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CONCENTRATION-COMPACTNESS PRINCIPLE FOR EMBEDDING INTO MULTIPLE EXPONENTIAL SPACES ON UNBOUNDED DOMAINS

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Abstract. Let $\Omega \subset \mathbb{R}^n$ be a domain and let $\alpha < n-1$. We prove the Concentration-Compactness Principle for the embedding of the space $W_0^1 L^n \log^\alpha L(\Omega)$ into an Orlicz space corresponding to a Young function which behaves like $\exp(t^{n/(n-1-\alpha)})$ for large t. We also give the result for the embedding into multiple exponential spaces.

Our main result is Theorem 1.6 where we show that if one passes to unbounded domains, then, after the usual modification of the integrand in the Moser functional, the statement of the Concentration-Compactnes Principle is very similar to the statement in the case of a bounded domain. In particular, in the case of a nontrivial weak limit the borderline exponent is still given by the formula

$$P := (1 - \|\Phi(|\nabla u|)\|_{L^1(\mathbb{R}^n)})^{-1/(n-1)}.$$

Keywords: Sobolev space; Orlicz-Sobolev space; Moser-Trudinger inequality; sharp constant; concentration-compactness principle

MSC 2010: 46E35, 46E30, 26D10

1. Introduction

Throughout the paper Ω is a domain in $\mathbb{R}^n$, $n \geqslant 2$, and ω_{n-1} denotes the surface of the unit sphere. Furthermore, $W_0^{1,p}(\Omega)$ denotes the usual completion of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

The aim of this paper is to prove the Concentration-Compactness Principle for Orlicz-Sobolev spaces embedded into exponential and multiple exponential Orlicz

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spaces in the case of an unbounded domain. The case of a bounded domain was treated in papers [9], [5], [7] and the results are strong enough to be called generalizations of the corresponding well-known results for the space $W_0^{1,n}(\Omega)$. Let us also note that the author is aware of only one paper giving the full statement of the Concentration-Compactness Principle for the space $W_0^{1,n}(\Omega)$ in the case of an unbounded domain (paper [3]), however, the presented proof is not correct (the error rests upon the fact that if $u \in W_0^{1,n}(\Omega)$ and $\widetilde{\Omega} \subset \Omega$, then $u \in W^{1,n}(\widetilde{\Omega})$ but $u \notin W_0^{1,n}(\widetilde{\Omega})$ in general and thus the result for unbounded domains is not just an easy consequence of the result for bounded domains). Nevertheless, the Concentration-Compactness Principle usually consists of three or four statements and the most important one was proved in [16] (this part is also the most difficult to prove). In fact, the proof of the main result of [16] can be significantly simplified as shown in the present paper. Concerning the Orlicz-Sobolev setting, no result for unbounded domains has been published so far.

Let us proceed to a detailed introduction.

Sobolev case on a bounded domain. If Ω is bounded, then the famous Moser-Trudinger inequality [28] concerning a classical embedding theorem by Trudinger [33] states that

$$\sup_{\|\nabla u\|_{L^n(\Omega)}\leqslant 1} \int_\Omega \exp(K|u|^{n'})\,\mathrm{d}x \begin{cases} \leqslant C(n,K,\mathcal{L}_n(\Omega)) & \text{when } K \leqslant n\omega_{n-1}^{1/(n-1)}, \\ = \infty & \text{when } K > n\omega_{n-1}^{1/(n-1)}. \end{cases} \tag{1.1}$$

This result is often used when proving the existence of nontrivial weak solutions to the n-Laplace equation

$$-\operatorname{div}(|\nabla u|^{n-2}\nabla u) = f(x,u), \tag{1.2}$$

where the nonlinearity f has the growth of exponential type (see for example [2], [14], [15]).

An often used improvement of the Moser-Trudinger inequality is the following Concentration-Compactness Principle.

Theorem 1.1. *Let $n \in \mathbb{N}$, $n \geqslant 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $\{u_k\} \subset W_0^{1,n}(\Omega)$ be a sequence satisfying $\|\nabla u_k\|_{L^n(\Omega)} \leqslant 1$ for every $k \in \mathbb{N}$, let $u \in W_0^{1,n}(\Omega)$ and $\mu \in \mathcal{M}(\overline{\Omega})$. Assume that*

$$u_k \rightharpoonup u \quad \text{in } W_0^{1,n}(\Omega), \quad u_k \to u \quad \text{a.e. in } \Omega \quad \text{and} \quad |\nabla u_k|^n \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\overline{\Omega}).$$

(i) *If $u = 0$, $\mu = \delta_{x_0}$ for some $x_0 \in \overline{\Omega}$, and*

$$\int_\Omega \exp(n\omega_{n-1}^{1/(n-1)}|u_k|^{n/(n-1)})\,\mathrm{d}x \to c + \mathcal{L}_n(\Omega)$$

for some $c \in [0, \infty)$, *then*

$$\exp(n\omega_{n-1}^{1/(n-1)}|u_k|^{n/(n-1)}) \overset{*}{\rightharpoonup} c\delta_{x_0} + \mathcal{L}_n|_\Omega \quad \text{in } \mathcal{M}(\overline{\Omega}).$$

(ii) *If* $u = 0$ *and* μ *is not a Dirac mass concentrated at one point, then there exists* $p > 1$ *such that*

$$\exp(n\omega_{n-1}^{1/(n-1)}p|u_k|^{n/(n-1)}) \quad \text{is bounded in } L^1(\Omega).$$

(iii) *If* $u \neq 0$ *and* $p < P := (1 - \|\nabla u\|^n_{L^n(\Omega)})^{-1/(n-1)}$ (*where we read* $P = \infty$ *if* $\|\nabla u\|_{L^n(\Omega)} = 1$), *then*

$$\exp(n\omega_{n-1}^{1/(n-1)}p\,|u_k|^{n/(n-1)}) \quad \text{is bounded in } L^1(\Omega).$$

Moreover, in both the cases (ii) *and* (iii),

$$\exp(n\omega_{n-1}^{1/(n-1)}|u_k|^{n/(n-1)}) \to \exp(n\omega_{n-1}^{1/(n-1)}|u|^{n/(n-1)}) \quad \text{in } L^1(\Omega).$$

The statement of Theorem 1.1 comes from [27], Theorem I.6 and Remark I.18. However, the proof of Theorem 1.1 (iii) in the case $n \geqslant 3$ is valid only for $p \leqslant \widetilde{P} := (1 - \|\nabla u^\sharp\|^n_{L^n(\Omega)})^{-1/(n-1)}$, where $u^\sharp$ denotes the Schwarz symmetral of u. One has $\widetilde{P} \leqslant P$ and it may happen that $\widetilde{P} < P$ in general. The correct proof of Theorem 1.1 (iii) is given in [8].

The Concentration-Compactness Principle is used in the proof that the supremum in the Moser-Trudinger inequality with $K = n\omega_{n-1}^{1/(n-1)}$ is attained (see [4]) and Theorem 1.1 (iii) also plays an important role when studying (1.2) with a nonlinearity having the so called critical growth (see for example [14], [15]) and when studying the multiplicity of weak solutions (see for example [17]). In these cases, the Moser-Trudinger inequality is not powerful enough.

Let us also note that the upper bound of p in Theorem 1.1 (iii) is sharp. Indeed, in [8], an example is given showing that we cannot have $p = P$ in Theorem 1.1 (iii). It is interesting to compare this result with the Moser-Trudinger inequality (1.1), where the supremum is finite also for the borderline exponent.

Sobolev case on an unbounded domain. If Ω is not bounded, then $W_0^{1,n}(\Omega)$ is embedded into $L^p(\Omega)$ for $p \in [n, \infty)$ only and thus it is natural to state the Moser-Trudinger inequality with a suitable part of the Taylor expansion corresponding to the function exp subtracted. We set

$$\Upsilon(t) = \exp(t) - \sum_{j=0}^{n-2} \frac{t^j}{j!}.$$

Versions of inequality (1.1) for unbounded domains were studied in [1], [26], [15] and [31].

Since any function from $W_0^{1,n}(\Omega)$ can be extended by zero outside Ω to obtain a function from $W_0^{1,n}(\mathbb{R}^n) = W^{1,n}(\mathbb{R}^n)$, the result is often stated for the space $W^{1,n}(\mathbb{R}^n)$: If $M \geqslant 0$, then

$$(1.3) \qquad \sup_{\substack{\|\nabla u\|_{L^n(\mathbb{R}^n)}\leqslant 1\\ \|u\|_{L^n(\mathbb{R}^n)}\leqslant M}} \int_\Omega \Upsilon(K|u|^{n/(n-1)}) \begin{cases} \leqslant C(n,K,M) & \text{when } K \leqslant n\omega_{n-1}^{1/(n-1)}, \\ = \infty & \text{when } K > n\omega_{n-1}^{1/(n-1)}. \end{cases}$$

This result is again often used when proving the existence of a nontrivial weak solution to the n-Laplace equation.

Let us also note that in the literature, the finiteness of the supremum in (1.3) is often proved only for $K < n\omega_{n-1}^{1/(n-1)}$. But for example from careful estimates in [3], Proof of Theorem 1.2, it can be seen that the supremum is finite also for $K = n\omega_{n-1}^{1/(n-1)}$.

The Concentration-Compactness Principle is now formulated as follows:

Theorem 1.2. *Let $n \in \mathbb{N}$, $n \geqslant 2$ and $M > 0$. Let $\{u_k\} \subset W^{1,n}(\mathbb{R}^n)$ be a sequence satisfying $\|\nabla u_k\|_{L^n(\mathbb{R}^n)} \leqslant 1$ and $\|u_k\|_{L^n(\mathbb{R}^n)} \leqslant M$ for every $k \in \mathbb{N}$. Let $u \in W^{1,n}(\mathbb{R}^n)$ and $\mu \in \mathcal{M}(\mathbb{R}^n)$. Assume that*

$$u_k \rightharpoonup u \text{ in } W^{1,n}(\mathbb{R}^n), \quad u_k \to u \quad \text{a.e. in } \mathbb{R}^n \quad \text{and} \quad |\nabla u_k|^n \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\mathbb{R}^n).$$

Let us set

$$A_\infty = \lim_{R\to\infty} \limsup_{k\to\infty} \int_{\mathbb{R}^n\setminus B(R)} |\nabla u_k|^n \,\mathrm{d}x.$$

Then $A_\infty \in [0,1]$, $\mu(\mathbb{R}^n) \leqslant 1 - A_\infty$ and we have:

(i) *If $u = 0$, $\mu = \delta_{x_0}$ for some $x_0 \in \mathbb{R}^n$, and*

$$\int_{B(x_0,\varrho)} \Upsilon(n\omega_{n-1}^{1/(n-1)}|u_k|^{n/(n-1)}) \,\mathrm{d}x \to c$$

for some $c \in [0,\infty)$ and for some $\varrho > 0$, then

$$\Upsilon(n\omega_{n-1}^{1/(n-1)}|u_k|^{n/(n-1)}) \overset{*}{\rightharpoonup} c\delta_{x_0} \quad \text{in } \mathcal{M}(\mathbb{R}^n),$$

while for every $p > 0$ and every $\varrho > 0$

$$\Upsilon(n\omega_{n-1}^{1/(n-1)}p|u_k|^{n/(n-1)}) \quad \text{is bounded in } L^1(\mathbb{R}^n \setminus B(x_0,\varrho)).$$

(ii) *If* $u = 0$ *and* $A_\infty = 1$, *then for every* $p > 1$ *and every open bounded set* $\Omega_0 \subset \mathbb{R}^n$

$$\Upsilon(n\omega_{n-1}^{1/(n-1)} p|u_k|^{n/(n-1)}) \quad \textit{is bounded in } L^1(\Omega_0).$$

(iii) *If* $u = 0$, μ *is not a Dirac mass concentrated at one point,* $A_\infty < 1$ *and*

$$p < P := \left(\max\left\{A_\infty, \max_{x\in\mathbb{R}^n} \mu(\{x\})\right\}\right)^{-1/(n-1)} \tag{1.4}$$

(with the convention $0^{-1/(n-1)} = \infty$*), then*

$$\Upsilon(n\omega_{n-1}^{1/(n-1)} p|u_k|^{n/(n-1)}) \quad \textit{is bounded in } L^1(\mathbb{R}^n).$$

(iv) *If* $u \neq 0$ *and* $p < P := (1 - \|\nabla u\|^n_{L^n(\mathbb{R}^n)})^{-1/(n-1)}$ *(where we read* $P = \infty$ *if* $\|\nabla u\|_{L^n(\mathbb{R}^n)} = 1$*), then*

$$\Upsilon(n\omega_{n-1}^{1/(n-1)} p|u_k|^{n/(n-1)}) \quad \textit{is bounded in } L^1(\mathbb{R}^n).$$

Moreover, in cases (ii), (iii) *and* (iv), *we have for every open bounded set* $\Omega_0 \subset \mathbb{R}^n$

$$\Upsilon(n\omega_{n-1}^{1/(n-1)} |u_k|^{n/(n-1)}) \to \Upsilon(n\omega_{n-1}^{1/(n-1)} |u|^{n/(n-1)}) \quad \textit{in } L^1(\Omega_0)$$

and in case (i), *we have for every open bounded set* $\Omega_0 \subset \mathbb{R}^n$ *and every* $\varrho > 0$

$$\Upsilon(n\omega_{n-1}^{1/(n-1)} |u_k|^{n/(n-1)}) \to \Upsilon(n\omega_{n-1}^{1/(n-1)} |u|^{n/(n-1)}) \quad \textit{in } L^1(\Omega_0 \setminus B(x_0, \varrho)).$$

Some parts of Theorem 1.2 were stated in [3] (with the proof which is not correct) and used to show that if $n = 2$ and Ω is a stripe $\{(x_1, x_2) \in \mathbb{R}^2\colon\ -1 < x_1 < 1\}$, then there is a version of the result [4] for inequality (1.3). Theorem 1.2 (iv) is given in [16], the proof is obtained by a minor modification of the proof of Theorem 1.1 (iii) given in [8]. Paper [16] further gives an application of Theorem 1.2 (ii) to the n-Laplace equation.

Notice that the maximum $\max_{x\in\mathbb{R}^n} \mu(\{x\})$ is actually attained, since $\mu(\{x\})$ exceeds any fixed positive number at finite number of points only.

The case of $W^{1,n}(\mathbb{R}^n)$ (or $W_0^{1,n}(\Omega)$ with Ω not being bounded) admits a new phenomenon, Theorem 1.2 (ii). This phenomenon is called the Concentration-Compactness Principle at infinity and it was introduced in [12].

The whole Theorem 1.2 is just a consequence of our general result Theorem 1.6 concerning the Orlicz-Sobolev setting.

Some notes concerning the sharpness of the upper bounds of p in Theorem 1.2 (ii), (iii) and (iv) are given in the last section.

Orlicz-Sobolev case on a bounded domain. First, let us recall some well known results concerning embeddings into exponential and multiple exponential spaces. If $l \in \mathbb{N}$ and $\alpha < n-1$, we set

$$\gamma = \frac{n}{n-1-\alpha} > 0, \quad B = 1 - \frac{\alpha}{n-1} = \frac{n}{(n-1)\gamma} > 0 \tag{1.5}$$

$$\text{and} \quad K_{l,n,\alpha} = \begin{cases} B^{1/B} n\omega_{n-1}^{\gamma/n} & \text{for } l = 1, \\ B^{1/B}\omega_{n-1}^{\gamma/n} & \text{for } l \geqslant 2. \end{cases}$$

The space $W_0L^n \log^\alpha L(\Omega)$ of the Sobolev type, modeled on the Zygmund space $L^n \log^\alpha L(\Omega)$, is continuously embedded into an Orlicz space with the Young function that behaves like $\exp(t^\gamma)$ for large t (see [24] and [22]). Moreover, it is shown in [22] (see also [13] and [21]) that in the limiting case $\alpha = n-1$ we have the embedding into a double exponential space, i.e., the space $W_0L^n \log^{n-1} L \log^\alpha \log L(\Omega)$, $\alpha < n-1$, is continuously embedded into the Orlicz space with the Young function that behaves like $\exp(\exp(t^\gamma))$ for large t. Furthermore, in the limiting case $\alpha = n-1$ we have the embedding into a triple exponential space and so on. The borderline case is always $\alpha = n-1$ and for $\alpha > n-1$ we have the embedding into $L^\infty(\Omega)$. It is well-known that the Zygmund space $L^n \log^\alpha L(\Omega)$ coincides with the Orlicz space $L^\Phi(\Omega)$, where

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^n \log^\alpha(t)} = 1,$$

the space $L^n \log^{n-1} L \log^\alpha \log L(\Omega)$ coincides with $L^\Phi(\Omega)$ where

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^n \log^{n-1}(t) \log^\alpha(\log(t))} = 1,$$

and so on. For other results concerning these spaces and their precise definitions we refer the reader to [21], [20], [19], [18], [23] and [29].

The following notation is useful when dealing with multiple logarithmic and multiple exponential spaces. Let us write

$$\log_{[1]}(t) = \log(t) \quad \text{and} \quad \log_{[j]}(t) = \log(\log_{[j-1]}(t)) \quad \text{for } j \geqslant 2,\ j \in \mathbb{N}$$

and

$$\exp_{[1]}(t) = \exp(t) \quad \text{and} \quad \exp_{[j]}(t) = \exp(\exp_{[j-1]}(t)) \quad \text{for } j \geqslant 2,\ j \in \mathbb{N}.$$

Let $l \in \mathbb{N}$ and $\alpha < n-1$. Then we have the above mentioned embedding results for any Young function Φ satisfying

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^n\bigl(\prod_{j=1}^{l-1}\log_{[j]}^{n-1}(t)\bigr)\log_{[l]}^{\alpha}(t)} = 1 \tag{1.6}$$

(for $l=1$ we read (1.6) as $\lim\limits_{t\to\infty}\Phi(t)/(t^n\log_{[1]}^{\alpha}(t)) = 1$). As Ω is bounded, all Young functions satisfying (1.6) give us the same Orlicz-Sobolev space.

Now, let us recall the generalized Moser-Trudinger inequality.

Theorem 1.3. *Let $l \in \mathbb{N}$, $n \in \mathbb{N}$, $n \geqslant 2$, $\alpha < n-1$, $K \geqslant 0$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let Φ be a Young function satisfying* (1.6).

(i) If $u \in W_0L^{\Phi}(\Omega)$, then

$$\int_\Omega \exp_{[l]}(K|u|^{\gamma})\,\mathrm{d}x < \infty.$$

(ii) If $K < K_{l,n,\alpha}$, then

$$\sup_{u\in W_0L^{\Phi}(\Omega),\|\Phi(\nabla u)\|_{L^1(\Omega)}\leqslant 1}\int_\Omega \exp_{[l]}(K|u|^{\gamma})\,\mathrm{d}x \leqslant C(l,n,\alpha,\Phi,\mathcal{L}_n(\Omega),K).$$

(iii) If $K > K_{l,n,\alpha}$, then

$$\sup_{u\in W_0L^{\Phi}(\Omega),\|\Phi(\nabla u)\|_{L^1(\Omega)}\leqslant 1}\int_\Omega \exp_{[l]}(K|u|^{\gamma})\,\mathrm{d}x = \infty.$$

The first assertion follows from [22], Remarks 3.11 (iv). The other two assertions follow from [25], Theorem 1.1 and Theorem 1.2 (cases $l=1$ and $l=2$) and [11], Theorem 1.1 and Theorem 1.2 (case $l \geqslant 3$). It is also shown in [25] and [11] that if $K = K_{l,n,\alpha}$, then the finiteness of the supremum depends on the choice of Φ.

Finally, we recall the Concentration-Compactness Principle given in [10], [5] and [7].

Theorem 1.4. *Let $l \in \mathbb{N}$, $n \in \mathbb{N}$, $n \geqslant 2$, $\alpha < n-1$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let Φ be a Young function satisfying* (1.6). *Let $\{u_k\} \subset W_0L^{\Phi}(\Omega)$ be a sequence satisfying $\|\Phi(|\nabla u_k|)\|_{L^1(\Omega)} \leqslant 1$ for every $k \in \mathbb{N}$, let $u \in W_0L^{\Phi}(\Omega)$ and $\mu \in \mathcal{M}(\overline{\Omega})$. Assume that*

$$u_k \rightharpoonup u \quad \text{in } W_0L^{\Phi}(\Omega), \quad u_k \to u \quad \text{a.e. in } \Omega \quad \text{and} \quad \Phi(|\nabla u_k|) \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\overline{\Omega}).$$

(i) *If* $u = 0$, $\mu = \delta_{x_0}$ *for some* $x_0 \in \overline{\Omega}$, *and*

$$\int_\Omega \exp_{[l]}(K_{l,n,\alpha}|u_k|^\gamma)\,\mathrm{d}x \to c + \exp_{[l]}(0)\mathcal{L}_n(\Omega)$$

for some $c \in [0,\infty)$, *then*

$$\exp_{[l]}(K_{l,n,\alpha}|u_k|^\gamma) \overset{*}{\rightharpoonup} c\delta_{x_0} + \exp_{[l]}(0)\mathcal{L}_n|_\Omega \quad \text{in } \mathcal{M}(\overline{\Omega}).$$

(ii) *If* $u = 0$ *and* μ *is not a Dirac mass concentrated at one point, then there exists* $p > 1$ *such that*

$$\exp_{[l]}(K_{l,n,\alpha}p|u_k|^\gamma) \quad \text{is bounded in } L^1(\Omega).$$

(iii) *If* $u \neq 0$ *and* $p < P := (1 - \|\Phi(|\nabla u|)\|_{L^1(\Omega)})^{-\gamma/n}$ *(where we read* $P = \infty$ *if* $\|\Phi(|\nabla u|)\|_{L^1(\Omega)} = 1$*), then*

$$\exp_{[l]}(K_{l,n,\alpha}p|u_k|^\gamma) \quad \text{is bounded in } L^1(\Omega).$$

Moreover, in both the cases (ii) *and* (iii),

$$\exp_{[l]}(K_{l,n,\alpha}|u_k|^\gamma) \to \exp_{[l]}(K_{l,n,\alpha}|u|^\gamma) \quad \text{in } L^1(\Omega).$$

Some notes concerning the sharpness of Theorem 1.4 (iii) are given in [7], where it is shown that if Φ satisfies (1.6) and some additional growth assumptions, then we cannot have $p = P$ in Theorem 1.4 (iii).

Orlicz-Sobolev case on an unbounded domain. In this case, the results depend also on the behavior of the Young function Φ for small arguments. Let us suppose that Φ satisfies an additional assumption

$$\frac{1}{C}t^n \leqslant \Phi(t) \leqslant Ct^n \quad \text{for } t \in \Big[0, \frac{1}{C}\Big). \tag{1.7}$$

Next, let

$$\exp_{[l]}(t) = \sum_{j=0}^{\infty} a_j t^j$$

be the Taylor expansion of the the function $\exp_{[l]}$. We set

$$\Upsilon(t) = \sum_{j \in [n/\gamma,\infty)\cap\mathbb{N}} a_j t^j.$$

The following result comes from [6].

Theorem 1.5. *Let $l \in \mathbb{N}$, $n \geqslant 2$ and $\alpha < n-1$. Suppose that a Young function $\Phi\colon [0,\infty) \mapsto [0,\infty)$ satisfies* (1.6) *and* (1.7). *Let $u \in WL^{\Phi}(\mathbb{R}^n)$.*

(i) *If $K \geqslant 0$ then*

$$\int_{\mathbb{R}^n} \Upsilon(K|u|^{\gamma})\,\mathrm{d}x < \infty.$$

(ii) *If $0 \leqslant K < K_{l,n,\alpha}$, $\|\Phi(|\nabla u|)\|_{L^1(\mathbb{R}^n)} \leqslant 1$ and $\|\Phi(|u|)\|_{L^1(\mathbb{R}^n)} \leqslant M$ for some $M \geqslant 0$, then*

$$\int_{\mathbb{R}^n} \Upsilon(K|u|^{\gamma})\,\mathrm{d}x \leqslant C(l,n,\alpha,\Phi,M,K).$$

(iii) *If $K > K_{l,n,\alpha}$, then there is a sequence $\{u_k\} \subset WL^{\Phi}(\mathbb{R}^n)$ such that $\|\Phi(\nabla u_k)\|_{L^1(\mathbb{R}^n)} \leqslant 1$ for every $k \in \mathbb{N}$, $\|\Phi(u_k)\|_{L^1(\mathbb{R}^n)} \to 0$ and*

$$\int_{\mathbb{R}^n} \Upsilon(K|u_k|^{\gamma})\,\mathrm{d}x \overset{k\to\infty}{\longrightarrow} \infty.$$

It is also shown in [6] that if $K = K_{l,n,\alpha}$, then the boundedness depends on the choice of Φ. Let us note that in the original statement of Theorem 1.5 in [6], the assumptions $\|\Phi(|\nabla u|)\|_{L^1(\mathbb{R}^n)} \leqslant 1$ and $\|\Phi(|u|)\|_{L^1(\mathbb{R}^n)} \leqslant M$ are replaced by estimates of the Luxemburg norms $\|\nabla u\|_{L^{\Phi}(\mathbb{R}^n)} \leqslant 1$ and $\|u\|_{L^{\Phi}(\mathbb{R}^n)} \leqslant M$. Our version is still valid since the Δ_2-condition implies that $\|\Phi(|\nabla u|)\|_{L^1(\mathbb{R}^n)} \leqslant 1$ if and only if $\|\nabla u\|_{L^{\Phi}(\mathbb{R}^n)} \leqslant 1$, and the boundedness of $\|\Phi(|u|)\|_{L^1(\mathbb{R}^n)}$ is equivalent to the boundedness of $\|u\|_{L^{\Phi}(\mathbb{R}^n)}$.

Now, let us state the main result of this paper.

Theorem 1.6. *Let $l \in \mathbb{N}$, $n \in \mathbb{N}$, $n \geqslant 2$, $\alpha < n-1$ and $M > 0$. Let Φ be a Young function satisfying* (1.6) *and* (1.7). *Let $\{u_k\} \subset WL^{\Phi}(\mathbb{R}^n)$ be a sequence satisfying $\|\Phi(|\nabla u_k|)\|_{L^1(\mathbb{R}^n)} \leqslant 1$ and $\|\Phi(|u_k|)\|_{L^1(\mathbb{R}^n)} \leqslant M$ for every $k \in \mathbb{N}$. Let $u \in WL^{\Phi}(\mathbb{R}^n)$ and $\mu \in \mathcal{M}(\mathbb{R}^n)$. Assume that*

$$u_k \rightharpoonup u \quad \text{in } WL^{\Phi}(\mathbb{R}^n), \quad u_k \to u \quad \text{a.e. in } \mathbb{R}^n \quad \text{and} \quad \Phi(|\nabla u_k|) \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\mathbb{R}^n).$$

Let us set

$$A_{\infty} = \lim_{R\to\infty} \limsup_{k\to\infty} \int_{\mathbb{R}^n \setminus B(R)} \Phi(|\nabla u_k|)\,\mathrm{d}x.$$

Then $A_{\infty} \in [0,1]$, $\mu(\mathbb{R}^n) \leqslant 1 - A_{\infty}$ and we have:

(i) *If $u = 0$, $\mu = \delta_{x_0}$ for some $x_0 \in \mathbb{R}^n$, and*

$$\int_{B(x_0,\varrho)} \Upsilon(K_{l,n,\alpha}|u_k|^{\gamma})\,\mathrm{d}x \to c$$

for some $c \in [0,\infty)$ *and some* $\varrho > 0$, *then*

$$\Upsilon(K_{l,n,\alpha}|u_k|^{\gamma}) \overset{*}{\rightharpoonup} c\delta_{x_0} \quad \text{in } \mathcal{M}(\mathbb{R}^n),$$

while for every $p > 0$ *and* $\varrho > 0$

$$\Upsilon(K_{l,n,\alpha}p\,|u_k|^{\gamma}) \quad \text{is bounded in } L^1(\mathbb{R}^n \setminus B(x_0,\varrho)).$$

(ii) *If* $u = 0$ *and* $A_\infty = 1$, *then for every* $p > 1$ *and every open bounded set* $\Omega_0 \subset \mathbb{R}^n$

$$\Upsilon(K_{l,n,\alpha}p\,|u_k|^{\gamma}) \quad \text{is bounded in } L^1(\Omega_0).$$

(iii) *If* $u = 0$, μ *is not a Dirac mass concentrated at one point,* $A_\infty < 1$ *and*

(1.8)
$$p < P := \sup\Big\{\tau \geqslant 1\colon\ \Phi(\tau^{1/\gamma}t) \leqslant \frac{1}{\max\{A_\infty, \max_{x\in\mathbb{R}^n}\mu(x)\}}\Phi(t) \text{ for every } t \geqslant 0\Big\},$$

then

$$\Upsilon(K_{l,n,\alpha}p\,|u_k|^{\gamma}) \quad \text{is bounded in } L^1(\mathbb{R}^n).$$

(iv) *If* $u \neq 0$ *and* $p < P := (1 - \|\Phi(|\nabla u|)\|_{L^1(\mathbb{R}^n)})^{-1/(n-1)}$ (where we read $P = \infty$ if $\|\Phi(|\nabla u|)\|_{L^1(\mathbb{R}^n)} = 1$), then

$$\Upsilon(K_{l,n,\alpha}p\,|u_k|^{\gamma}) \quad \text{is bounded in } L^1(\mathbb{R}^n).$$

Moreover, in cases (ii), (iii) and (iv), we have for every open bounded set $\Omega_0 \subset \mathbb{R}^n$

$$\Upsilon(K_{l,n,\alpha}|u_k|^{\gamma}) \to \Upsilon(K_{l,n,\alpha}|u|^{\gamma}) \quad \text{in } L^1(\Omega_0)$$

and in case (i), we have for every open bounded set $\Omega_0 \subset \mathbb{R}^n$ and every $\varrho > 0$

$$\Upsilon(K_{l,n,\alpha}|u_k|^{\gamma}) \to \Upsilon(K_{l,n,\alpha}|u|^{\gamma}) \quad \text{in } L^1(\Omega_0 \setminus B(x_0,\varrho)).$$

In this case we are not going to give a detailed discussion concerning the sharpness of the upper bounds of p in Theorem 1.6. The discussion concerning the sharpness of Theorem 1.2 implies that the upper bounds of p cannot be improved in general in Theorem 1.6.

Theorem 1.2 is obtained from Theorem 1.6 by setting $\Phi(t) = t^n$. Indeed, in this case we have

$$\|\Phi(|\nabla u|)\|_{L^1(\mathbb{R}^n)} = \||\nabla u|^n\|_{L^1(\mathbb{R}^n)} = \|\nabla u\|^n_{L^n(\mathbb{R}^n)}$$

and it is also easy to check that (1.8) turns to (1.4).

Let us note that the Δ_2-condition implies that the number P in (1.8) satisfies $P > 1$.

The paper is organized as follows. Section 3 is devoted to preparation for the proof of Theorem 1.6. The proof of Theorem 1.6 is given in the fourth section. In the last section we give some comments concerning assumption (1.7) and we also study the sharpness of the upper bounds of the exponents in Theorem 1.2.

Notice that our proof of Theorem 1.6 (iv) is much simpler than the proof of the corresponding $W^{1,n}$-result given in [16]. In [16], the proof is obtained by a minor modification (based on the so called radial lemma) of the proof of Theorem 1.1 (iii) (the original proof comes from [8] and is quite long). In fact, it is enough to combine just the statement of Theorem 1.1 (iii) and the radial lemma.

2. Preliminaries

The n-dimensional Lebesgue measure is denoted by $\mathcal{L}_n$. Further, $\mathcal{L}_n|_\Omega$ is its restriction to Ω, i.e., $\mathcal{L}_n|_\Omega(A) = \mathcal{L}_n(A \cap \Omega)$ for every measurable set $A \subset \mathbb{R}^n$. If u is a measurable function on Ω, then by $u = 0$ (or $u \neq 0$) we mean that u is equal (or not equal) to the zero function a.e. on Ω.

By $\mathcal{M}(\mathbb{R}^n)$ we denote the set of all Radon measures on $\mathbb{R}^n$. We write $\mu_j \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\mathbb{R}^n)$ if $\int_{\mathbb{R}^n} \psi \, \mathrm{d}\mu_j \to \int_{\mathbb{R}^n} \psi \, \mathrm{d}\mu$ for every test-function $\psi \in C_0(\mathbb{R}^n)$ ($C_0(\mathbb{R}^n)$ denotes the set of continuous functions with compact support). It is well known that each sequence bounded in $L^1(\mathbb{R}^n)$ contains a subsequence converging weakly* in $\mathcal{M}(\mathbb{R}^n)$.

By $B(x_0, R)$ we denote the open Euclidean ball in $\mathbb{R}^n$ centered at x_0 with the radius $R > 0$. If $x_0 = 0$ we simply write $B(R)$.

By C we denote a generic positive constant which may depend on l, n, α and Φ. This constant may vary from expression to expression as usual. Sometimes we say that for every $\varepsilon > 0$ something is true. In such a case the constants C may depend also on fixed $\varepsilon > 0$.

Properties of $\exp_{[l]}$. The following result comes from [6], Lemma 2.1.

Lemma 2.1. *Let $l \in \mathbb{N}$. The Taylor coefficients of the function $\exp_{[l]}$ satisfy*

$$a_j \geqslant 0 \quad \text{for each } j \in \mathbb{N}.$$

Young functions and Orlicz spaces. A function $\Phi\colon [0,\infty) \to [0,\infty)$ is a Young function if Φ is increasing, convex, $\Phi(0) = 0$ and $\lim\limits_{t\to\infty} \Phi(t)/t = \infty$.

Denote by $L^\Phi(A, \mathrm{d}\mu)$ the Orlicz space corresponding to a Young function Φ on a set A with a measure μ. If $\mu = \mathcal{L}_n$ we simply write $L^\Phi(A)$. The space $L^\Phi(A, \mathrm{d}\mu)$

is equipped with the Luxemburg norm

$$\|u\|_{L^\Phi(A,\,\mathrm{d}\nu)} = \inf\Big\{\lambda > 0\colon \int_A \Phi\Big(\frac{|u(x)|}{\lambda}\Big)\,\mathrm{d}\nu(x) \leqslant 1\Big\}. \tag{2.1}$$

For an introduction to Orlicz spaces see e.g. [30].

Δ_2-condition. In this paper, we say that a function Φ satisfies the Δ_2-condition, if there is $C_\Delta > 1$ such that

$$\Phi(2t) \leqslant C_\Delta \Phi(t) \quad \text{for every } t \geqslant 0.$$

Using the Δ_2-condition one easily proves that for any $\eta > 0$ we can find $\varepsilon > 0$ that

$$\Phi((1+\varepsilon)t) \leqslant (1+\eta)\Phi(t) \quad \text{for every } t \geqslant 0. \tag{2.2}$$

It is not difficult to check the Δ_2-condition for our Young functions satisfying (1.6) and (1.7).

Orlicz-Sobolev spaces. Let A be an nonempty open set in $\mathbb{R}^n$ and let Φ be a Young function satisfying (1.6). In this subsection we consider Orlicz spaces only with the Lebesgue measure. We define the Orlicz-Sobolev space $WL^\Phi(A)$ as the set

$$WL^\Phi(A) := \{u\colon\ u, |\nabla u| \in L^\Phi(A)\}$$

equipped with the norm

$$\|u\|_{WL^\Phi(A)} := \|u\|_{L^\Phi(A)} + \|\nabla u\|_{L^\Phi(A)},$$

where ∇u is the gradient of u and we use its Euclidean norm in $\mathbb{R}^n$.

We put $W_0L^\Phi(A)$ for the closure of $C_0^\infty(A)$ in $WL^\Phi(A)$. We write $u_k \rightharpoonup u$ in $WL^\Phi(A)$, if

$$\int_A \frac{\partial u_k}{\partial x_i} v\,\mathrm{d}x \to \int_A \frac{\partial u}{\partial x_i} v\,\mathrm{d}x \quad \text{and} \quad \int_A u_k v\,\mathrm{d}x \to \int_A uv\,\mathrm{d}x$$

for every $v \in L^\Psi(A)$ and $i \in \{1,\dots,n\}$. Here, Ψ is the associated Young function to Φ.

Non-increasing rearrangement. The non-increasing rearrangement u^* of a measurable function u on Ω is

$$u^*(t) = \inf\{s > 0\colon\ \mathcal{L}_n(\{x \in \Omega\colon\ |u(x)| > s\}) \leqslant t\}, \quad t > 0.$$

We also define the non-increasing radially symmetric rearrangement $u^\sharp$ by

$$u^\sharp(x) = u^*\Big(\frac{\omega_{n-1}}{n}|x|^n\Big) \quad \text{for } x \in B(R), \quad \mathcal{L}_n(B(R)) = \mathcal{L}_n(\Omega).$$

For an introduction to these rearrangements see e.g. [32]. We need the Pólya-Szegő inequality (see [32], Theorem 1.C).

Theorem 2.2. *Let Φ be a Young function and let u be a Lipschitz continuous function decaying at infinity ($\mathcal{L}_n(\{x \in \mathbb{R}^n\colon\ |u(x)| > t\}) < \infty$ for all $t > 0$). Then*

$$\int_{\mathbb{R}^n} \Phi(|\nabla u(x)|)\,\mathrm{d}x \geqslant \int_{\mathbb{R}^n} \Phi(|\nabla u^\sharp(x)|)\,\mathrm{d}x.$$

Tools from the measure theory. Let us recall [5], Lemma 2.3.

Lemma 2.3. *Let $l \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^n$ be an open bounded set, $\{u_k\}$ a sequence of measurable functions and let $u_k \to u$ a.e. in Ω. Suppose that there are $K, \delta, \gamma, C_1 > 0$ such that*

$$\|\exp_{[l]}(K(1+\delta)|u_k|^\gamma)\|_{L^1(\Omega)} < C_1 \quad \text{for all } k \in \mathbb{N}. \tag{2.3}$$

Let F be an even continuous function such that

$$\sup_{t\in(t_0,\infty)} \frac{|F(t)|}{\exp_{[l]}(K|t|^\gamma)} < \infty \quad \text{for some } t_0 > 0.$$

Then

$$F(u_k) \overset{k\to\infty}{\longrightarrow} F(u) \quad \text{in the } L^1(\Omega)\text{-norm.}$$

In particular,

$$\exp_{[l]}(K|u_k|^\gamma) \overset{k\to\infty}{\longrightarrow} \exp_{[l]}(K|u|^\gamma) \quad \text{in the } L^1(\Omega)\text{-norm.}$$

Next, we need a suitable estimate of a radially symmetric function $u \in L^\Phi(\mathbb{R}^n)$ on large spheres. This result comes from [6], Lemma 2.9.

Lemma 2.4. *Let $u \in L^\Phi(\mathbb{R}^n)$ with $\|u\|_{L^\Phi(\mathbb{R}^n)} \leqslant \widetilde{M}$ for some $\widetilde{M} > 0$. Suppose that u is non-negative, radially symmetric and non-increasing with respect to $|x|$. Then there are $R_s > 0$ and $C_s > 0$ independent of u such that*

$$u(x) \leqslant C_s\widetilde{M}\frac{1}{|x|} \quad \text{for } |x| > R_s.$$

Case of $u \neq 0$. The proof of Theorem 1.6 (iv) is obtained by combining the radial estimate from Lemma 2.4 with Theorem 1.4 (iii) (the version of the result for a bounded domain and functions vanishing on the boundary). First of all we need the following observation.

Remark 3.1. Theorem 1.4 (iii) can be extended to the case when $\Omega = \mathbb{R}^n$ and functions $\{u_k\} \subset WL^\Phi(\mathbb{R}^n)$ satisfy an additional assumption

$$\mathcal{L}_n(\{u_k \neq 0\}) \leqslant C \quad \text{for every } k \in \mathbb{N}. \tag{3.1}$$

This can be seen from the proof of Theorem 1.4 (iii) given in [7]. Indeed, major part of the proof is based on estimates of the growth of $u_k^\sharp$ on $\Omega^\sharp$ and the same estimates also hold in our case on the ball $B(R)$ such that $\mathcal{L}_n(B(R)) = C$ (hence $\{u_k^\sharp \neq 0\} \subset B(R)$ for every $k \in \mathbb{N}$). Finally, in the proof of Theorem 1.4 (iii), one uses some basic properties of the weak convergence in $W_0L^\Phi(\Omega)$ (such as the weak lower semicontinuity of the modular of the gradient), but these properties are also valid for the space $WL^\Phi(\mathbb{R}^n)$.

Case of $u = 0$.

Lemma 3.2. *Let l, n, α, M, Ω, Φ, $\{u_k\}$, u, μ and A_∞ be the same as in Theorem 1.6. Suppose that $u = 0$. Let $N \subset \mathbb{R}^n$ be a compact set. Let us define open bounded sets $N_\theta = \{x \in \mathbb{R}^n;\ \mathrm{dist}(x, N) < \theta\}$, $\theta > 0$.*

(i) *If $\mu(N) < 1$ and*

$$p < P := \sup\Big\{\tau \geqslant 1\colon\ \Phi(\tau^{1/\gamma} t) \leqslant \frac{1}{\mu(N)}\Phi(t) \text{ for every } t \geqslant 0\Big\}$$

(with the convention that $P = \infty$ for $\mu(N) = 0$), then there is $\theta > 0$ such that

$$\|\Upsilon(K_{l,n,\alpha} p\, |u_k|^\gamma)\|_{L^1(N_\theta)} \quad \text{is bounded.}$$

(ii) *If $A_\infty < 1$ and*

$$p < P := \sup\Big\{\tau \geqslant 1\colon\ \Phi(\tau^{1/\gamma} t) \leqslant \frac{1}{A_\infty}\Phi(t) \text{ for every } t \geqslant 0\Big\}$$

(with the convention that $P = \infty$ for $A_\infty = 0$), then there is $R > 0$ such that

$$\|\Upsilon(K_{l,n,\alpha} p\, |u_k|^\gamma)\|_{L^1(\mathbb{R}^n \setminus B(R))} \quad \text{is bounded.}$$

P r o o f. The proof is obtained by suitably modifying [10], proof of Lemma 3.1. We are going to give a detailed proof of Lemma 3.2 (i), then we give a sketch of the proof of Lemma 3.2 (ii).

First, suppose that $0 < \mu(N) < 1$. Fix $p < P$. Next we fix $p_1, p_2 \in (p, P)$ such that $p_1 < p_2$. Let $m \in \mathbb{N}$ be so large that $2^m \geqslant 2p_1^{1/\gamma}$. Since $p_1 < p_2$ and $\mu(N) > 0$, we can find $\sigma > 0$ so small that

$$\frac{1}{\mu(N)} p_1^{1/\gamma} \leqslant \frac{1-3\sigma}{\mu(N)+2\sigma} p_2^{1/\gamma}. \tag{3.2}$$

The definition of the sets N_θ, $\theta > 0$, and the countable additivity of measures imply that we can find $0 < a < b < c$ such that

$$\mu(N_b \setminus N_a) \leqslant \frac{\sigma}{2C_\Delta^m} \quad \text{and} \quad \mu(N_c) < \mu(N) + \sigma. \tag{3.3}$$

We are going to construct $\{v_k\} \subset WL^\Phi(\mathbb{R}^n)$ with the following properties

$$\begin{gathered} v_k = p_1^{1/\gamma} u_k \quad \text{in } N_a, \quad v_k = 0 \quad \text{in } \mathbb{R}^n \setminus N_b, \\ \|\Phi(|\nabla v_k|)\|_{L^1(\mathbb{R}^n)} \leqslant 1, \quad \|\Phi(|v_k|)\|_{L^1(\mathbb{R}^n)} \leqslant C. \end{gathered}$$

Let $\psi \in C_0(\mathbb{R}^n)$ be a test-function satisfying $0 \leqslant \psi \leqslant 1$, $\psi \equiv 1$ on N_b and $\psi \equiv 0$ on $\mathbb{R}^n \setminus N_c$. Hence

$$\int_{N_b} \Phi(|\nabla u_k|) \leqslant \int_{N_c} \psi \Phi(|\nabla u_k|) \overset{k\to 0}{\longrightarrow} \int_{N_c} \psi \,\mathrm{d}\mu \leqslant \mu(N_c) \leqslant \mu(N) + \sigma$$

and thus there is $k_1 \in \mathbb{N}$ such that

$$\int_{N_b} \Phi(|\nabla u_k|) \leqslant \mu(N) + 2\sigma \quad \text{for } k \geqslant k_1. \tag{3.4}$$

Next, using the definition of P, $p_2 < P$, the estimate $\Phi(st) \leqslant s\Phi(t)$ for $s \in [0,1]$ and $t \geqslant 0$ (this estimate easily follows from the definition of a Young function) and (3.2) we obtain

$$\Phi(p_1^{1/\gamma} t) \leqslant \Big(\frac{p_1}{p_2}\Big)^{1/\gamma} \Phi(p_2^{1/\gamma} t) \leqslant \mu(N) \frac{1-3\sigma}{\mu(N)+2\sigma} \frac{1}{\mu(N)} \Phi(t) \quad \text{for every } t \geqslant 0$$

and thus (3.4) implies

$$\int_{N_b} \Phi(p_1^{1/\gamma} |\nabla u_k|) \leqslant 1 - 3\sigma \quad \text{for } k \geqslant k_1. \tag{3.5}$$

In a similar way to that applied when obtaining (3.4) we can use (3.3) to find $k_2 \geqslant k_1$ such that

$$\int_{N_b \setminus N_a} \Phi(|\nabla u_k|) \leqslant \frac{\sigma}{C_\Delta^m} \quad \text{for } k \geqslant k_2. \tag{3.6}$$

Now, we can define v_k. Fix $\psi \in C^1(\mathbb{R}^n)$ such that $0 \leqslant \psi \leqslant 1$, $\psi \equiv 1$ on N_a and $\psi \equiv 0$ on $\mathbb{R}^n \setminus N_b$. We set $v_k = p_1^{1/\gamma} \psi u_k$. We are going to apply Theorem 1.5 (ii). Thus, we need to check its assumptions. First, by the Δ_2-condition we have for every $k \in \mathbb{N}$

$$\int_{\mathbb{R}^n} \Phi(|v_k|) \leqslant \int_{\mathbb{R}^n} \Phi(p_1^{1/\gamma}|u_k|) \leqslant C \int_{\mathbb{R}^n} \Phi(|u_k|) \leqslant C.$$

Next, let us prove that there is $k_3 \geqslant k_2$ such that

$$\int_{\mathbb{R}^n} \Phi(|\nabla v_k|) \leqslant 1 \quad \text{for } k \geqslant k_3. \tag{3.7}$$

We have

$$\int_{\mathbb{R}^n} \Phi(|\nabla v_k|) = \int_{N_a} \Phi(|\nabla v_k|) + \int_{N_b \setminus N_a} \Phi(|\nabla v_k|) = I_1 + I_2.$$

By (3.5) we have

$$I_1 = \int_{N_a} \Phi(|\nabla v_k|) = \int_{N_a} \Phi(p_1^{1/\gamma}|\nabla u_k|) \leqslant 1 - 3\sigma.$$

Next, we set $T = \max\limits_{N_b} |\nabla \psi|$. We obtain

$$\Phi(|\nabla v_k|) \leqslant \Phi(p_1^{1/\gamma}\psi|\nabla u_k| + p_1^{1/\gamma}|u_k||\nabla\psi|) \leqslant \Phi(p_1^{1/\gamma}|\nabla u_k| + p_1^{1/\gamma}T|u_k|).$$

Hence

$$\begin{aligned} I_2 = \int_{N_b \setminus N_a} \Phi(|\nabla v_k|) &= \int_{(N_b \setminus N_a)\cap\{|\nabla u_k| > T|u_k|\}} + \int_{(N_b \setminus N_a)\cap\{|\nabla u_k| \leqslant T|u_k|\}} \\ &\leqslant \int_{N_b \setminus N_a} \Phi(2p_1^{1/\gamma}|\nabla u_k|) + \int_{N_b \setminus N_a} \Phi(2p_1^{1/\gamma}T|u_k|) = J_1 + J_2. \end{aligned}$$

By (3.6), the choice of m and by the Δ_2-condition, we have $J_1 \leqslant \sigma$ for $k \geqslant k_2$. Furthermore, as $u_k \rightharpoonup 0$ in $WL^\Phi(N_b)$, we obtain $u_k \to 0$ in $L^\Phi(N_b)$ ($L^\Phi(N_b)$ is compactly embedded into $WL^\Phi(N_b)$) and thus $J_2 \leqslant \sigma$ for k sufficiently large. Thus (3.7) follows and we can use Theorem 1.5 (ii) with $K = (p/p_1)K_{l,n,\alpha} < K_{l,n,\alpha}$ to obtain for $k > k_3$

$$\begin{aligned}\|\Upsilon(K_{l,n,\alpha}p|u_k|^\gamma)\|_{L^1(N_a)} &= \|\Upsilon(Kp_1|u_k|^\gamma)\|_{L^1(N_a)}\\ &\leqslant \|\Upsilon(K|v_k|^\gamma)\|_{L^1(\mathbb{R}^n)} \leqslant C.\end{aligned}$$

Furthermore, we can use Theorem 1.5 (i) to show that

$$\|\Upsilon(K_{l,n,\alpha}p|u_k|^\gamma)\|_{L^1(N_a)} \leqslant C \quad \text{for } k = 1, \ldots, k_3.$$

Hence we are done in the case $0 < \mu(N) < 1$.

If $\mu(N) = 0$, the proof has to be modified a bit, as σ cannot be defined by (3.2). However, since Φ satisfies the Δ_2-condition, there is plainly $\sigma \in (0, 1/4)$ so small that

$$\Phi(p_1^{1/\gamma}t) \leqslant \frac{1-3\sigma}{2\sigma}\Phi(t) \quad \text{for every } t \geqslant 0.$$

This ensures that (3.4) still implies (3.5).

The proof of Lemma 3.2 (ii) is similar to the above. In this case we fix $R_0 > 0$ so large that $\int_{\mathbb{R}^n\setminus B(R_0)} \Phi(|u_k|)$ is very close to A_∞ for k large, then we pick $R_1, R_2 \in (0, R_0)$ such that $R_1 < R_2$, $\mu(B(R_2) \setminus B(R_1))$ is very small and $\mu(B(R_0) \setminus B(R_1))$ is very small. We set $v_k = p_1^{1/\gamma}\psi u_k$, where the function $\psi \in C^1(\mathbb{R}^n)$ is chosen so that $0 \leqslant \psi \leqslant 1$, $\psi \equiv 0$ on $B(R_1)$ and $\psi \equiv 1$ on $\mathbb{R}^n \setminus B(R_2)$. □

4. Proof of Theorem 1.6

P r o o f of the estimate $\mu(\mathbb{R}^n) \leqslant 1 - A_\infty$. We can use the countable additivity of measures to see that it is enough to show that $\mu(B(R)) \leqslant 1 - A_\infty$ for arbitrary $R > 0$. Thus, the proof is an easy exercise based on the assumptions $\|\Phi(|\nabla u_k|)\|_{L^1(\mathbb{R}^n)} \leqslant 1$ for every $k \in \mathbb{N}$, $\Phi(|\nabla u_k|) \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\overline{\Omega})$, the definition of A_∞ and a suitably chosen test-function. □

P r o o f of Theorem 1.6 (i). First, let us prove the assertion concerning the boundedness. Fix $p > 0$ and $\varrho > 0$. In our case we have $A_\infty = 0$ and thus, by Lemma 3.2 (ii), we can find $R > 0$ so large that

$$\|\Upsilon(K_{l,n,\alpha}p\,|u_k|^\gamma)\|_{L^1(\mathbb{R}^n\setminus B(R))} \quad \text{is bounded.}$$

Next, let us define a compact set $N = \overline{B}(2R) \setminus B(x_0, \varrho)$. We plainly have $\mu(N) = 0$ and thus we can use Lemma 3.2 (i) to obtain that

$$\|\Upsilon(K_{l,n,\alpha}p\,|u_k|^\gamma)\|_{L^1(N)} \quad \text{is bounded.}$$

Hence

$$\|\Upsilon(K_{l,n,\alpha}p\,|u_k|^{\gamma})\|_{L^1(\mathbb{R}^n\setminus B(x_0,\varrho))} \quad \text{is bounded.}$$

It remains to prove the assertion concerning the convergence in measures. From Lemma 2.3, $u_k \rightharpoonup 0$, $\Upsilon(0)=0$ and from the previous results it follows that if $G \subset \mathbb{R}^n$ is an open bounded set, then

$$\eta > 0 \Rightarrow \int_{G\setminus B(x_0,\eta)} \Upsilon(K_{l,n,\alpha}|u_k|^{\gamma}) \overset{k\to\infty}{\longrightarrow} 0. \tag{4.1}$$

Now, (4.1) and the assumptions imply

$$\eta > 0 \Rightarrow \int_{B(x_0,\eta)} \Upsilon(K_{l,n,\alpha}|u_k|^{\gamma}) \overset{k\to\infty}{\longrightarrow} c. \tag{4.2}$$

Fix an arbitrary test function $\psi \in C_0(\mathbb{R}^n)$ and let $\varepsilon > 0$. Then there is $\eta > 0$ such that

$$|\psi(x) - \psi(x_0)| < \frac{\varepsilon}{2\max(c,1)} \quad \text{whenever } |x - x_0| < \eta. \tag{4.3}$$

We have

$$\begin{aligned} I &:= \left|\int_{\mathbb{R}^n} \psi \,\mathrm{d}(c\delta_{x_0}) - \int_{\mathbb{R}^n} \psi\Upsilon(K_{l,n,\alpha}|u_k|^{\gamma})\right| = \left|c\psi(x_0) - \int_{\mathbb{R}^n} \psi\Upsilon(K_{l,n,\alpha}|u_k|^{\gamma})\right| \\ &\leqslant \int_{\mathbb{R}^n\setminus B(x_0,\eta)} |\psi|\Upsilon(K_{l,n,\alpha}|u_k|^{\gamma}) + \int_{B(x_0,\eta)} |\psi - \psi(x_0)|\Upsilon(K_{l,n,\alpha}|u_k|^{\gamma}) \\ &\quad + |\psi(x_0)|\left|c - \int_{B(x_0,\eta)} \Upsilon(K_{l,n,\alpha}|u_k|^{\gamma})\right| = I_1 + I_2 + I_3. \end{aligned}$$

From (4.1), compactness of the support of ψ and $\sup_{\mathbb{R}^n}|\psi| < \infty$ we see that there is $k_1 \in \mathbb{N}$ such that $I_1 < \varepsilon$ for $k > k_1$. Further, using (4.2) and (4.3) we obtain

$$\begin{aligned} I_2 &= \int_{B(x_0,\eta)} |\psi - \psi(x_0)|\Upsilon(K_{l,n,\alpha}|u_k|^{\gamma}) \\ &\leqslant \frac{\varepsilon}{2\max(c,1)} \int_{B(x_0,\eta)} \Upsilon(K_{l,n,\alpha}|u_k|^{\gamma}) \overset{k\to\infty}{\longrightarrow} \frac{\varepsilon}{2}\frac{c}{\max(c,1)} \leqslant \frac{\varepsilon}{2}. \end{aligned}$$

Therefore we can find $k_2 > k_1$ such that $I_2 < \varepsilon$ for $k > k_2$. Finally, from (4.2) and $|\psi(x_0)| < \infty$ we obtain $k_3 > k_2$ such that $I_3 < \varepsilon$ for $k > k_3$. Hence we have $I < 3\varepsilon$ for k large enough and we are done. $\square$

P r o o f of Theorem 1.6 (ii). Since $A_\infty = 1$, for every $R > 0$ we have $\mu(B(R)) = 0$ and thus the assertion easily follows from Lemma 3.2 (i). $\square$

Proof of Theorem 1.6 (iii). Fix $p < P$. By Lemma 3.2 (ii), we can find $R > 0$ so large that

$$\|\Upsilon(K_{l,n,\alpha}p\,|u_k|^\gamma)\|_{L^1(\mathbb{R}^n\setminus B(R))} \quad \text{is bounded.}$$

Next, for every point $x \in \overline{B}(R)$, we can use Lemma 3.2 (i) to find a radius $r_x > 0$ such that

$$\|\Upsilon(K_{l,n,\alpha}p\,|u_k|^\gamma)\|_{L^1(B(x,r_x))} \quad \text{is bounded.}$$

Since $\overline{B}(R)$ is compact, the result follows. $\square$

Proof of Theorem 1.6 (iv). Fix $p < P$. By the equimeasurability of u_k and $u_k^\sharp$ we have

$$\int_{\mathbb{R}^n} \Upsilon(K_{l,n,\alpha}p\,|u_k|^\gamma) = \int_{\mathbb{R}^n} \Upsilon(K_{l,n,\alpha}p\,|u_k^\sharp|^\gamma).$$

Next, since Φ satisfies the Δ_2-condition and since we assume that $\|\Phi(|u_k|)\|_{L^1(\mathbb{R}^n)} \leqslant M$ for every $k \in \mathbb{N}$, there is $\widetilde{M} > 0$ such that $\|u_k\|_{L^\Phi(\mathbb{R}^n)} \leqslant \widetilde{M}$ for every $k \in \mathbb{N}$, and thus we can apply Lemma 2.4 to obtain $R_s > 0$ and $C_s > 0$ such that

$$u_k^\sharp(x) \leqslant C_s\widetilde{M}\frac{1}{|x|} \quad \text{for every } |x| > R_s \text{ and every } k \in \mathbb{N}. \tag{4.4}$$

Let us set $R = \max\{R_s, C_s\widetilde{M}\}$. We have

$$\begin{aligned}\int_{\mathbb{R}^n} \Upsilon(K_{l,n,\alpha}p\,|u_k^\sharp|^\gamma) &= \int_{B(R)} \Upsilon(K_{l,n,\alpha}p\,|u_k^\sharp|^\gamma) + \int_{\mathbb{R}^n\setminus B(R)} \Upsilon(K_{l,n,\alpha}p\,|u_k^\sharp|^\gamma)\\ &= I_1 + I_2.\end{aligned}$$

First, we estimate I_1. Let us fix $\widetilde{p} \in (p, P)$. Next we define $v_k = \max\{|u_k| - t_k, 0\}$, where t_k are such that $u_k^\sharp(x) = t_k$ for $|x| = R$. Further, by (4.4) we have $t_k \leqslant T := C_s\widetilde{M}/R$. Since we can apply Theorem 1.4 (iii) to v_k (see Remark 3.1), we have

$$\begin{aligned} I_1 &\leqslant \int_{B(R)} \Upsilon(K_{l,n,\alpha}p\,|u_k^\sharp|^\gamma) \leqslant \int_{B(R)} \exp_{[l]}(K_{l,n,\alpha}p\,|u_k^\sharp|^\gamma)\\ &\leqslant \int_{B(R)} \exp_{[l]}(K_{l,n,\alpha}p\,|v_k^\sharp + T|^\gamma)\\ &\leqslant \int_{B(R)\cap\{p\,|v_k^\sharp+T|^\gamma\leqslant\widetilde{p}\,|v_k^\sharp|^\gamma\}} + \int_{B(R)\cap\{p\,|v_k^\sharp+T|^\gamma\geqslant\widetilde{p}\,|v_k^\sharp|^\gamma\}}\\ &\leqslant \int_{B(R)} \exp_{[l]}(K_{l,n,\alpha}\widetilde{p}\,|v_k^\sharp|^\gamma) + \int_{B(R)} \exp_{[l]}(K_{l,n,\alpha}p\,|CT + T|^\gamma)\\ &\leqslant C + C = C.\end{aligned}$$

It remains to estimate I_2. From (4.4), $C_s\widetilde{M}/R \leqslant 1$, and the Lebesgue Monotone Convergence Theorem (the Taylor coefficients a_j are non-negative by Lemma 2.1) we obtain

$$\begin{aligned}(4.5)\qquad &\int_{\mathbb{R}^n\setminus B(R)} \sum_{j>n/\gamma} a_j K_{l,n,\alpha}^j p^j |u_k^\sharp|^{\gamma j}\,\mathrm{d}x \\ &\leqslant \omega_{n-1} \sum_{j>n/\gamma} a_j K_{l,n,\alpha}^j p^j C_s^{\gamma j} \widetilde{M}^{\gamma j} \int_R^\infty y^{n-1-\gamma j}\,\mathrm{d}y \\ &= C \sum_{j>n/\gamma} a_j K_{l,n,\alpha}^j p^j C_s^{\gamma j} \widetilde{M}^{\gamma j} R^{n-\gamma j} \\ &\leqslant CR^n \sum_{j=0}^\infty a_j K_{l,n,\alpha}^j p^j = CR^n \exp_{[l]}(K_{l,n,\alpha}p) = C.\end{aligned}$$

If $n/\gamma \in \mathbb{N}$, then we also need to estimate the summand corresponding to this index. By (1.7) we have

$$\int_{\mathbb{R}^n\setminus B(R)} |u_k^\sharp|^n \leqslant C \int_{\mathbb{R}^n\setminus B(R)} \Phi(|u_k^\sharp|) \leqslant C \int_{\mathbb{R}^n} \Phi(|u_k^\sharp|) \leqslant CM = C$$

and thus

$$(4.6)\qquad \int_{\mathbb{R}^n\setminus B(R)} a_{n/\gamma} K_{l,n,\alpha}^{n/\gamma} p^{n/\gamma} |u_k^\sharp|^n \leqslant C a_{n/\gamma} K_{l,n,\alpha}^{n/\gamma} p^{n/\gamma} = C.$$

Therefore we have from (4.5) and (4.6)

$$I_2 = \int_{\mathbb{R}^n\setminus B(R)} \Upsilon(K_{l,n,\alpha} p\, |u_k^\sharp|^\gamma) = \int_{\mathbb{R}^n\setminus B(R)} \sum_{j\geqslant n/\gamma} a_j K_{l,n,\alpha}^j p^j |u_k^\sharp|^{\gamma j} \leqslant C$$

and we are done. □

P r o o f of the results concerning the L^1-convergence. The results concerning the L^1-convergence of $\Upsilon(K_{l,n,\alpha}|u_k|^\gamma)$ follow from the results concerning the boundedness and from Lemma 2.3. □

5. Concluding remarks

Some comments concerning the assumption (1.7)**.** This assumption comes from paper [6], where Theorem 1.5 is proved and applied to some PDEs. It can be seen that the proof of Theorem 1.5 uses only the first of the two inequalities in (1.7), i.e. $(1/C)t^n \leqslant \Phi(t)$, $t \in [0, 1/C]$ (this inequality is used in the proof of Lemma 2.4, while the inequality $\Phi(t) \leqslant Ct^n$, $t \in [0, 1/C]$ is used in the proof of the result concerning PDEs). In this paper, we also use assumption (1.7) to ensure the Δ_2-condition. Thus, a careful inspection of our proofs shows that (1.7) can be replaced by a bit weaker assumption

$$\Phi(t) \geqslant \frac{1}{C} t^n, \quad t \in \Big[0, \frac{1}{C}\Big], \quad \text{and} \quad \Phi \text{ satisfies the } \Delta_2\text{-condition.}$$

It can be also seen that we can replace (1.7) by

$$\Phi(t) \geqslant \frac{1}{C} t^p, \quad t \in \Big[0, \frac{1}{C}\Big], \quad \text{and} \quad \Phi \text{ satisfies the } \Delta_2\text{-condition}$$

for some $p > 1$, as long as we state our results for $\widetilde{\Upsilon}(t) = \sum\limits_{j \geqslant p/\gamma} a_j t^j$ instead of Υ.

Sharpness of the upper bounds of p **in Theorem 1.2.** To show that we cannot have $p = P$ in Theorem 1.2 (iv), it is enough to use the sequence of compactly supported functions $\{u_k\}$ given in [8], Proof of Proposition 2.1. In fact, we cannot use the result from [8] directly, since the paper [8] studies the integrability with respect to the function exp, while our function Υ is a bit smaller. However, it can be seen that for any fixed $D > 0$ we have

$$\mathrm{e}^{-k} \Upsilon((D + k^{(n-1)/n})^{n/(n-1)}) \stackrel{k\to\infty}{\longrightarrow} \infty. \tag{5.1}$$

Indeed, we have $\Upsilon(t) = \exp(t) - \sum\limits_{j=0}^{n-2} t^j/j\,!$ and

$$\mathrm{e}^{-k} \exp((D + k^{(n-1)/n})^{n/(n-1)}) \stackrel{k\to\infty}{\longrightarrow} \infty,$$

while for every $p > 0$

$$\mathrm{e}^{-k}((D + k^{(n-1)/n})^{n/(n-1)})^p \leqslant \mathrm{e}^{-k} C k^p \stackrel{k\to\infty}{\longrightarrow} 0.$$

From (5.1) it can be seen that the construction from [8] still works.

Now, suppose that we have the situation from Theorem 1.2 (i). To see that on no neighborhood of x_0 we can have a better exponent than the one given by Moser-Trudinger inequality (1.1), it is enough to fix $R > 0$ sufficiently small and to use the Moser functions

$$m_k(x) = \begin{cases} \omega_{n-1}^{-1/n} \log^{1/n'}(k) & \text{for } |x| \in [0, k^{-1}R], \\ \omega_{n-1}^{-1/n} \log^{-1/n}(k) \log(R/|x|) & \text{for } |x| \in [k^{-1}R, R] \end{cases}$$

defined for every $n \in \mathbb{N}$. Again, it is not important that we work with Υ instead of exp.

The sharpness for unbounded sets in the situation of Theorem 1.2 (ii) is obtained considering the sequence $\{m_k(\cdot - x_k)\}$, where $\{x_k\} \subset \mathbb{R}^n$ is a suitable sequence satisfying $|x_k| \to \infty$.

Finally, suppose that we have the situation of Theorem 1.2 (iii) with at least one of the quantities A_∞ and $\max\limits_{x \in \mathbb{R}^n} \mu(\{x\})$ being positive (otherwise we have $P = \infty$). Now, if $A_\infty > 0$ then we can use the sequence $\{A_\infty m_k(\cdot - x_k)\}$, where $|x_k| \to \infty$, to see that we cannot have

$$\Upsilon(n\omega_{n-1}^{1/(n-1)} A_\infty^{-1/(n-1)} |u_k|^{n/(n-1)}) \quad \text{is bounded in } L^1(\mathbb{R}^n \setminus B(R))$$

for any fixed $R > 0$.

In the case $\max\limits_{x \in \mathbb{R}^n} \mu(\{x\}) > 0$, let us suppose that $\max\limits_{x \in \mathbb{R}^n} \mu(\{x\}) = \mu(\{0\})$. Now, it is easy to see that the sequence $\{\mu(\{0\}) m_k\}$ can be used to show that we cannot have

$$\Upsilon(n\omega_{n-1}^{1/(n-1)} \mu^{-1/(n-1)}(\{0\}) |u_k|^{n/(n-1)}) \quad \text{is bounded in } L^1(B(\varrho))$$

for any fixed $\varrho > 0$.

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