frac{1}{2}-\varepsilon+\alpha a; \\ m_2(\alpha,\beta) &= \left(\frac{1}{2}-\varepsilon+\alpha a\right)\left(\frac{1}{2}-\varepsilon+\beta a\right) - b^2(\alpha+\beta)^2; \\ m_3(\alpha,\beta,\gamma) &= \left(\frac{1}{2}-\varepsilon+\gamma d\right) m_2(\alpha,\beta) + 2c^2 b(\alpha+\beta)(\alpha+\gamma)(\beta+\gamma) \\ &\quad - c^2(m_1(\alpha)(\beta+\gamma)^2 + m_1(\beta)(\alpha+\gamma)^2). \end{aligned} \tag{11}$$

Our intuition says that if the difference quotients α, β, and γ, are bounded and if k is sufficiently large then the system synchronizes as the linear part does. These ideas are explicitly stated and proved in the following theorem.

Theorem 4.2. Suppose that the following quotients

$$\begin{aligned} \alpha = \alpha(x_1,q_2,t) &= \frac{f_1(x_1,t)-f_1(q_1,t)}{x_1-q_1}, \\ \beta = \beta(x_2,q_2,t) &= \frac{f_2(x_2,t)-f_2(q_2,t)}{x_2-q_2}, \\ \gamma = \gamma(y,w,t) &= \frac{h(y,t)-h(w,t)}{y-w}, \end{aligned}$$

are bounded. Then if k is sufficiently large, there is bounded generalized synchronization for system (7), with a one-dimensional synchronization manifold $\mathcal{A}_t$ that can be seen as a graph over the subspace spanned by $(1,1,1)^T$, or with a two-dimensional synchronization manifold $\mathcal{A}_t$ that can be seen as a graph over the subspace spanned by $(1,-1,0)^T$, and $(1,1,1)^T$.

Proof. If we specify a concrete value for λ in (11), the expressions became much simpler. We will choose $\lambda = k/2$, that will give the one-dimensional manifold, and $\lambda = 2k$ that will give a two dimensional manifold. Those values were chosen arbitrarily, given that they growth linearly with k. For the first value of λ the four expressions in (10) are given by

$$a = \frac{2}{5k},\ b = \frac{2}{5k},\ c = \frac{1}{5k},\ d = \frac{1}{5k}.$$

So the minors in (11) become

$$\begin{aligned} m_1(\alpha) =& \frac{1}{2} - \varepsilon + \alpha\frac{2}{5k};\\ m_2(\alpha,\beta) =& \left(\frac{1}{2} - \varepsilon + \alpha\frac{2}{5k}\right)\left(\frac{1}{2} - \varepsilon + \beta\frac{2}{5k}\right) - \frac{4}{25k^2}(\alpha+\beta)^2;\\ m_3(\alpha,\beta,\gamma) =& \left(\frac{1}{2} - \varepsilon + \frac{\gamma}{5k}\right) m_2(\alpha,\beta) + \frac{4}{125k^2}(\alpha+\beta)(\alpha+\gamma)(\beta+\gamma)\\ &- \frac{1}{25k^2}\left(m_1(\alpha)(\beta+\gamma)^2 + m_1(\beta)(\alpha+\gamma)^2\right). \end{aligned}$$

Now it is clear that if α, β and γ are bounded, we can find a sufficiently large k, and a sufficiently small ε, that make the last expressions always positive. For these values of k and ε, Ω is positive definite and the result follows from the last Theorem.

When $\lambda = 2k$ the proof is similar, in this case four expressions in (10) are given by

$$a = \frac{1}{4k},\ b = -\frac{1}{8k},\ c = \frac{1}{8k},\ d = -\frac{1}{4k}.$$

So the minors in (11) become

$$\begin{aligned} m_1(\alpha) =& \frac{1}{2} - \varepsilon + \frac{\alpha}{4k};\\ m_2(\alpha,\beta) =& \left(\frac{1}{2} - \varepsilon + \frac{\alpha}{4k}\right)\left(\frac{1}{2} - \varepsilon + \frac{\beta}{4k}\right) - \frac{1}{64k^2}(\alpha+\beta)^2;\\ m_3(\alpha,\beta,\gamma) =& \left(\frac{1}{2} - \varepsilon - \frac{\gamma}{4k}\right) m_2(\alpha,\beta) + \frac{1}{256k^2}(\alpha+\beta)(\alpha+\gamma)(\beta+\gamma)\\ &- \frac{1}{68k^2}\left(m_1(\alpha)(\beta+\gamma)^2 + m_1(\beta)(\alpha+\gamma)^2\right). \end{aligned}$$

Using the same reasoning of the case when $\lambda = k/2$ we can establish the desired result. □

5. IDENTICAL SYNCHRONIZATION IF $F_1 = F_2$

In this section we consider the special case where the perturbations are identical in each oscillator, i. e. we assume that $f_1 = f_2 = f$. In fact, when the nonlinear perturbation is identical on both oscillators, the qualitative behavior of the perturbed system is relatively simple.

This symmetry allows us to find an explicit Lyapunov function for the system. First we notice that in this case the subspace spanned by $(1,1,1)^T$ and the two-dimensional subspace spanned by $(1,1,1)^T$ and $(1,-1,0)^T$, are not invariant. However the two-dimensional subspace orthogonal to $(1,-1,0)^T$, the subspace $\{x_1 = x_2\}$, is still invariant. Actually, we can give conditions under which it attracts all the solutions.

If we let $z = x_1 - x_2$, and if $x_1 \neq x_2$, then

$$\begin{aligned} \dot{z} &= -k(x_1 - x_2) + \frac{f(x_1,t) - f(x_2,t)}{x_1 - x_2}(x_1 - x_2)\\ &= -(k - a(x_1,x_2,t))z_1, \end{aligned}$$

with $a(x_1, x_2, t) = (f(x_1, t) - f(x_2, t))/(x_1 - x_2)$. So if $a < k_1 < k$, for $k_1 \in \mathbb{R}$, and for all x_1, x_2, t, $x_1 \neq x_2$, then $z(t) \to 0$ as $t \to +\infty$, which is equivalent to say that $\{x_1 = x_2\}$ is a synchronization manifold.

Theorem 5.1. If $f_1 = f_2 = f$ and for some $k_1 \in \mathbb{R}$

$$\frac{f(x_1, t) - f(x_2, t)}{x_1 - x_2} < k_1 < k$$

for all x_1, x_2, t, $x_1 \neq x_2$, then the system (7) synchronizes where $\{x_1 = x_2\}$ is a two-dimensional synchronization manifold.

This is what we can call identical synchronization. In this case, the asymptotic behavior of one oscillator can be determined by the asymptotic behavior of the other. Notice that in this situation we can prove that all the orbits, bounded or not, converge to the synchronization manifold. On the other hand, the two-dimensional manifold given by Theorem 4.2 is a graph over the subspace spanned by $(1, -1, 0)^T$ and $(1, 1, 1)^T$, so in general it is other kind of synchronization.

6. ABOUT THE RANGES THAT α, β AND γ CAN ASSUME

In Theorem 4.2 we saw that if α, β and γ are bounded then we can assure the existence of a sufficiently large k in order Theorem 4.1 holds. In this section we try to find optimal values to bound the parameters α, β, γ and k. As we will see, it is not easy to give analytic results about these bounds. However, we will be able to give some numerical results that will give us the necessary insight about them.

In order to make some graphical representations we must consider some restrictions on the parameters. We start by the case where $f_1 = f_2 = f$ and $h = 0$. In this case Ω only depends on ε, α, λ and k. On the other hand, the ε only introduces an arbitrary small perturbation on Ω, which means that if Ω is positive definite for $\varepsilon = 0$ then it is also positive definite for ε sufficiently small. So, in the following figures we make $\varepsilon = 0$. In Figure 1, on the left, we make $k = 1$ and draw the region in the plane $\lambda - \alpha$ where Ω is positive definite. This figure was drawn using the expressions in (11).

We only draw this figure for $\lambda \in (0, 3k)$, this is the relevant interval in Russel Smith's condition. For $\lambda \in (0, k)$, if the range of values of α falls in the shaded region for this λ then we have an one-dimensional manifold accordingly to Theorem 4.1. On the other hand, for $\lambda \in (k, 3k)$, if the range of values of α falls in the shaded region for this λ then we have a two-dimensional manifold accordingly to the same Theorem.

This picture can give us an idea of the kind of ranges α can span in order to guarantee synchronization. Notice that in general we must choose a different λ for each interval. When k grows, this shaded region has a similar shape but tends to become larger in the α direction, this is essentially what makes Theorem 4.2 come true.

We can draw an analogous figure for the case where there is only a perturbation on de medium and no perturbation on the oscillators, $f_1 = f_2 = 0$ (see Figure 1 on the right). In this case we can show a shaded region on the plane $\lambda - \gamma$. In this scenario we can make similar observations as we did for the last case.

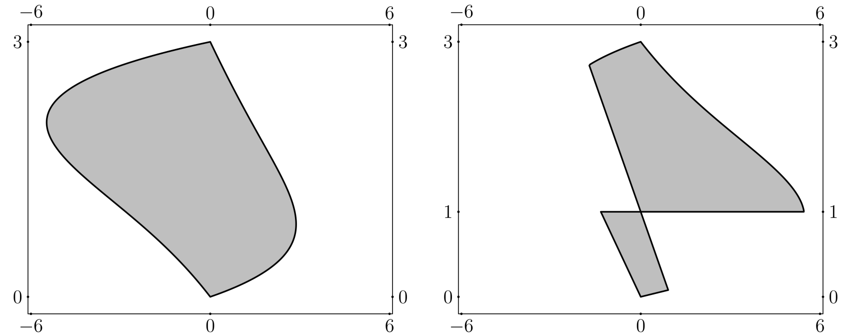
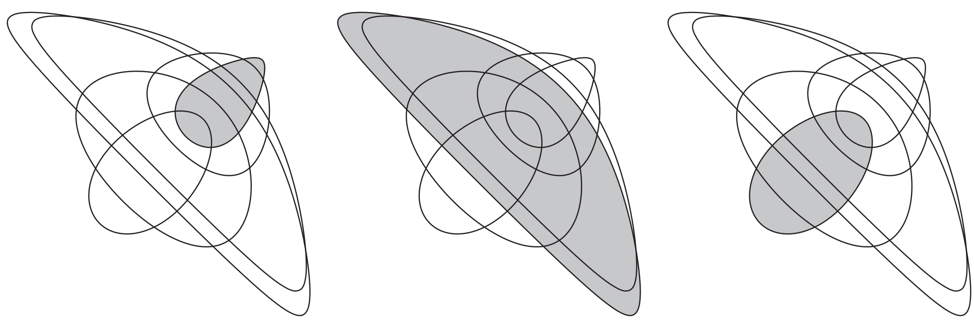


Fig. 1. On the left, the region where Ω is positive definite in the plane $\lambda - \alpha$ and on the right, the region where Ω is positive definite in the plane $\lambda - \gamma$.

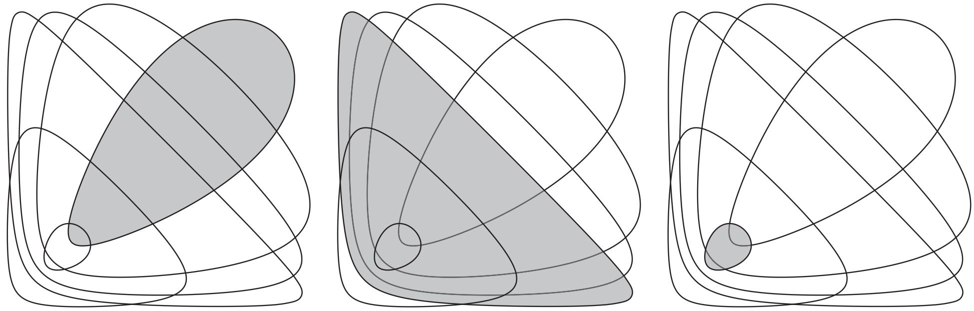
When the nonlinear perturbations on the oscillators are non-identical, the treatment of the problem is not so simple since we need one extra dimension. In Figure 2(a) − 2(c) we make $h = 0$ and draw the shaded region where Ω is positive definite, on the $\alpha - \beta$ plane, for $k = 1$ and a sample of values of λ in the open interval $(0, k)$.



(a) $\lambda = 0.1$ (b) $\lambda = 0.5$ (c) $\lambda = 0.9$

Fig. 2. Several examples of the domains $D_{k,\lambda}$, for $k = 1$ and $\lambda \in (0, 1)$. The contours represents the border of the domains, beyond those explicitly represented, for $\lambda \in \{0.2, 0.4, 0.7\}$.

On the other hand, in Figure 3(a) − 3(c) we consider a sample of values of λ in the open interval $(k, 3k)$. We would like to do a slightly deeper study of this case. From now on we will consider $\gamma = 0$. First of all, this is the natural choice to obtain a two dimensional bifurcation diagram. This fact is an important to explain the underlying idea involving the generalized synchronization. On the other hand, in [4] it is considered a system with identical oscillators and the general case, that we study in here, it is



(a) $\lambda = 1.1$ (b) $\lambda = 2.1$ (c) $\lambda = 2.9$

Fig. 3. Several examples of the domains $D_{k,\lambda}$, for $k = 1$ and $\lambda \in (1, 3)$. The contours represents the border of the domains, beyond those explicitly represented, for $\lambda \in \{1.6, 1.9, 2.5\}$.

produced via nonlinear pertubations that do not have any effect on the medium. To simplify the notation, we write $m_3(\alpha, \beta, 0) = m_3(\alpha, \beta)$. For fixed values of k and λ, we define $D_{k,\lambda}$ as the set where Ω is positive definite:

$$D_{k,\lambda} = \{(\alpha, \beta) \in \mathbb{R}^2 \colon m_1(\alpha) > 0 \wedge m_2(\alpha, \beta) > 0 \wedge m_3(\alpha, \beta) > 0\}.$$

In the figures we see that the domains $D_{k,\lambda}$ are not contained on each other for different values of λ. In general, given two intervals of values where the range of α and β are contained, we obtain a rectangle and the system synchronizes if this rectangle is a subset of one of this shaded region for some value of k and λ. Again, these regions tend to get larger as k increases. This is what makes Theorem 4.2 possible.

Notice also that if α, β and γ are contained in a compact set that is contained in the interior of the shaded area for some k and λ then we can find a sufficiently small ε in order that Russel Smith's condition holds, so there is essentially no loss of generality in considering these shaded regions for $\varepsilon = 0$.

7. CONVEXITY AND REGULARITY OF THE BOUNDARY OF $D_{K,\lambda}$

We now give some analytical results about the domains $D_{k,\lambda}$. We prove that they are convex and in which cases they have a smooth boundary. In a forthcoming paper we will use these properties to construct an algorithm to find the largest area isothetic rectangle, i. e. a rectangle with its sides parallel to the axis, that is contained in one of these regions. This allows us to, given a value of k, find concrete bounds on the nonlinearities, where synchronization occurs.

Theorem 7.1. Fix a value for λ, ε and k. The region $D_{k,\lambda}$ where Ω is positive definite is convex.

P r o o f. Fix a value for λ, ε and k and consider the set $D_{k,\lambda}$. Now consider (α_1, β_1) and (α_2, β_2) in $D_{k,\lambda}$. We will show that if $\xi \in (0,1)$ and

$$(\alpha, \beta) = (1-\xi)(\alpha_1, \beta_1) + \xi(\alpha_2, \beta_2)$$

then $(\alpha, \beta) \in D_{k,\lambda}$. Notice that a point (α, β) is in $D_{k,\lambda}$ if and only if (8) holds. Given $\binom{X}{y}$ and $\binom{Q}{w}$ in $\mathbb{R}^3$,

$$\begin{aligned}
&\begin{pmatrix} X-Q \\ y-w \end{pmatrix}^T \left[\left(\frac{1}{2}-\varepsilon\right) I - P \begin{pmatrix} (1-\xi)\alpha_1 + \xi\alpha_2 & 0 & 0 \\ 0 & (1-\xi)\beta_1 + \xi\beta_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] \begin{pmatrix} X-Q \\ y-w \end{pmatrix} \\
&\quad = (1-\xi) \begin{pmatrix} X-Q \\ y-w \end{pmatrix}^T \left[\left(\frac{1}{2}-\varepsilon\right) I - P \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \beta_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] \begin{pmatrix} X-Q \\ y-w \end{pmatrix} \\
&\quad + \xi \begin{pmatrix} X-Q \\ y-w \end{pmatrix}^T \left[\left(\frac{1}{2}-\varepsilon\right) I - P \begin{pmatrix} \alpha_2 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] \begin{pmatrix} X-Q \\ y-w \end{pmatrix} \geq 0.
\end{aligned}$$

We conclude that (α, β) satisfies (8), so it belongs to $D_{k,\lambda}$. □

With the next set of lemmas we prepare the proof of the smoothness of the boundary of the sets $D_{k,\lambda}$. Considering $\varepsilon = 0$, the set $D_{k,\lambda}$ can be seen as the intersection of three regions where each one of the following polynomials are positive:

$$\begin{aligned}
m_1(\alpha) &= \frac{1}{2} + \alpha a; \\
m_2(\alpha, \beta) &= \left(\frac{1}{2} + \alpha a\right)\left(\frac{1}{2} + \beta a\right) - b^2(\alpha+\beta)^2; \\
m_3(\alpha, \beta) &= \frac{1}{2} m_2(\alpha, \beta) + c^2 \left((2b-a)(\alpha+\beta)\alpha\beta - \frac{\alpha^2+\beta^2}{2}\right),
\end{aligned} \tag{12}$$

with the parameters given by (10). In general, if we intersect three regular domains with a smooth boundary, the intersection may produce singular points. In the case we have here, the three sets are sequentially contained in each other in a very specific way.

The behavior of the first minor $m_1(\alpha)$ is very simple. From the equations (12) we see that $m_1(\alpha) > 0$ is generically a semi-plane. If $a \neq 0$ i.e. if $\lambda \neq \frac{3\pm\sqrt{5}}{2}k$ then the boundary of this semi-plane is given by $\alpha = -\frac{1}{2a}$. If $\lambda = \frac{3\pm\sqrt{5}}{2}k$, then $m_1(\alpha) > 0$ is the whole plane.

The second minor is a quadratic form. First we notice that the region $m_2(\alpha, \beta) > 0$ is non-empty, since it clearly contains a neighborhood of the origin, so the conic section associated is non-degenerate. After expanding the expression in (12), the quadratic matrix of $m_2(\alpha, \beta)$, M, is given by

$$M = \begin{bmatrix} -b^2 & \frac{a^2-2b^2}{2} \\ \frac{a^2-2b^2}{2} & -b^2 \end{bmatrix}.$$

Without to much effort we can show that the determinant of M, is given by

$$|M| = a^2(4b^2 - a^2)/4 = \frac{(\lambda - 2k)(k^2 - 3k\lambda + \lambda^2)^2}{64\lambda^3(3k-\lambda)^3(\lambda-k)^3},$$

that is positive if $\lambda \in (0,k)$ or $\lambda \in (2k,3k)$ and negative if $\lambda \in (k,2k)$. This means that $m_2(\alpha,\beta)=0$ is an ellipse if $\lambda \in (0,k)$ or $\lambda \in (2k,3k)$, is a hyperbola if $\lambda \in (k,2k)$ and is a parabola if $\lambda = 2k$. It is straightforward to see that the line $\alpha = -1/2a$, provided that $\lambda \neq \frac{3\pm\sqrt{5}}{2}k$, is tangent to the curve $m_2(\alpha,\beta)=0$.

We define $D_{k,\lambda,m_i} = \{(\alpha,\beta) \in \mathbb{R}^2 \colon m_i(\alpha,\beta) > 0\}$ for $i=1,2,3$. A very simple argument shows that $D_{k,\lambda,m_2} \subset D_{k,\lambda,m_1}$. Indeed, it is true that $m_1(0,0) > 0$ and $m_2(0,0) > 0$. Due to the tangency stated above and that D_{k,λ,m_2} is a union of convex components, we can therefore state the following lemma.

Lemma 7.2. If $\lambda \in (0,k)\backslash\left\{\frac{3-\sqrt{5}}{2}k\right\}$ then D_{k,λ,m_2} is an interior of an ellipse and $D_{k,\lambda,m_2} \subset D_{k,\lambda,m_1}$. If $\lambda \in (k,2k)$ then D_{k,λ,m_2} has two connected components bounded by the hyperbola. In this case one of these components is contained in D_{k,λ,m_1}. If $\lambda \in (2k,3k)\backslash\left\{\frac{3+\sqrt{5}}{2}k\right\}$ then D_{k,λ,m_2} is again an interior of an ellipse and $D_{k,\lambda,m_2} \subset D_{k,\lambda,m_1}$. If $\lambda = \frac{3\pm\sqrt{5}}{2}k$ then D_{k,λ,m_2} is the region between the two lines $\alpha+\beta = \pm\frac{1}{2b}$ and D_{k,λ,m_1} is the whole plane, so clearly $D_{k,\lambda,m_2} \subset D_{k,\lambda,m_1}$.

When we move ourselves to understand what are the geometric and analytical consequences to put in scene the third minor m_3, we acknowledge that the problem becomes much more difficult. We start by study the points where the boundary of D_{k,λ,m_2} and D_{k,λ,m_3} intersect, i. e., the solution of the following nonlinear system

$$\begin{cases} \frac{a(\alpha+\beta)}{2} + a^2\alpha\beta - b^2(\alpha+\beta)^2 + \frac{1}{4} = 0 \\ (2b-a)\alpha\beta(\alpha+\beta) - \frac{\alpha^2+\beta^2}{2} = 0. \end{cases} \tag{13}$$

This system is not easily solvable, so we consider the change of variables $\alpha = \xi - \eta$ and $\beta = \xi + \eta$. This transforms the system (13) into

$$\begin{cases} (a^2-4b^2)\xi^2 + a\xi - a^2\eta^2 + \frac{1}{4} = 0 \\ 2(2b-a)\xi^3 - 2(2b-a)\eta^2\xi - \xi^2 - \eta^2 = 0. \end{cases} \tag{14}$$

To solve this system we start by looking for solutions with $\eta = 0$ and we find

$$(\alpha_0,\beta_0) = \left(\frac{1}{2(2b-a)}, \frac{1}{2(2b-a)}\right).$$

Notice that $2b-a = \frac{1}{2(k-\lambda)} \neq 0$. Then, solving the first equation in order to η^2 and substituting it in the second equation yields

$$8(2b-a)b^2\xi^3 - 4b(a-b)\xi^2 - \frac{2b-a}{2}\xi - \frac{1}{4} = 0.$$

Now we have one root of this equation, $\xi = \frac{1}{2(2b-a)}$ and we can factorize the second equation and obtain

$$8b^2\left(\xi - \frac{1}{2(2b-a)}\right)\left(\xi + \frac{1}{4b}\right)^2(2b-a) = 0.$$

For $\xi = -\frac{1}{4b}$, we obtain $\eta^2 = \frac{a(a-4b)}{4b^2a^2}$ and two more solutions

$$(\alpha_1, \beta_1) = \left(\frac{-a - \sqrt{a(a-4b)}}{4ab}, \frac{-a + \sqrt{a(a-4b)}}{4ab} \right),$$

$$(\alpha_2, \beta_2) = \left(\frac{-a + \sqrt{a(a-4b)}}{4ab}, \frac{-a - \sqrt{a(a-4b)}}{4ab} \right),$$

for $a > 0$, i.e. for $\lambda \in \left(\frac{3-\sqrt{5}}{2}k, k\right)$ and $\lambda \in \left(\frac{3+\sqrt{5}}{2}k, 3k\right)$. Actually, some computations show that $a - 4b > 0$ for $\lambda \in (0, 3k)\backslash\{k\}$.

The solution (α_0, β_0) belongs to the line $\alpha = \beta$ and the solutions (α_1, β_1) and (α_2, β_2) are the reflection of each other from the same line. This is not at all surprising because the system is indeed symmetric about this line. Symmetry is by all means an important property and is established in the next lemma.

Lemma 7.3. The domain $D_{k,\lambda}$ is symmetric in relation to the line $\alpha = \beta$.

Now we prove that at (α_i, β_i), for $i \in \{0, 1, 2\}$, the algebraic varieties $m_2(\alpha, \beta) = 0$ and $m_3(\alpha, \beta) = 0$ are indeed tangent to each other. A straightforward computation shows that

$$\nabla m_3(\alpha_0, \beta_0) = \left(\frac{1}{2} - \frac{c^2}{4b(2b-a)} \right) \nabla m_2(\alpha_0, \beta_0),$$

which means that at (α_0, β_0) the gradients are parallel. Necessarilly the algebraic varieties are tangent at (α_0, β_0). Following the same method, for $i = 1, 2$ we have

$$\nabla m_3(\alpha_i, \beta_i) = \left(\frac{1}{2} + \frac{c^2}{2ab} \right) \nabla m_2(\alpha_i, \beta_i)$$

and so at the points (α_1, β_1) and (α_2, β_2) the algebraic varieties are also tangent to each other. We conclude that:

Lemma 7.4. The algebraic varieties $m_2(\alpha, \beta) = 0$ and $m_3(\alpha, \beta) = 0$ are tangent at their intersection points and are not singular.

Next, we will study the regularity of curve $m_3(\alpha, \beta) = 0$. In the coordinates ξ and η, m_2 and m_3 are given by

$$m_2(\xi, \eta) = (a^2 - 4b^2)\xi^2 + a\xi - a^2\eta^2 + \frac{1}{4},$$

$$m_3(\xi, \eta) = \frac{1}{2} m_2(\xi, \eta) + 2c^2(2b-a)\xi^3 - 2c^2(2b-a)\eta^2\xi - c^2\xi^2 - c^2\eta^2.$$

In order to find the points where m_3 is non-singular (see [1], p. 33), we have to find the points where the gradient $\nabla m_3(\xi, \eta) = (0, 0)$. The last condition is equivalent to have

$$\begin{cases} (a^2 - 4b^2 - 2c^2)\xi + 2c^2(2b-a)(3\xi^2 - \eta^2) + \frac{a}{2} = 0 \\ (a^2 + 2c^2)\eta + 4c^2(2b-a)\eta\xi = 0. \end{cases} \tag{15}$$

Take notice that along the line $\eta = 0$ the solutions of the system (15) are the solutions of

$$\frac{\partial m_3}{\partial \xi}(\xi, 0) = 6c^2(2b-a)\xi^2 + (a^2 - 4b^2 - 2c^2)\xi + \frac{a}{2} = 0, \tag{16}$$

and we get a singular point if a solution of (16) is also solution of $m_3(\xi, 0) = 0$, i.e.

$$m_3(\xi, 0) = 2c^2(2b-a)\xi^3 + \frac{1}{2}(a^2 - 4b^2 - 2c^2)\xi^2 + \frac{a}{2}\xi + \frac{1}{8} = 0.$$

The roots of this polynomial are

$$\xi_1 = -\frac{1}{2(a-2b)}, \quad \xi_\pm = \frac{a + 2b \pm \sqrt{(a+2b)^2 + 8c^2}}{8c^2}$$

It is obvious that the two solutions of (16) are between the last three roots. Then the solutions of $\nabla m_3(\xi, \eta) = (0, 0)$ along the line $\eta = 0$ are the values of ξ where two of the last roots coincide. Since $c \neq 0$ then $\xi_+ \neq \xi_-$ for all $\lambda \in (0, 3k) \backslash \{k\}$. On the other hand, after some computations we find that

$$b_1 = b_\pm \Leftrightarrow c^2 + 2b(a-2b) = 0 \Leftrightarrow \lambda = \left(\frac{7 \pm 2\sqrt{11}}{5}\right) k.$$

Looking now for solutions where $\eta \neq 0$ and going back to the system (15), the second equation gives us immediately

$$\xi_s = \frac{a^2 + 2c^2}{4c^2(a-2b)}. \tag{17}$$

Using this information in the first equation of system (15) we get

$$2(a-2b)c^2\eta^2 - \frac{(a^2+2c^2)(a^2+8b^2+10c^2)}{8(a-2b)c^2} + \frac{a}{2} = 0.$$

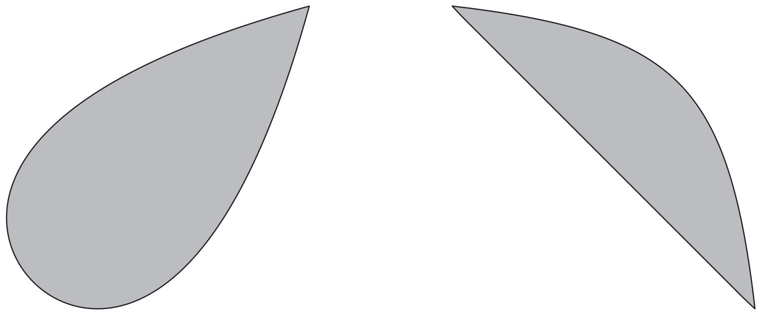
It is not surprising that the value of ξ_s found in (17) will produce the two symmetric values for η

$$\eta_\pm = \pm\sqrt{\frac{(a^2+2c^2)(a^2-8b^2-10c^2)}{16c^2(a-2b)^2} + \frac{a}{4c^2(a-2b)}}.$$

This is of course result of the symmetry of the domains stated in lemma 7.3. The change of variables just produced a change of the axis of symmetry of the domains. This symmetry implies that $m_3(\xi_s, \eta_-) = m_3(\xi_s, \eta_+)$. Again, for the points $(\xi_s, \eta_\pm)$ belong to $\partial D_{k,\lambda,m_3}$ it is necessary that $m_3(\xi_s, \eta_+) = 0$. Writing the last equation in the variables (k, λ) we get[1]

$$\begin{aligned} m_3(\xi_s, \eta_+) &= -\frac{(3k^2 - 8k\lambda + 3\lambda^2)^2(2k^4 - 8k^3\lambda + 12k^2\lambda^2 - 6k\lambda^3 + \lambda^4)}{8\lambda^2(k-\lambda)^4(\lambda-3k)^2} \\ &= 0. \end{aligned}$$

[1] The following computations are elementar but to long to be made by hand. We used a computer algebra system.



(a) $\lambda = \frac{7-2\sqrt{11}}{5}k$ (b) $\lambda = \frac{4-\sqrt{7}}{3}k$

Fig. 4. Representation of the non-regular cases of $D_{k,\lambda}$.

So the solutions of the equation $m_3(\xi_s, \eta_+) = 0$ will be also solution of

$$3k^2 - 8k\lambda + 3\lambda^2 = 0 \quad \vee \quad 2k^4 - 8k^3\lambda + 12k^2\lambda^2 - 6k\lambda^3 + \lambda^4 = 0.$$

Again, with a computer algebra system, its possible to show that the only solutions of the last equations are

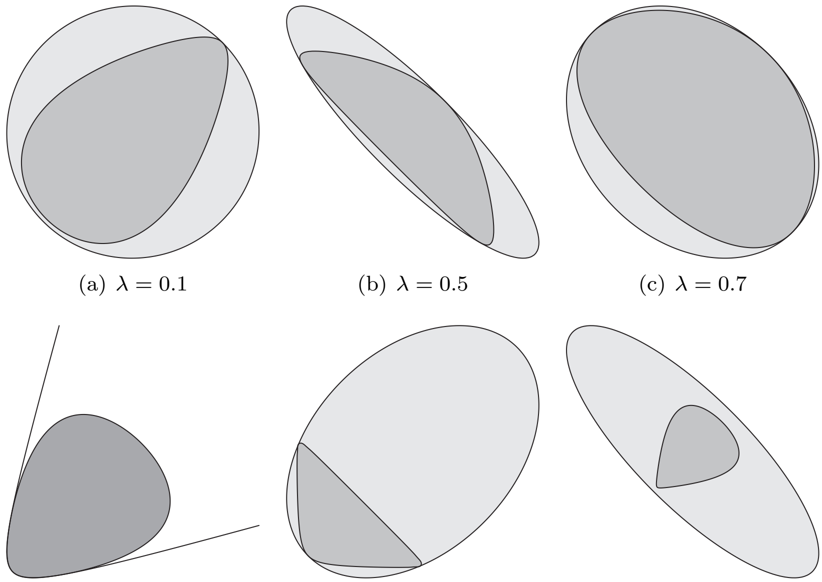
$$\lambda = \left(\frac{4 \pm \sqrt{7}}{3}\right) k.$$

In Figures 4(a) − 4(b) we represent two examples of the values where $\partial D_{k,\lambda}$ is not smooth. In all other cases, the implicit function theorem give us the guarantee that the domains are of class C^∞.

So far, we have proved that our domain is convex and we have shown in which situations they are regular. We have one last property that comes out a little bit surprising. As stated before, the origin is contained in all domains D_{k,λ,m_1}, D_{k,λ,m_2} and D_{k,λ,m_3}. We already showed that at these points the borders of the domains are tangent to each other. This proves that the intersections do not produce singularities. What is more surprising is the fact that the connected component of D_{k,λ,m_3} that contains the origin is totally contained in D_{k,λ,m_2}. This can be seen through the Figures 5(a) − 5(f). The next theorem states precisely this.

Theorem 7.5. If $\lambda \in (0, 3k)\backslash \left\{k, \frac{4\pm\sqrt{7}}{3}k, \frac{7\pm2\sqrt{11}}{5}k\right\}$ the boundary of $D_{k,\lambda}$ is C^∞. Moreover, for the connected component of D_{k,λ,m_3}, call it D^0_{k,λ,m_3}, that contains $(0,0)$ we have $D^0_{k,\lambda,m_3} = D_{k,\lambda}$.

Proof. We have already shown that the algebraic variety $m_3 = 0$ is C^∞. As we have seen from the previous observations, there are two distinct situations: $\partial D_{k,\lambda,m_3}$ and $\partial D_{k,\lambda,m_2}$ intersect in one or three points. We concentrate first in the case when the intersection occurs in three distinct points. Using the coordinates (ξ, η), when $\xi = 0$



(a) $\lambda = 0.1$ (b) $\lambda = 0.5$ (c) $\lambda = 0.7$

(d) $\lambda = 1.3$ (e) $\lambda = 2.15$ (f) $\lambda = 2.75$

Fig. 5. Relation between $m_2(\alpha, \beta) \geq 0$ and the third minor.

and $m_2(\xi, \eta) = 0$ we have $\eta = \pm\frac{1}{2a}$. We obtain a point of intersection $(\xi^*, \eta^*) = (0, \frac{1}{2a})$ and another one that is symmetric with this one. On the other hand, the intersection point (ξ_1, η_1) is given by

$$(\xi_1, \eta_1) = \left(-\frac{1}{4b}, \frac{\sqrt{a(a-4b)}}{2ab}\right).$$

Knowing that $-\frac{1}{4b} = -\frac{(k-\lambda)(3k-\lambda)\lambda}{k^2}$ and that the value of ξ for (α_0, β_0) is $\frac{1}{2(2b-a)} = k-\lambda$ it is easy to see that we have both

$$\begin{cases} -\frac{1}{4b} < 0 \wedge \frac{1}{2(2b-a)} > 0, & \lambda \in (0, k) \\ -\frac{1}{4b} > 0 \wedge \frac{1}{2(2b-a)} < 0, & \lambda \in (k, 3k). \end{cases}$$

A simple computation shows that

$$m_3(\xi^*, \eta^*) = m_3\left(0, \frac{1}{2a}\right) = -\frac{c^2}{4a^4} < 0. \tag{18}$$

Putting all this together, we have shown that the point (ξ^*, η^*) is between (α_0, β_0) and (α_1, β_1) and that $(\xi^*, \eta^*) \notin D_{k,\lambda,m_3}$. This shows that for ξ between $-\frac{1}{2a}$ and $\frac{1}{2(2b-a)}$, the connected component of D_{k,λ,m_3} that contains $(0,0)$ does not get out D_{k,λ,m_2}. By symmetry, we know that the same situation happens between the points (α_0, β_0) and (α_2, β_2). A similar argument, done along the line $\eta = 0$ could be used to show that this is exactly the same situation between (α_2, β_2) and (α_1, β_1).

When there is only one intersection point, a similar argument could be done to show that the connected component of D_{k,λ,m_3} containing the point $(0, 0)$ is totally contained in D_{k,λ,m_2}. □

With this theorem we just finish the geometric characterization of the domains $D_{k,\lambda}$. The property that it exhibits, the inclusion

$$D^0_{k,\lambda,m_3} \subset D_{k,\lambda,m_2} \subset D_{k,\lambda,m_1}$$

is by all means remarkable. A complete understanding of this phenomena remains open and is certainly important enough to be object of future work.

A. PROOF OF THEOREM OF GENERALIZED SYNCHRONIZATION

In order to keep this paper as self contained as possible, we present a proof of Theorem of Generalized Synchronization (theorem 2.3). To simplify the notation, we will consider a simplified version of system (1) written as

$$x' = f(x, t), x \in \mathbb{R}^N,$$

where $N = nm+p$. Without loss of generality, we assume that the origin is an amenable point. Indeed, if $\left(\overline{x}_0, \overline{t}_0\right)$ is an amenable point, consider the change of variables $\tilde{x}(t) = x(t) - \overline{x}(t)$, with $\overline{x}(t) = x\left(t; \overline{x}_0, \overline{t}_0\right)$. Then $x(t; t_0, x_0)$ is solution of the equation (1) if and only if $\tilde{x}(t; t_0, x_0)$, with $\tilde{x}_0 = x_0 - \overline{x}_0$, is solution of the equation

$$\dot{\tilde{x}}(t) = f(\tilde{x}(t) + \overline{x}(t), t) - \dot{\overline{x}}(t) := \tilde{f}\left(\tilde{x}(t), t\right). \tag{19}$$

The next lemma establishes the relation among the amenable points of both equations.

Lemma A.1. The point $(\tilde{x}_0, t_0)$ is amenable for equation (19) if and only if $(x_0, t_0) = (\tilde{x}_0 + \overline{x}(t_0), t_0)$ is an amenable point for equation (1). In particular have

$$\tilde{\mathcal{A}}_{t_0} = \mathcal{A}_{t_0} - \overline{x}(t_0).$$

Proof. By the inequality $\|A \pm B\|^2 \leq 2\|A\|^2 + 2\|B\|^2$, using the change of variables $\tilde{x}(t) = x(t) - \overline{x}(t)$ it follows the inequality

$$\int_{-\infty}^{t_0} e^{2\lambda t}\|x(t; x_0, t_0)\|^2 \,\mathrm{d}t \leq 2\int_{-\infty}^{t_0} e^{2\lambda t}\|\tilde{x}(t; \tilde{x}_0, t_0)\|^2 \,\mathrm{d}t + 2\int_{-\infty}^{t_0} e^{2\lambda t}\|\overline{x}(t)\|^2 \,\mathrm{d}t.$$

This shows that if $(\tilde{x}_0, t_0)$ is an amenable point for equation (19) then $(x_0, t_0) = (\tilde{x}_0 + \overline{x}_0, t_0)$ is amenable for equation (1). In the opposite direction we have

$$\int_{-\infty}^{t_0} e^{2\lambda t} \|\tilde{x}(t; \tilde{x}_0, t_0)\|^2 \, \mathrm{d}t \leq 2 \int_{-\infty}^{t_0} e^{2\lambda t} \|x(t; x_0, t_0)\|^2 \, \mathrm{d}t + 2 \int_{-\infty}^{t_0} e^{2\lambda t} \|\overline{x}(t)\|^2 \, \mathrm{d}t,$$

which is enough to show the reciprocal implication. □

From the previous lemma we have that $\mathcal{A}_{t_0}$ is a graph of a function over $\mathcal{V}_-$ if and only if $\tilde{\mathcal{A}}_{t_0}$ also is. On the other hand, if the origin is an equilibrium point, i. e. $f(0, t) = 0$ for all $t \in \mathbb{R}$, then any $(0, t)$ with $t \in \mathbb{R}$ is also an amenable point for equation (1). From this point on, we will assume this in all further developments. The next lemma gives a characterization of the quadratic form V on the amenable points.

Lemma A.2. If the Russel Smith's condition (4) is valid then, given an amenable point (α_0, t_0), any other point (α_1, t_0) is amenable if and only if for all $t \in \mathbb{R}$ we have $V(x_1(t; \alpha_1, t_0) - x_0(t; \alpha_0, t_0)) < 0$.

Proof. Integrating (2) in the interval (α, τ) we have

$$e^{2\lambda\tau} V(x_1(\tau) - x_0(\tau)) \leq e^{2\lambda\alpha} V(x_1(\alpha) - x_0(\alpha)) - 2\varepsilon \int_{\alpha}^{\tau} e^{2\lambda t} \|x_1(t) - x_0(t)\|^2 \, \mathrm{d}t, \tag{20}$$

with $x_1(t) = x_1(t; \alpha_1, t_0)$ and $x_0(t) = x_0(t; \alpha_0, t_0)$. Assuming that (α_1, t_0) is amenable observe that

$$\int_{-\infty}^{t_0} e^{2\lambda t} \|x_1(t) - x_0(t)\|^2 \, \mathrm{d}t \leq 2 \int_{-\infty}^{t_0} e^{2\lambda t} \|x_1(t)\|^2 \, \mathrm{d}t + 2 \int_{-\infty}^{t_0} e^{2\lambda t} \|x_0(t)\|^2 \, \mathrm{d}t < \infty.$$

Then there is a sequence $t_n \to -\infty$ such that $e^{2\lambda t} \|x_1(t_n) - x_0(t_n)\|^2 \to 0$. Assigning $\alpha = t_n$ in (20) and if we let $n \to \infty$ we get

$$e^{2\lambda\tau} V(x_1(\tau) - x_0(\tau)) \leq -2\varepsilon \int_{-\infty}^{\tau} e^{2\lambda t} \|x_1(t) - x_0(t)\|^2 \, \mathrm{d}t$$

which means that $V(x_1(t) - x_0(t)) < 0$ for all $t \in \mathbb{R}$. Reciprocally, again by (20), we have

$$0 \leq 2\varepsilon \int_{-\infty}^{t_0} e^{2\lambda t} \|x_1(t) - x_0(t)\|^2 \, \mathrm{d}t \leq -e^{2\lambda t_0} V(x_1(t_0) - x_0(t_0)).$$

Then necessarily $\int_{-\infty}^{t_0} e^{2\lambda t}\|x_1(t)-x_0(t)\|^2\,\mathrm{d}t<\infty$, whereby

$$\int_{-\infty}^{t_0} e^{2\lambda t}\|x_1(t)\|^2\,\mathrm{d}t \le 2\int_{-\infty}^{t_0} e^{2\lambda t}\|x_1(t)-x_0(t)\|^2\,\mathrm{d}t + 2\int_{-\infty}^{t_0} e^{2\lambda t}\|x_0(t)\|^2\,\mathrm{d}t \le \infty.$$

This proves that (α_1, t_0) is amenable. □

By the symmetry of matrix P, we know that $\mathbb{R}^N$ admits an orthonormal base made by eigenvectors of this matrix. We represent it by

$$v_1^-,\ldots,v_k^-,v_{k+1}^+,\ldots,v_N^+.$$

Hence $M=\left[v_1^-\ldots v_k^- v_{k+1}^+\ldots v_N^+\right]$ is an orthogonal matrix and therefore

$$Q=M^TPM=\operatorname{diag}\left\{\lambda_1^-,\ldots,\lambda_k^-,\lambda_{k+1}^+,\ldots,\lambda_N^+\right\}.$$

Thus, to represent V by the matrix Q we have to produce the change of variables

$$M^TX=\Xi=\begin{pmatrix}\xi_1\\ \vdots\\ \xi_N\end{pmatrix}.$$

We define the projection of $\mathbb{R}^N$ over $\mathcal{V}_-$ by

$$\pi_-\colon \mathbb{R}^N\to\mathcal{V}_-$$
$$\pi_-(X)=(\xi_1\ldots\xi_k\,0\ldots0)^T.$$

In the variables Ξ, the quadratic form V is given by

$$\begin{aligned}V(X)=X^TPX=&X^TMM^TPMM^TX\\ =&(M^TX)^TM^TPM(M^TX)=\lambda_1^-\xi_1^2+\cdots+\lambda_N^+\xi_N^2.\end{aligned}$$

This observation is useful to show the following lemma.

Lemma A.3. There are $\delta>0$ and $\overline{\lambda}$ such that, for all $X\in\mathbb{R}^N$

$$\overline{\lambda}\left[\delta V(X)+\|\pi_-(X)\|^2\right]>\|\Xi\|^2\ge\|\pi_-(X)\|^2.$$

P r o o f. For a sufficiently small δ for which is valid the inequality

$$-1<\delta\lambda_i^-<0<\delta\lambda_j^+$$

and another $\overline{\lambda}$ that also check the inequalities

$$\overline{\lambda}>\frac{1}{1+\delta\lambda_i^-}\quad\text{e}\quad\overline{\lambda}>\frac{1}{\delta\lambda_j^+}$$

for $i = 1, \dots, k$ e $j = k+1, \dots, n$, we will get successively

$$\begin{aligned}
&\overline{\lambda}\left[\delta V(X) + \|\pi_-(X)\|^2\right] \\
&\quad = \overline{\lambda}\left[(1+\delta\lambda_1^-)\xi_1^2 + \cdots + (1+\delta\lambda_k^-)\xi_k^2 + \delta\lambda_{k+1}^+\xi_{k+1}^2 + \cdots + \delta\lambda_N^+\xi_N^2\right] \\
&\quad > \xi_1^2 + \cdots + \xi_k^2 + \xi_{k+1}^2 + \cdots + \xi_N^2 = \|\Xi\|^2 \\
&\quad \geq \xi_1^2 + \cdots + \xi_k^2 = \|\pi_-(X)\|^2,
\end{aligned}$$

which is the point that we wanted to prove. □

The inequality proved above is fundamental to show the next result.

Lemma A.4. Given $t_0 \in \mathbb{R}$ and the correspondent manifold $\mathcal{A}_{t_0}$, the function $\pi_-\colon \mathcal{A}_{t_0} \to \pi_-(\mathcal{A}_{t_0}) \subset \mathcal{V}_-$ is one-to-one, continuous and globally Lipchitz.

P r o o f. For the usual topology we know that a projection in a vector space is a continuous map. Given two amenable points $x_1 \neq x_2$ in $\mathcal{A}_{t_0}$, by lemma A.2 we get $V(x_1 - x_2) < 0$. By lemma A.3, with the correspondence $M^T x_i = \Xi_i$, we obtain

$$\overline{\lambda}\|\pi_-(x_1 - x_2)\|^2 > \|\Xi_1 - \Xi_2\|^2 \geq \|\pi_-(x_1 - x_2)\|^2,$$

which is enough to assure $\pi_-(x_1) \neq \pi_-(x_2)$. Otherwise we would have either $\Xi_1 = \Xi_2$ and $x_1 = x_2$. On the other hand, by the same inequality we also get

$$\|\pi_-(x_1 - x_2)\| \leq \|\Xi_1 - \Xi_2\| \leq \|M\|\|X_1 - X_2\|.$$

This means that $\pi_-\colon \mathcal{A}_{t_0} \to \mathcal{V}_-$ is $\|M\|$-Lipchitz. □

To proceed in the direction of our goal we need to introduce the Wazewski's Topological Principle (see [10] for full details). To do that we need to set up some concepts.

Definition A.5. Let X be a topological space and $A \subset X$. A continuous map $r\colon X \to A$ such that $r(a) = a$ for all $a \in A$ is called a *retraction*. The set A is called a *retract* of X if there exists a retraction $r\colon X \to A$.

A classical result of Algebraic Topology shows that in $\mathbb{R}^n$ the border of unit disk $\partial D^n = \{x \in \mathbb{R}^n\colon |x| = 1\}$ is not a retraction of $D^n = \{x \in \mathbb{R}^n\colon |x| \leq 1\}$ (see [2], p. 114 for full details). Consider a continuous vector field f on a open set $A \subset \mathbb{R}^n$ and a Cauchy problem

$$\begin{cases} \dot{x} = f(x,t) \\ x(t_0) = x_0, \end{cases} \tag{21}$$

for which the existence and uniqueness of solutions holds. Let $x(t; t_0, x_0)$ be the flux of f and Ω an open set in $\mathbb{R}^n \times \mathbb{R}$.

Definition A.6. Apoint $(t_0, x_0) \in \partial\Omega$ is called an *ingress point* for the equation (21) if there is $\varepsilon > 0$ such that $(x(t, t_0, x_0), t) \in \Omega$ for all $t \in (t_0, t_0 + \varepsilon]$. Furthermore, if $(x(t; t_0, x_0), t) \notin \overline{\Omega}$ for any $t \in (t_0 - \varepsilon, t_0)$ then (t_0, x_0) is called a *strict ingress point.*

We represent by Ω_i and Ω_{si}, respectively, the sets of ingress and strictly ingress points. We are in conditions to state the so called Wazewski's Topological Principle as presented in [10].

Theorem A.7. (Wazewski's Topological Principle) Assuming that $\Omega_i = \Omega_{si}$, let $S \subset \Omega \cup \Omega_i$ such that $S \cap \Omega_i$ is a retract of Ω_i and $S \cap \Omega_i$ is not a retract of S. Then there is a point $(t_0, x_0) \in S \cap \Omega$ such that the respective solution of (21) verifies $(t, x(t; t_0, x_0)) \in \Omega$ for all $t \in (\alpha(t_0, x_0), t_0]$, where $\alpha(t_0, x_0)$ is the lower bound of the maximal interval of existence for the solution $x(t; t_0, x_0)$.

We are now in condition to complete the prove that $\mathcal{A}_{t_0}$ is a graph over $\mathcal{V}_-$.

Lemma A.8. Consider the sets $\mathcal{A}_{t_0}$ e $\mathcal{V}_-$ given before. The map $\pi_-\colon \mathcal{A}_{t_0} \to \mathcal{V}_-$ is onto.

Proof. Let C be the cone associated to V

$$C = \left\{x \in \mathbb{R}^N : V(x) < 0\right\} = \left\{x \in \mathbb{R}^N : \lambda_1^- \xi_1^2 + \cdots + \lambda_N^+ \xi_N^2 < 0\right\},$$

and Ω the subset of $\mathbb{R}^N \times \mathbb{R}$ defined by

$$\Omega = \left\{(x, t) \in \mathbb{R}^N \times \mathbb{R} \colon V(x) < 0\right\}.$$

If $\Omega_{t_0} = \{(x, t) \in \Omega \colon t = t_0\}$, $(x_0, t_0) \in \partial\Omega$ and $x_0 = 0$ then $(x_0, t_0) \notin \Omega_i$. In alternative, if $(x_0, t_0) \in \partial\Omega$ and $x_0 \neq 0$, by (2) we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\{e^{2\lambda t} V(x(t; x_0, t_0))\right\}\bigg|_{t=t_0} \leq -2e^{2\lambda t_0} \varepsilon \|x_0\|^2 < 0.$$

Hence, in a neighborwood t_0, with $t < t_0$ we have $V(x(t)) > 0$ and $x(t) \notin \Omega$. In a neighborhood of t_0, with $t > t_0$ we have $V(x(t)) < 0$ e $x(t) \in \Omega$. From these points we get

$$\Omega_i = \Omega_{si} = \partial\Omega \setminus \{(0, t) \colon t \in \mathbb{R}\}. \tag{22}$$

Given $\overline{\xi} \in \mathcal{V}_-$, our task is to find a $x_0 \in \mathcal{A}_{t_0}$ such that

$$\pi_-(x_0) = \overline{\xi} = \left(\overline{\xi_1}, \dots, \overline{\xi_k}, 0, \dots, 0\right).$$

To apply the Wazewski's Topological Principle we define the set

$$\begin{aligned} S &= \left\{(x, t_0) \in \mathbb{R}^N \times \mathbb{R} \colon\ \pi_-(x) = \overline{\xi} \text{ e } V(x) \leq 0\right\} \\ &= \Big\{(x, t_0) \in \mathbb{R}^N \times \mathbb{R} \colon\ \lambda_{k+1}^+ \xi_{k+1}^2 + \cdots + \lambda_N^+ \xi_N^2 \\ &\qquad\qquad \leq -\lambda_1^- \overline{\xi}_1^2 - \cdots - \lambda_k^- \overline{\xi}_k^2 \wedge \pi_-(x) = \overline{\xi}\Big\} \end{aligned}$$

It is easy to show that S and D^{N-k} are homeomorphic. On the other hand we have that

$$\begin{aligned} S \cap \partial\Omega = \Big\{(x, t_0) \in \mathbb{R}^N \times \mathbb{R} \colon\ &\lambda_{k+1}^+ \xi_{k+1}^2 + \cdots + \lambda_N^+ \xi_N^2 \\ &= -\lambda_1^- \overline{\xi}_1^2 - \cdots - \lambda_k^- \overline{\xi}_k^2 \wedge \pi_-(x) = \overline{\xi}\Big\} \end{aligned}$$

is homeomorphic to $S^{N-k-1} = \partial D^{N-k}$. Therefore, $S \cap \partial\Omega$ is not a rectract of S. Hence, the set of ingress point may be written as

$$\Omega_i = \left\{ (x, t_0) \in \mathbb{R}^N \times \mathbb{R} \colon \lambda_1^- \xi_1^2 + \cdots + \lambda_N^+ \xi_N^2 = 0 \wedge \sum_{j=1}^{N} \xi_k^2 > 0 \right\}.$$

Is easy to show that $S \cap \Omega_i = S \cap \partial\Omega$. We will now show that $S \cap \Omega_i$ is a retract of Ω_i. Following the same arguments given in [5], adapted to our framework, it is easy to find a retraction of $r_1 \colon \Omega_i \to \partial(\Omega_{t_0} \setminus \{(0, t_0)\})$. Consider the set $T = \{x \in \partial\Omega_{t_0} \colon V(\pi_-(x)) = V(\bar{\xi})\}$. Is direct to show that $r_2 \colon \partial(\Omega_{t_0} \setminus \{(0, t_0)\}) \to T$ defined by

$$r_2(x) = \frac{V(\bar{\xi})}{V(\pi_-(x))} x$$

is a retraction. Defining $\pi_+ := I - \pi_-$, the set T can also be given by the equalities

$$V(\pi_-(x)) = V(\bar{\xi}) \text{ e } V(\pi_+(x)) = -V(\bar{\xi}).$$

The first defines a set that is diffeomorphous to $\mathbb{S}^{k-1} \subset \mathcal{V}_-$ and the second a set diffeomorphous to $\mathbb{S}^{N-k-1} \subset \mathcal{V}_+$. Therefore T is given by the cartesian product $T_1 \times T_2$ diffeomorphic to $\mathbb{S}^{k-1} \times \mathbb{S}^{N-k-1}$. Finally we may define a retraction $r_3 \colon T \to S \cap \partial\Omega_i$ by $r_3(x) := \bar{\xi} + \pi_+(x)$. Follows immediately that $r_3 \circ r_2 \circ r_1$ is a retraction of Ω_i in $S \cap \Omega_i$.

By Wazewski's Topological Principle there is a point $(x_0, t_0) \in S \cap \Omega$ such that $x(t; x_0, t_0) \in \Omega$ for all $t \in \mathbb{R}$. This equivalent to say that $V(x(t; x_0, t_0)) < 0$ for all $t \in \mathbb{R}$. By lemma A.2 the point (x_0, t_0) is amenable and $\pi_-(x_0) = \bar{\xi}$. □

To finish the proof of Theorem 2.3 we still need to show that the amenable manifold $\mathcal{A}_t$ is the assymptotic limit of the bounded orbits.

Lemma A.9. If $x(t)$ for all $t > 0$ is a bounded solution of (1) then

$$d(x(t), \mathcal{A}_t) \xrightarrow[t \to +\infty]{} 0.$$

P r o o f. By assuming that the system (1) is T-periodic, the Poincaré stroboscopic map $\mathcal{P} \colon \mathbb{R}^N \to \mathbb{R}^N$

$$\mathcal{P}(x_0) = x(T; x_0, 0)$$

is well defined. Because the sequence $\{x(cT; 0, x_0)\}_{c \in \mathbb{N}}$ is bounded, its ω-limit, represented by the set A, is non-empty, compact and invariant for the Poincaré map. Consider a solution $y(t) = y(t; y_0, 0)$ such that $y_0 \in A$. As $y(t)$ is contained in a compact set

$$\{x(t; A, 0) \colon t \in [0, T]\},$$

then $y(t)$ is bounded and therefore $(y_0, 0)$ is an amenable point. With an analogous argument we may show that the ω-limit of the sequence $\{x(cT + t)\}_{c \in \mathbb{N}}$ is a subset of $\mathcal{A}_t$ for all $t \in \mathbb{R}$. In order to obtain a contradiction, suppose that there is a sequence $t_c \to +\infty$ such that

$$d\left(x(t_c), \mathcal{A}_t\right) > \varepsilon > 0.$$

Let $t_c = l_c + h_c T$, with $l_c \in [0,T]$ and $h_c \in \mathbb{Z}$. Because $\{l_c\}$ and $\{x(t_c)\}$ are both bounded we may assume the existence of $l \in [0,T]$ and $P \in \mathbb{R}^N$ such that $l_c \to l$ and $x(t_c) \to P$. We will then get

$$\begin{aligned}\|x(h_cT+l)-P\| &\le \|x(t_c-l_c+l)-x(t_c)\| + \|x(t_c)-P\| \\ &\le \max_{t>0}\|x'(t)\|\|l_c-l\| + \|x(t_c)-P\| \xrightarrow[k\to+\infty]{} 0.\end{aligned}$$

Then necessarilly either $x(h_cT+l) \to P$ and $P \in \mathcal{A}_l$. On the other hand, by the time periodicity of the system (1), we have $\mathcal{A}_{t_0+T} = \mathcal{A}_{t_0}$ and therefore

$$0 < \varepsilon < d\left(x(t_c), \mathcal{A}_{t_c}\right) = d\left(x(t_c), \mathcal{A}_{l_c}\right).$$

However, we also have

$$\begin{aligned}d\left(x(t_c), \mathcal{A}_{l_c}\right) &< \|x(l_c;P,l)-x(t_c)\| \\ &\le \|x(l_c;P,l)-P\| + \|P-x(t_c)\| \xrightarrow[k\to+\infty]{} 0\end{aligned}$$

which results in a contradiction. Then $d(x(t), \mathcal{A}_t) \xrightarrow[t\to+\infty]{} 0$. □

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