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CO-RANK AND BETTI NUMBER OF A GROUP

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Abstract. For a finitely generated group, we study the relations between its rank, the maximal rank of its free quotient, called co-rank (inner rank, cut number), and the maximal rank of its free abelian quotient, called the Betti number. We show that any combination of the group's rank, co-rank, and Betti number within obvious constraints is realized for some finitely presented group (for Betti number equal to rank, the group can be chosen torsion-free). In addition, we show that the Betti number is additive with respect to the free product and the direct product of groups. Our results are important for the theory of foliations and for manifold topology, where the corresponding notions are related with the cut-number (or genus) and the isotropy index of the manifold, as well as with the operations of connected sum and direct product of manifolds.

Keywords: co-rank; inner rank; fundamental group

MSC 2010: 20E05, 20F34, 14F35

In this paper, the relation between the rank $\operatorname{rk} G$, co-rank $\operatorname{corank}(G)$, and the Betti number $b(G)$ of a finitely generated group G is studied. The latter two values bound the isotropy index $i(G)$ of G: $\operatorname{corank}(G) \leqslant i(G) \leqslant b(G)$, see [2], [5], [11]. These notions have important applications in the theory of manifolds, where they are called the first non-commutative Betti number $b_1'(M) = \operatorname{corank}(\pi_1(M))$, the first Betti number $b_1(M) = b(\pi_1(M))$, and the isotropy index of the manifold $h(M) = i(\pi_1(M))$, where $\pi_1(M)$ is the fundamental group of the manifold M. Namely, for any $n \geqslant 4$, a group is the fundamental group of a smooth closed connected n-manifold if and only if it is finitely presented. In the theory of 2- and 3-manifolds, co-rank of the fundamental group coincides with the cut-number, a generalization of the genus for closed surfaces [7], [13]. In the theory of foliations of Morse forms, $b_1'(M)$ and $h(M)$ define the topology of the foliation [5], [6], the form's cohomology class [3], and the types of its singularities [4].

For a finitely generated abelian group $G = \mathbb{Z}^n \oplus T$, where T is finite, its torsion-free rank, Prüfer rank, or (first) Betti number, is defined as $b(G) = \operatorname{rk}(G/T) = n$. The latter term extends to finitely generated groups by $b(G) = b(G^{ab}) = \operatorname{rk}(G^{ab}/\operatorname{T}(G^{ab}))$, where $G^{ab} = G/[G,G]$ is the abelianization and $\operatorname{T}(\cdot)$, the torsion subgroup. In other words:

Definition 1. Let G be a finitely generated group. The *Betti number* $b(G)$ is the maximum rank of a free abelian quotient group of G, i.e., the maximum rank of a free abelian group A such that there exists an epimorphism $\varphi\colon G \twoheadrightarrow A$.

The term came from geometric group theory, where G^{ab} is called the first homology group $H_1(G)$. A non-commutative analog of Betti number can be defined as follows:

Definition 2 ([7], [8])**.** Let G be a finitely generated group. The *co-rank* $\operatorname{corank}(G)$, see [8], is the maximum rank of a free quotient group of G, i.e., the maximum rank of a free group F such that there exists an epimorphism $\varphi\colon G \twoheadrightarrow F$.

The same notion is called inner rank $\operatorname{IN}(G)$ in [7] or $\operatorname{Ir}(G)$ in [9], or the first non-commutative Betti number $b_1'(G)$ in [1].

The notion of co-rank is also in a way dual to that of rank, which is the minimum rank of a free group allowing an epimorphism onto G. In contrast to rank, co-rank is algorithmically computable for finitely presented groups [10], [12].

For example, $\operatorname{corank}(\mathbb{Z}^n) = 1$, while $b(\mathbb{Z}^n) = n$. For a finite group G, $\operatorname{corank}(G) = b(G) = 0$; the same holds for $G = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$, even though it is infinite and contains F_2 and thus free subgroups of all ranks up to countable. Obviously, for any finitely generated group, $\operatorname{corank}(G) \leqslant b(G) \leqslant \operatorname{rk} G$ and $b(G) \geqslant 1$ implies $\operatorname{corank}(G) \geqslant 1$. In this paper we show that these are the only constraints between these values:

Theorem 3. *Let $0 \leqslant c, b, r \in \mathbb{Z}$. Then there exists a finitely generated group G with $\operatorname{corank}(G) = c$, $b(G) = b$, and $\operatorname{rk} G = r$ if and only if*

$$c = b = 0 \quad \text{or} \quad 1 \leqslant c \leqslant b \leqslant r;$$

the group can be chosen to be finitely presented and, if $b = r$, torsion-free.

Lemma 4. *Let G_1, G_2 be finitely generated groups. Then for the Betti number of the free product and of the direct product,*

$$b(G_1 * G_2) = b(G_1 \times G_2) = b(G_1) + b(G_2).$$

P r o o f. Obviously, $(G_1 * G_2)^{ab} = (G_1 \times G_2)^{ab}$. Denote $G = G_1 \times G_2$. Since epimorphisms $G_i \twoheadrightarrow \mathbb{Z}^{b(G_i)}$ onto free abelian groups can be extended to an epimorphism of $G_1 \times G_2 \twoheadrightarrow \mathbb{Z}^{b(G_2)} \times \mathbb{Z}^{b(G_2)} = \mathbb{Z}^{b(G_2)+b(G_2)}$, we have $b(G) \geqslant b(G_1) + b(G_2)$.

Let us now show that $b(G) \leqslant b(G_1)+b(G_2)$. Consider the natural homomorphisms $\psi_1\colon G_1 \to G_1 \times 1 \subseteq G$, $\psi_2\colon G_1 \to 1 \times G_2 \subseteq G$. Then ψ_i and an epimorphism onto a free abelian group

$$G_i \xrightarrow{\psi_i} G = G_1 \times G_2 \twoheadrightarrow A = \mathbb{Z}^{b(G)}$$

induces a homomorphism $\varphi_i\colon G_i \to A$. Since $A_i = \varphi_i(G_i) \subseteq A$ are free abelian groups, $\operatorname{rk} A_i \leqslant b(G_i)$. Since $G = \langle \psi_1(G_1), \psi_2(G_2)\rangle$, we have $A = \langle A_1, A_2\rangle$; in particular, $b(G) = \operatorname{rk} A \leqslant \operatorname{rk} A_1 + \operatorname{rk} A_2$. □

P r o o f of Theorem 3. For $1 \leqslant c \leqslant b \leqslant r$, consider $G = \mathbb{Z}^{b_1} * \ldots * \mathbb{Z}^{b_c} * \mathbb{Z}_2^{r-b}$ such that $\sum_{i=1}^{c} b_i = b$. By Proposition 6.4 in [9], $\operatorname{corank}(G_1 * G_2) = \operatorname{corank}(G_1) + \operatorname{corank}(G_2)$, so $\operatorname{corank}(G) = \sum_{i=1}^{c} \operatorname{corank}(\mathbb{Z}^{b_i}) = c$. By Lemma 4, $b(G) = \sum_{i=1}^{c} b(\mathbb{Z}^{b_i}) = b$, and by the Grushko-Neumann theorem, $\operatorname{rk} G = r$. □

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