Ahmed Sani; Hafida Laasri

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EVOLUTION EQUATIONS GOVERNED BY LIPSCHITZ CONTINUOUS NON-AUTONOMOUS FORMS

Ahmed Sani, Agadir, Hafida Laasri, Wuppertal

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Abstract. We prove L^2-maximal regularity of the linear non-autonomous evolutionary Cauchy problem

$$\dot{u}(t) + A(t)u(t) = f(t) \quad \text{for a.e. } t \in [0,T], \quad u(0) = u_0,$$

where the operator $A(t)$ arises from a time depending sesquilinear form $\mathfrak{a}(t,\cdot,\cdot)$ on a Hilbert space H with constant domain V. We prove the maximal regularity in H when these forms are time Lipschitz continuous. We proceed by approximating the problem using the frozen coefficient method developed by El-Mennaoui, Keyantuo, Laasri (2011), El-Mennaoui, Laasri (2013), and Laasri (2012). As a consequence, we obtain an invariance criterion for convex and closed sets of H.

Keywords: sesquilinear form; non-autonomous evolution equation; maximal regularity; convex set

MSC 2010: 35K90, 35K45, 47D06

1. Introduction

In this paper we study non-autonomous evolutionary linear Cauchy problems

$$\dot{u}(t) + \mathcal{A}(t)u(t) = f(t), \quad u(0) = u_0, \tag{1.1}$$

where the operators $\mathcal{A}(t)$, $t \in [0,T]$, arise from sesquilinear forms on Hilbert spaces. More precisely, throughout this work H and V are two separable Hilbert spaces. The scalar products and the corresponding norms on H and V will be denoted by $(\cdot|\cdot)$, $(\cdot|\cdot)_V$, $\|\cdot\|$ and $\|\cdot\|_V$, respectively. We assume that $V \underset{d}{\hookrightarrow} H$; i.e., V is densely embedded into H and

$$\|u\| \leqslant c_H \|u\|_V, \quad u \in V \tag{1.2}$$

for some constant $c_H > 0$.

Let V' denote the antidual of V. The duality between V' and V is denoted by $\langle\cdot,\cdot\rangle$. As usual, we identify H with H'. It follows that $V \hookrightarrow H \cong H' \hookrightarrow V'$ and so V is identified with a subspace of V'. These embeddings are continuous and

$$(1.3) \qquad \|f\|_{V'} \leqslant c_H \|f\|, \quad f \in V'$$

with the same constant c_H as in (1.2) (see, e.g., [6]).

With a non-autonomous form

$$\mathfrak{a}\colon [0,T] \times V \times V \mapsto \mathbb{C}$$

such that $\mathfrak{a}(t,\cdot,\cdot)$ is sesquilinear for all $t \in [0,T]$, $\mathfrak{a}(\cdot,u,v)$ is measurable for all $u,v \in V$,

$$|\mathfrak{a}(t,u,v)| \leqslant M\|u\|_V\|v\|_V, \quad t \in [0,T],\ u,v \in V$$

and

$$\operatorname{Re}\mathfrak{a}(t,u,u) + \omega\|u\|^2 \geqslant \alpha\|u\|_V^2, \quad t \in [0,T],\ u \in V$$

for some $\alpha > 0$, $M \geqslant 0$ and $\omega \in \mathbb{R}$, for each $t \in [0,T]$ we can associate a unique operator $\mathcal{A}(t) \in \mathcal{L}(V,V')$ such that $\mathfrak{a}(t,u,v) = \langle \mathcal{A}(t)u, v\rangle$ for all $u,v \in V$. It is a known fact that $-\mathcal{A}(t)$ with domain V generates a holomorphic semigroup $(\mathcal{T}_t(s))_{s\geqslant 0}$ on V'. Observe that $\|\mathcal{A}(t)u\|_{V'} \leqslant M\|u\|_V$ for all $u \in V$ and all $t \in [0,T]$. It is worth mentioning that the mapping $t \mapsto \mathcal{A}(t)$ is strongly measurable by the Dunford-Pettis Theorem in [1] since the spaces are assumed to be separable and $t \mapsto \mathcal{A}(t)$ is weakly measurable. Thus $t \mapsto \mathcal{A}(t)u$ is Bochner integrable on $[0,T]$ with values in V' for all $u \in V$.

The following well known maximal regularity result is due to J. L. Lions.

Theorem 1.1. *Given $f \in L^2(0,T;V')$ and $u_0 \in H$, there is a unique solution $u \in \mathrm{MR}(V,V') := L^2(0,T;V) \cap H^1(0,T;V')$ of*

$$(1.4) \qquad \dot{u}(t) + \mathcal{A}(t)u(t) = f(t), \quad u(0) = u_0.$$

Note that $\mathrm{MR}(V,V') \underset{d}{\hookrightarrow} C([0,T];H)$ (see [17], page 106), so the condition $u(0) = u_0$ in (1.4) makes sense and the solution is continuous on $[0,T]$ with values in H. The proof of Theorem 1.1 can be given by an application of the Lions Representation Theorem in [13] (see also [17], page 112, and [20], Chapter 3) or by Galerkin's method in [7], XVIII Chapter 3, page 620. We refer also to an alternative proof given by Tanabe in [19], Section 5.5.

In Section 3, we give another proof by using the approach of frozen coefficient developed in [8], [12] and [11], from which we derive the criterion for invariance of convex closed sets established (see [4]) and also the recent result given in [3] for Lipschitz continuous forms.

Let $\Lambda := (0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n+1} = T)$ be a subdivision of $[0,T]$. We approximate (1.1) by (1.5), obtained when the generators $\mathcal{A}(t)$ are frozen on the interval $[\lambda_k, \lambda_{k+1}[$. More precisely, let $\mathcal{A}_\Lambda \colon [0,T] \to \mathcal{L}(V,V')$ be given by

$$\mathcal{A}_\Lambda(t) := \begin{cases} \mathcal{A}_k & \text{for } \lambda_k \leqslant t < \lambda_{k+1}, \\ \mathcal{A}_n & \text{for } t = T, \end{cases}$$

with

$$\mathcal{A}_k x := \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} \mathcal{A}(r) u \, \mathrm{d}r, \quad u \in V, \ k = 0, 1, \ldots, n.$$

Note that the integral on the right hand side makes sense since the mapping $t \mapsto \mathcal{A}(t)$ is, as mentioned above, strongly Bochner-integrable.

We show (see Theorem 3.2) that for all $u_0 \in H$ and $f \in L^2(0,T;V')$ the non-autonomous problem

$$\dot{u}_\Lambda(t) + \mathcal{A}_\Lambda(t) u_\Lambda(t) = f(t), \quad u_\Lambda(0) = u_0 \tag{1.5}$$

has a unique solution $u_\Lambda \in \mathrm{MR}(V,V')$ which converges in $\mathrm{MR}(V,V')$ as $|\Lambda| \to 0$, and $u := \lim\limits_{|\Lambda| \to 0} u_\Lambda$ solves (1.4) uniquely.

Let $\mathcal{C}$ be a closed convex subset of the Hilbert space H and let $P \colon H \to \mathcal{C}$ be the orthogonal projection onto $\mathcal{C}$. As a consequence of Theorem 3.2 we obtain: If $u_0 \in \mathcal{C}$, $P(V) \subset V$ and

$$\operatorname{Re} \mathfrak{a}(t, Pv, v - Pv) \geqslant 0 \tag{1.6}$$

for almost every $t \in [0,T]$ and for all $v \in V$, then $u(t) \in \mathcal{C}$ for all $t \in [0,T]$, where u is the solution of (1.4) with $f = 0$. In the autonomous case condition (1.6) is also necessary for the invariance of $\mathcal{C}$, see [15]. More recently, for $f \neq 0$ the invariance of $\mathcal{C}$ under the solution of (1.4) was proved by Arendt, Dier and Ouhabaz in [4] provided that

$$\operatorname{Re} \mathfrak{a}(t, Pv, v - Pv) \geqslant \langle f(t), v - Pv \rangle \tag{1.7}$$

for almost every $t \in [0,T]$ and for all $v \in V$.

Theorem 1.1 establishes L^2-maximal regularity of the Cauchy problem (1.4) in V' assuming only that $t \mapsto \mathfrak{a}(t,u,v)$ is measurable for all $u, v \in V$. However, in applications to boundary value problems, only the part $A(t)$ of $\mathcal{A}(t)$ in H does realize the boundary conditions in question. Thus one is interested in L^2-maximal regularity in H:

Problem 1.2. If $f \in L^2(0,T;H)$ and $u_0 \in V$, does the solution of (1.5) belong to $\mathrm{MR}(V,H) := L^2(0,T;V) \cap H^1(0,T;H)$?

This Problem 1.2 was asked (for $u_0 = 0$) by Lions [13], page 68, and is, to our knowledge, still open. Note that if $\mathfrak{a}$ (or equivalently $\mathcal{A}$) is a step function and symmetric the answer to Problem 1.2 is affirmative. In fact, for $u_0 \in V$ and $f \in L^2(0,T;H)$ the solution u_Λ of (1.5) belongs to $\mathrm{MR}(V,H) \cap C([0,T];V)$ (see Section 3). Thus, for symmetric forms Problem 1.2 can be reformulated as follows:

Problem 1.3. If $f \in L^2(0,T;H)$ and $u_0 \in V$, does the solution of (1.5) converge in $\mathrm{MR}(V,H)$ as $|\Lambda| \to 0$?

For general forms, an affirmative answer of Problem 1.2 is given under an additional regularity assumption (with respect to t) on $\mathfrak{a}(t,\cdot,\cdot)$. For symmetric forms, Lions proved L^2-maximal regularity in H for $u_0 = 0$ (respectively for $u_0 \in D(A(0))$) provided $\mathfrak{a}(\cdot,u,v) \in C^1[0,T]$ (respectively $\mathfrak{a}(\cdot,u,v) \in C^2[0,T]$) for all $u,v \in V$, (see [13], page 68 and page 94). Moreover, a combination of [13], Theorem 1.1, page 129, and [13], Theorem 5.1, page 138, shows that if $\mathfrak{a}(\cdot,u,v) \in C^1[0,T]$ for all $u,v \in V$, then (1.5) has L^2-maximal regularity in H. Bardos [5] gave also an affirmative answer to Problem (1.2) under the assumptions that the domains of both $A(t)^{1/2}$ and $A(t)^{*1/2}$ coincide with V and that $\mathcal{A}(.)^{1/2}$ is continuously differentiable with values in $\mathcal{L}(V,V')$. We mention also a result of Ouhabaz and Spina [16] and Ouhabaz and Haak [9]. They proved L^2-maximal regularity for (possibly non-symmetric) forms such that $\mathfrak{a}(\cdot,u,v) \in C^\alpha[0,T]$ for all $u,v \in V$ and some $\alpha > 1/2$. The result in [16] concerns the case $u_0 = 0$ and the one in [9] concerns the case u_0 in the real-interpolation space $(H, D(A(0)))_{1/2,2}$.

In Section 4, we are concerned with a recent result obtained in [3]. Assume that the sesquilinear form $\mathfrak{a}$ can be written as $\mathfrak{a}(t,u,v) = \mathfrak{a}_1(t,u,v) + \mathfrak{a}_2(t,u,v)$ where $\mathfrak{a}_1$ is symmetric, bounded (i.e. $\mathfrak{a}_1(t,u,v) \leqslant M_1\|u\|\|v\|$, $M_1 \geqslant 0$) and coercive as above and piecewise Lipschitz-continuous on $[0,T]$ with Lipschitz constant L_1, whereas $\mathfrak{a}_2\colon [0,T] \times V \times H \mapsto \mathbb{C}$ satisfies $|\mathfrak{a}_2(t,u,v)| \leqslant M_2\|u\|_V\|v\|_H$ and $\mathfrak{a}_2(\cdot,u,v)$ is measurable for all $u \in V$, $v \in H$. Furthermore, let $B\colon [0,T] \to \mathcal{L}(H)$ be strongly measurable such that $\|B(t)\|_{\mathcal{L}(H)} \leqslant \beta_1$ for all $t \in [0,T]$ and $0 < \beta_0 \leqslant (B(t)g \mid g)_H$ for $g \in H$, $\|g\|_H = 1$, $t \in [0,T]$. Then, the following result is proved in [3], Corollary 4.3:

Theorem 1.4. *Let $u_0 \in V$, $f \in L^2(0,T;H)$. Then there exists a unique $u \in \mathrm{MR}(V,H)$ satisfying*

$$\dot{u}(t) + B(t)\mathcal{A}(t)u(t) = f(t) \quad \text{a.e.,} \quad u(0) = u_0.$$

Moreover,

$$\|u\|_{\mathrm{MR}(V,H)} \leqslant C[\|u_0\|_V + \|f\|_{L^2(0,T;H)}], \tag{1.8}$$

where the constant C depends only on β_0, β_1, M_1, M_2, α, T, L_1 and γ.

In the special case where $B = I$ and $\mathfrak{a} = \mathfrak{a}_1$ (or equivalently $\mathfrak{a}_2 = 0$) we prove that Problem 1.3 has an affirmative answer.

We emphasize that our result on approximation may be applied to concrete linear evolution equations. For example, to the evolution equation governed by the elliptic operator in nondivergence form on a domain Ω with time depending coefficients

$$\begin{cases} \dot{u}(t) - \sum\limits_{i,j} \partial_i a_{ij}(t,\cdot)\partial_j u(t) = f(t), \\ u(0) = u_0 \in H^1(\Omega) \end{cases}$$

with an appropriate Lipschitz continuity property on the coefficients with respect to t and boundary conditions such as Niemann or non-autonomous Robin boundary conditions.

2. Preliminary

Consider a *continuous* and *H-elliptic* sesquilinear form $\mathfrak{a}\colon V \times V \mapsto \mathbb{C}$. This means, respectively,

$$|\mathfrak{a}(u,v)| \leqslant M\|u\|_V\|v\|_V \quad \text{for some } M \geqslant 0 \text{ and all } u,v \in V, \tag{2.1}$$

$$\operatorname{Re}\mathfrak{a}(u) + \omega\|u\|^2 \geqslant \alpha\|u\|_V^2 \quad \text{for some } \alpha > 0,\ \omega \in \mathbb{R} \text{ and all } u \in V. \tag{2.2}$$

Here and in the following we shortly write $\mathfrak{a}(u)$ for $\mathfrak{a}(u,u)$. The operator $\mathcal{A} \in \mathcal{L}(V,V')$ associated with $\mathfrak{a}$ on V' is defined by

$$\langle \mathcal{A}u, v\rangle = \mathfrak{a}(u,v), \quad u,v \in V.$$

Seen as an unbounded operator on V' with domain $D(\mathcal{A}) = V$, the operator $-\mathcal{A}$ generates a holomorphic C_0-semigroup $\mathcal{T}$ on V'. The semigroup is bounded on a sector if $\omega = 0$, in which case $\mathcal{A}$ is an isomorphism. Denote by A the part of $\mathcal{A}$ on H; i.e.,

$$D(A) := \{u \in V;\ \mathcal{A}u \in H\},$$
$$Au = \mathcal{A}u.$$

It is a known fact that $-A$ generates a holomorphic C_0-semigroup T on H and $T = \mathcal{T}_{|H}$ is the restriction of the semigroup generated by $-\mathcal{A}$ to H. Then A is the operator *induced* by $\mathfrak{a}$ on H. We refer to [10], [14] and [19], Chapter 2.

Remark 2.1. The sesquilinear form $\mathfrak{a}$ satisfies condition (2.2) if and only if the form a_ω given by

$$\mathfrak{a}_\omega(u,v) := \mathfrak{a}(u,v) + \omega(u \,|\, v)$$

is coercive. Moreover, if $\mathcal{T}_\omega$ and $\mathcal{A}_\omega$ denote, respectively, the semigroup and the operator associated with $\mathfrak{a}_\omega$, then $\mathcal{T}_\omega(t) = \mathrm{e}^{-\omega t}\mathcal{T}(t)$ and $\mathcal{A}_\omega = \omega + \mathcal{A}$ for all $t \geqslant 0$. Then it is possible to choose, without loss of generality, $\mathfrak{a}$ coercive (i.e. $\omega = 0$).

The following maximal regularity results are well known: If $u_0 \in H$, $f \in L^2(a,b;V')$ then the function

$$u(t) = \mathcal{T}(t)u_0 + \int_a^t \mathcal{T}(t-r)f(r)\,\mathrm{d}r$$

belongs to $L^2(a,b;V) \cap H^1(a,b;V')$ and is the unique solution of the autonomous initial value problem

$$\dot{u}(t) + \mathcal{A}u(t) = f(t), \quad \text{for a.e. } t \in [a,b] \subset [0,T], \quad u(a) = u_0. \tag{2.3}$$

Recall that the maximal regularity space

$$\mathrm{MR}(a,b;V,V') := L^2(a,b;V) \cap H^1(a,b;V') \tag{2.4}$$

is continuously embedded in $C([a,b],H)$ and if $u \in \mathrm{MR}(a,b;V,V')$ then the function $\|u(\cdot)\|^2$ is absolutely continuous on $[a,b]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(\cdot)\|^2 = 2\,\mathrm{Re}\langle \dot{u}, u\rangle, \tag{2.5}$$

see e.g. [17], Chapter III, Proposition 1.2, or [19], Lemma 5.5.1. For $[a,b] = [0,T]$ we shortly denote $\mathrm{MR}(a,b;V,V')$ by $\mathrm{MR}(V,V')$ in (2.4).

Furthermore, if $(f,u_0) \in L^2(a,b;H) \times V$ then the solution u of (2.3) belongs to the maximal regularity space

$$\mathrm{MR}(a,b;D(A),H) := L^2(a,b;D(A)) \cap H^1(a,b;H) \tag{2.6}$$

which is equipped with the norm $\|\cdot\|_{\mathrm{MR}}$ given for all $u \in \mathrm{MR}(a,b;D(A),H)$ by

$$\|u\|_{\mathrm{MR}}^2 := \int_a^b \|u(t)\|^2\,\mathrm{d}t + \int_a^b \|\dot{u}(t)\|^2\,\mathrm{d}t + \int_a^b \|Au(t)\|^2\,\mathrm{d}t. \tag{2.7}$$

The maximal regularity space $\mathrm{MR}(a,b;D(A),H)$ is continuously embedded into $C([a,b];V)$, (see [7], Example 1, page 577). If the form $\mathfrak{a}$ is symmetric, then for each $u \in \mathrm{MR}(a,b;D(A),H)$, the function $\mathfrak{a}(u(\cdot))$ belongs to $W^{1,1}(a,b)$ and the following product formula holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{a}(u(t)) = 2(Au(t) \mid \dot{u}(t)) \quad \text{for a.e. } t \in [a,b], \tag{2.8}$$

for the proof we refer to [2], Lemma 3.1.

The next lemma gives a locally uniform estimate for the solution of the autonomous problem. This estimate will play an important role in the study of the convergence in Theorem 5.1.

Lemma 2.2 ([2], Theorem 3.1)**.** *Let $\mathfrak{a}$ be a continuous and H-elliptic sesquilinear form. Assume the form $\mathfrak{a}$ is symmetric. Let $f \in L^2(a,b;H)$ and $u_0 \in V$. Let $u \in \mathrm{MR}(a,b;D(A),H)$ be such that*

$$\dot{u}(t) + Au(t) = f(t), \quad \textit{for a.e. } t \in [a,b] \subset [0,T], \quad u(a) = u_0. \tag{2.9}$$

Then there exists a constant $c_1 > 0$ such that

$$\sup_{s\in[a,b]} \|u(s)\|_V^2 \leqslant c_1[\|u(a)\|_V^2 + \|f\|_{L^2(a,b;H)}^2] \tag{2.10}$$

where $c_1 = c_1(M,\alpha,\omega,T) > 0$ is independent of f, u_0 and $[a,b] \subset [0,T]$.

For the sake of completeness, we include here a simpler proof in the non restrictive case $\omega = 0$.

P r o o f. We use the same technique as in the proof of [2], Theorem 3.1. For simplicity and according to Remark 2.1 we may assume without loss of generality that $\omega = 0$ in (2.2). For almost every $t \in [a,b]$

$$(\dot{u}(t) \mid \dot{u}(t)) + (Au(t) \mid \dot{u}(t)) = (f(t) \mid \dot{u}(t)).$$

The rule formula (2.8) and the Cauchy-Schwartz inequality together with the Young inequality applied to the term on the right-hand side of the above equality imply that, for almost every $t \in [a,b]$,

$$\frac{1}{2}\|\dot{u}(t)\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{a}(u(t)) \leqslant \frac{1}{2}\|f(t)\|^2.$$

Integrating this inequality on $[a,t]$, it follows that

$$\int_a^t \|\dot{u}(s)\|^2\,\mathrm{d}s + \mathfrak{a}(u(t)) \leqslant \mathfrak{a}(u(a)) + \int_a^t \|f(s)\|^2\,\mathrm{d}s.$$

Thus, by (2.1) and (2.2),

$$\int_a^t \|\dot{u}(s)\|^2 \,\mathrm{d}s + \alpha\|u(t)\|_V^2 \leqslant M\|u(a)\|_V^2 + \|f\|^2_{L^2(a,b;H)} \tag{2.11}$$

for almost every $t \in [0,T]$. It follows that

$$\sup_{t\in[a,b]} \|u(t)\|_V^2 \leqslant \frac{1}{\alpha}(M\|u(a)\|_V^2 + \|f\|^2_{L^2(a,b;H)}), \tag{2.12}$$

which gives the desired estimate. □

Remark 2.3. Lemma 2.2 says that the constant c_1 in (2.12) depends only on M, α, ω and T, but does not depend on the subinterval $[a,b]$ or on $\mathfrak{a}$ itself.

3. Well-posedness in V'

Let H, V be the Hilbert spaces introduced in the previous sections. Let $T > 0$ and let

$$\mathfrak{a}\colon\ [0,T]\times V\times V \mapsto \mathbb{C}$$

be a *non-autonomous form*, i.e., $\mathfrak{a}(t,\cdot,\cdot)$ is sesquilinear for all $t \in [0,T]$, $\mathfrak{a}(\cdot,u,v)$ is measurable for all $u,v \in V$,

$$|\mathfrak{a}(t,u,v)| \leqslant M\|u\|_V\|v\|_V, \quad t\in[0,T],\ u,v\in V \tag{3.1}$$

and

$$\operatorname{Re}\mathfrak{a}(t,u,u) + \omega\|u\| \geqslant \alpha\|u\|_V^2, \quad t\in[0,T],\ u\in V \tag{3.2}$$

for some $\alpha > 0$, $M \geqslant 0$ and $\omega \in \mathbb{R}$.

We recall that for all $t \in [0,T]$ we denote by $\mathcal{A}(t) \in \mathcal{L}(V,V')$ the operator associated with the form $\mathfrak{a}(t,\cdot,\cdot)$ in V' and by $\mathcal{T}_t$ the analytic C_0-semigroup generated by $-\mathcal{A}(t)$ on V'. Consider the non-autonomous Cauchy problem

$$\dot{u}(t) + \mathcal{A}(t)u(t) = f(t) \quad \text{for a.e. } t\in[0,T], \quad u(0) = u_0. \tag{3.3}$$

In this section, we are interested in the well-posedness of (3.3) in V' with L^2-maximal regularity. The case where $\mathfrak{a}$ is independent of t is described in the previous section. The case where $\mathfrak{a}$ is a step function is also easy to describe. In fact, let $\Lambda = (0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n+1} = T)$ be a subdivision of $[0,T]$. Let

$$\mathfrak{a}_k\colon\ V\times V\to\mathbb{C} \quad \text{for } k = 0,1,\ldots,n$$

be a finite family of continuous and H-elliptic forms. The associated operators are denoted by $\mathcal{A}_k \in \mathcal{L}(V, V')$. Let $\mathcal{T}_k$ denote the C_0-semigroup generated by $-\mathcal{A}_k$ on V' for all $k = 0, 1 \ldots, n$. The function

$$\mathfrak{a}_\Lambda \colon [0, T] \times V \times V \to \mathbb{C} \tag{3.4}$$

defined by $\mathfrak{a}_\Lambda(t; u, v) := \mathfrak{a}_k(u, v)$ for $\lambda_k \leqslant t < \lambda_{k+1}$ and $\mathfrak{a}_\Lambda(T; u, v) := \mathfrak{a}_n(u, v)$ is strongly measurable on $[0, T]$. Let

$$\mathcal{A}_\Lambda \colon [0, T] \to \mathcal{L}(V, V')$$

be given by $\mathcal{A}_\Lambda(t) := \mathcal{A}_k$ for $\lambda_k \leqslant t < \lambda_{k+1}$, $k = 0, 1, \ldots, n$, and $\mathcal{A}_\Lambda(T) := \mathcal{A}_n$. For each subinterval $[a, b] \subset [0, T]$ such that

$$\lambda_{m-1} \leqslant a < \lambda_m < \ldots < \lambda_{l-1} \leqslant b < \lambda_l$$

we define the operators $\mathcal{P}_\Lambda(a, b) \in \mathcal{L}(V')$ by

$$\mathcal{P}_\Lambda(a, b) := \mathcal{T}_{l-1}(b - \lambda_{l-1})\mathcal{T}_{l-2}(\lambda_{l-1} - \lambda_{l-2}) \ldots \mathcal{T}_m(\lambda_{m+1} - \lambda_m)\mathcal{T}_{m-1}(\lambda_m - a), \tag{3.5}$$

and for $\lambda_{l-1} \leqslant a \leqslant b < \lambda_l$ by

$$\mathcal{P}_\Lambda(a, b) := \mathcal{T}_{l-1}(b - a). \tag{3.6}$$

It is easy to see that for all $u_0 \in H$ and $f \in L^2(a, b, V')$ the function

$$u_\Lambda(t) = \mathcal{P}_\Lambda(a, t)u_0 + \int_a^t \mathcal{P}_\Lambda(r, t)f(r)\,\mathrm{d}r \tag{3.7}$$

belongs to $\mathrm{MR}(a, b; V, V')$ and is the unique solution of the initial value problem

$$\dot{u}_\Lambda(t) + \mathcal{A}_\Lambda(t)u_\Lambda(t) = f(t), \quad \text{for a.e. } t \in [a, b] \subset [0, T], \quad u_\Lambda(a) = u_0.$$

The product given by (3.5)–(3.6) and also the existence of a limit of this product as $|\Lambda|$ converges to 0 uniformly on $[a, b] \subset [0, T]$, were studied in [8], [11] and [12]. This leads to a theory of product integral, comparable to that of the classical Riemann integral. The notion of product integral was introduced by V. Volterra at the end of the 19th century. We refer to A. Slavík [18] and the references therein for a discussion on the work of Volterra and for more details on the product integration theory.

Consider now the general case where $\mathfrak{a} \colon [0, T] \times V \times V \to \mathbb{C}$ is a non-autonomous form and let $\mathcal{A}(t) \in \mathcal{L}(V, V')$ be the associated operator with $\mathfrak{a}(t, \cdot, \cdot)$ on V'. We want to approximate $\mathfrak{a}$ and $\mathcal{A}$ by step functions. Let $\Lambda := (0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n+1} = T)$ be a subdivision of $[0,T]$ and let $a_\Lambda\colon\ [0,T]\times V\times V \to \mathbb{C}$ and $\mathcal{A}_\Lambda\colon\ [0,T] \to \mathcal{L}(V,V')$ be as above where $\mathcal{A}_k$ are associated with the sesquilinear forms

$$(3.8)\qquad \mathfrak{a}_k(u,v) := \frac{1}{\lambda_{k+1}-\lambda_k}\int_{\lambda_k}^{\lambda_{k+1}} \mathfrak{a}(r;u,v)\,\mathrm{d}r \quad \text{for } u,v\in V,\ k=0,1,\ldots,n.$$

Note that since $\mathfrak{a}_k$ satisfies (3.1) and (3.2), $k=0,1,\ldots,n$, we have for all $u\in V$

$$(3.9)\qquad \mathcal{A}_k u := \frac{1}{\lambda_{k+1}-\lambda_k}\int_{\lambda_k}^{\lambda_{k+1}} \mathcal{A}(r)u\,\mathrm{d}r.$$

Let $u_0\in H$ and $f\in L^2(0,T;V')$ and let $u_\Lambda \in \mathrm{MR}(V,V')$ denote the unique solution of

$$(3.10)\qquad \dot u_\Lambda(t) + \mathcal{A}_\Lambda(t)u_\Lambda(t) = f(t), \quad \text{for a.e. } t\in[0,T], \quad u_\Lambda(0)=u_0.$$

Recall that u_Λ is given explicitly by (3.5)–(3.7).

For simplicity and according to Remark 2.1, we may assume without loss of generality that $\omega=0$ in (3.2). In fact, let $u_\Lambda\in\mathrm{MR}(V,V')$ and $v_\Lambda(t) := \mathrm{e}^{-wt}u_\Lambda(t)$. Then u_Λ satisfies (3.10) if and only if v_Λ satisfies

$$(3.11)\qquad \dot v_\Lambda(t) + (\omega+\mathcal{A}_\Lambda(t))v_\Lambda(t) = \mathrm{e}^{-wt}f(t) \quad \text{for a.e. } t\in[0,T], \quad v_\Lambda(0)=u_0.$$

In the sequel $\omega=0$ will be our assumption. The following lemma is the key to the main result.

Lemma 3.1. *Let $u_0\in H$ and $f\in L^2(0,T;V')$. Let $u_\Lambda\in\mathrm{MR}(V,V')$ be the solution of* (3.10)*. Then there exists a constant $c_2>0$ independent of f,u_0 and Λ such that*

$$(3.12)\qquad \int_0^t \|u_\Lambda(s)\|_V^2\,\mathrm{d}s \leqslant c_2\Big[\int_0^t \|f(s)\|_{V'}^2\,\mathrm{d}s + \|u_0\|^2\Big]$$

for a.e. $t\in[0,T]$.

P r o o f. Since $u_\Lambda\in\mathrm{MR}(V,V')$, it follows from (2.5) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\Lambda(t)\|^2 &= 2\,\mathrm{Re}\langle \dot u_\Lambda(t), u_\Lambda(t)\rangle = 2\,\mathrm{Re}\langle f(t)-\mathcal{A}_\Lambda(t)u_\Lambda(t), u_\Lambda(t)\rangle\\ &= -2\,\mathrm{Re}\, a_\Lambda(t,u_\Lambda(t),u_\Lambda(t)) + 2\,\mathrm{Re}\langle f(t), u_\Lambda(t)\rangle\end{aligned}$$

for almost every $t \in [0,T]$. Integrating this equality on $(0,t)$, by coercivity of the form $\mathfrak{a}$ and the Cauchy-Schwartz inequality we obtain

$$\|u_\Lambda(t)\|^2 + 2\alpha \int_0^t \|u\|_\Lambda(s)_V^2 \,\mathrm{d}s \leqslant 2\int_0^t \|f(s)\|_{V'} \|u\|_\Lambda(s)_V \,\mathrm{d}s + \|u_0\|^2.$$

Inequality (3.12) follows from this estimate and the standard inequality

$$ab \leqslant \frac{1}{2}\Big(\frac{a^2}{\varepsilon} + \varepsilon b^2\Big), \quad \varepsilon > 0,\ a, b \in \mathbb{R}.$$

□

Let $|\Lambda| := \max\limits_{j=0,1,\ldots,n} (\lambda_{j+1} - \lambda_j)$ denote the mesh of the subdivision Λ of $[0,T]$. The main result of this section is

Theorem 3.2. *Let $f \in L^2(0,T;V')$ and $u_0 \in H$. Then the solution u_Λ of* (3.10) *converges weakly in* $\mathrm{MR}(V,V')$ *as $|\Lambda| \to 0$ and $u := \lim\limits_{|\Lambda|\to 0} u_\Lambda$ is the unique solution of* (1.4).

P r o o f. To prove that $\lim u_\Lambda$ exists as $|\Lambda| \to 0$, it suffices, by the compactness of bounded sets of $L^2(0,T,V)$, to show that it exists $u \in \mathrm{MR}(V,V')$ such that every convergent subsequence of u_Λ converges to u. We begin with the uniqueness.

Uniqueness: Let $u \in \mathrm{MR}(V,V')$ be a solution of (1.4) with $f = 0$ and $u(0) = 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 = 2\mathrm{Re}\langle \dot{\mathrm{u}}(\mathrm{t}), \mathrm{u}(\mathrm{t})\rangle = -2\,\mathrm{Re}\langle \mathcal{A}(\mathrm{t})\mathrm{u}(\mathrm{t}), \mathrm{u}(\mathrm{t})\rangle = -2\,\mathrm{Re}\,\mathfrak{a}(\mathrm{t}, \mathrm{u}(\mathrm{t}), \mathrm{u}(\mathrm{t})).$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 \leqslant -2\alpha\|u(t)\|_V^2$$

and since $u(0) = 0$, it follows that $u(t) = 0$ for a.e. $t \in [0,T]$.

Existence: Let $u_0 \in H$ and $f \in L^2(0,T;V')$. Let $u_\Lambda \in \mathrm{MR}(V,V')$ be the solution of (3.10). Since u_Λ is bounded in $L^2(0,T;V)$ by Lemma 3.1, we can assume (after passing to a subsequence) that $u_\Lambda \rightharpoonup u$ in $L^2(0,T;V)$ as $|\Lambda|$ goes to 0. Let now $g \in L^2(0,T;V)$. We have $\mathcal{A}_\Lambda^* g \to \mathcal{A}^* g$ in $L^2(0,T;V')$ (see [12], Lemma 2.3 and Lemma 3.1). Since

$$\int_0^T \langle \mathcal{A}_\Lambda(s)u_\Lambda(s), g(s)\rangle \,\mathrm{d}s = \int_0^T \langle u_\Lambda(s), \mathcal{A}_\Lambda^*(s)g(s)\rangle \,\mathrm{d}s,$$

it follows that

$$\int_0^T \langle \mathcal{A}_\Lambda(s)u_\Lambda(s), g(s)\rangle \,\mathrm{d}s \to \int_0^T \langle \mathcal{A}(s)u(s), g(s)\rangle \,\mathrm{d}s$$

or, in other words, $\mathcal{A}_\Lambda u_\Lambda \rightharpoonup \mathcal{A}u$ in $L^2(0,T;V')$ and so $\dot u_\Lambda$ converges weakly in $L^2(0,T;V')$ by (3.10).

Thus, letting $|\Lambda| \to 0$ in (3.10) shows that

$$\dot u(t) + \mathcal{A}(t)u(t) = f(t) \quad \text{for a.e. } t \in [0,T],$$

Since $\mathrm{MR}(V,V') \hookrightarrow C([0,T];H)$, we have also that $u_\Lambda \rightharpoonup u$ in $C([0,T];H)$ and in particular $u_\Lambda(0) \rightharpoonup u(0)$ in H, so that u satisfies (1.4). This completes the proof. $\square$

4. Invariance of convex sets

We use the same notation as in the previous sections. We consider a non-autonomous form $\mathfrak{a}\colon [0,T] \times V \times V \to \mathbb{C}$. Let $\mathcal{A}(t) \in \mathcal{L}(V,V')$ be the associated operator. In this section we give an other proof of a known invariance criterion for the non-autonomous homogeneous Cauchy-problem

$$\dot u(t) + \mathcal{A}(t)u(t) = 0 \quad \text{for a.e. } t \in [0,T], \quad u(0) = u_0. \tag{4.1}$$

Let $\mathcal{C}$ be a closed convex subset of the Hilbert space H and let $P\colon H \to \mathcal{C}$ be the orthogonal projection onto $\mathcal{C}$; i.e., for $x \in H$, Px is the unique element $x_{\mathcal{C}}$ in $\mathcal{C}$ such that

$$\mathrm{Re}(x - x_{\mathcal{C}} \,|\, y - x_{\mathcal{C}}) \leqslant 0 \quad \text{for all } y \in \mathcal{C}.$$

Recall that the closed convex set $\mathcal{C}$ is invariant for the Cauchy problem (4.1) (in the sense of [4], Definition 2.1) if for each $u_0 \in \mathcal{C}$ the solution u of (4.1) satisfies $u(t) \in \mathcal{C}$ for all $t \in [0,T]$. Recently, Arendt et al. [4] proved that $\mathcal{C}$ is invariant for the inhomogeneous Cauchy problem (1.4) provided that $PV \subset V$ and

$$\mathrm{Re}\,\mathfrak{a}(t, Pv, v - Pv) \geqslant \mathrm{Re}\langle f(t), v - Pv\rangle$$

for all $v \in V$ and for a.e. $t \in [0,T]$.

As a consequence of our approach, we obtain easily Theorem 2.2 in [4] for the homogeneous Cauchy problem from Theorem 3.2.

Theorem 4.1. *Let $\mathfrak{a}$ be a non-autonomous form on V. Let $\mathcal{C}$ be a closed convex subset of H. Then the convex set $\mathcal{C}$ is invariant for the Cauchy problem* (4.1) *provided that $PV \subset V$ and $\mathrm{Re}\,\mathfrak{a}(t, Pv, v - Pv) \geqslant 0$ for all $v \in V$ and a.e. $t \in [0,T]$.*

P r o o f. Let $u_0 \in \mathcal{C}$ and let $u_\Lambda \in \mathrm{MR}(V, V')$ be the solution of (4.1). The function u_Λ is given explicitly by (3.5)–(3.6). From Theorem 2.1 in [15] (or Theorem 2.2 in [14]), it follows easily that $u_\Lambda(t) \in \mathcal{C}$ if and only if $PV \subset V$ and

$$\operatorname{Re} \mathfrak{a}_k(Pv, v - Pv) \geqslant 0 \quad \text{for all } v \in V \text{ and } k = 0, 1, \ldots, n.$$

Recall that $\mathfrak{a}_k$ is given by (3.8). The above inequality holds if and only if $\operatorname{Re} \mathfrak{a}(t, Pv, v - Pv) \geqslant 0$ for a.e. $t \in [0, T]$. Let now u be the solution of (4.1). By Theorem 3.2 we have $u_\Lambda \rightharpoonup u$ in $\mathrm{MR}(V, V') \underset{d}{\hookrightarrow} C([0, T], H)$. The claim follows from the fact that the weak closure of the convex set $\mathcal{C}$ is equal to its norm closure. $\square$

Theorem 4.2. *Assume that the non-autonomous form $\mathfrak{a}$ is symmetric and accretive. The convex set $\mathcal{C}$ is invariant for the homogeneous Cauchy problem* (4.1) *provided that $PV \subset V$ and $\mathfrak{a}(t, Pv, Pv) \leqslant \mathfrak{a}(t, v, v)$ for a.e. $t \in [0, T]$.*

P r o o f. Let $u_\Lambda \in \mathrm{MR}(V, V')$ be the solution of (4.1). By Theorem 2.2 in [14], we have $u_\Lambda(t) \in \mathcal{C}$ if and only if $PV \subset V$ and

$$\mathfrak{a}_k(Pv, Pv) \geqslant \mathfrak{a}_k(v, v) \quad \text{for all } v \in V \text{ and } k = 0, 1, \ldots, n.$$

This inequality holds if and only if $\operatorname{Re} \mathfrak{a}(t, Pv, Pv) \geqslant \mathfrak{a}(t, v, v)$ for a.e. $t \in [0, T]$ and for all $v \in V$. The claim follows from the fact that u_Λ converge weakly in $C([0, T], H)$ to the solution of (4.1). $\square$

5. Well-posedness in H

Recall that V, H denote two separable Hilbert spaces and $\mathfrak{a}\colon [0, T] \times V \times V \to \mathbb{C}$ is a non-autonomous form introduced in the previous section. We adopt here the notation of Section 3. Assume that the form $\mathfrak{a}$ is symmetric, i.e.,

$$\mathfrak{a}(t, u, v) = \overline{\mathfrak{a}(t, v, u)}, \quad t \in [0, T],\ u, v \in V. \tag{5.1}$$

We consider the Hilbert space

$$\mathrm{MR}(V, H) := L^2(0, T; V) \cap H^1(0, T; H)$$

with the norm

$$\|u\|^2_{\mathrm{MR}(V,H)} := \|u\|^2_{L^2(0,T;V)} + \|u\|^2_{H^1(0,T;H)}.$$

Let Λ be a subdivision of $[0,T]$ and let $f \in L^2(0,T;H)$ and $u_0 \in V$. The solution u_Λ of (3.10) belongs to $\mathrm{MR}(V,H)$ and $u_\Lambda \in C([0,T],V)$. In fact, let $\mathcal{A}_k$ be given by (3.9) and let A_k be the part of $\mathcal{A}_k$ in H. Then it is not difficult to see that

$$u_{\Lambda|_{[\lambda_k,\lambda_{k+1}[}} \in \mathrm{MR}(\lambda_k,\lambda_{k+1};D(A_k),H), \quad k=0,1,2,\dots,n. \tag{5.2}$$

Note that on each interval $[\lambda_k,\lambda_{k+1}[$ the solution u_Λ coincides with the solution of the autonomous Cauchy problem

$$\dot{u}_k(t) + A_k u_k(t) = f(t) \quad \text{for a.e. } t \in (\lambda_k,\lambda_{k+1}), \quad u_k(\lambda_k) = u_{k-1}(\lambda_k) \in V$$

which belongs to $\mathrm{MR}(\lambda_k,\lambda_{k+1};D(A_k),H)$, see Section 2.

In addition we assume that $\mathfrak{a}$ is *Lipschitz continuous*, i.e., there exists a positive constant L such that

$$|\mathfrak{a}(t,u,v) - \mathfrak{a}(s,u,v)| \leqslant L|t-s| \|u\|_V \|v\|_V, \quad t,s \in [0,T], \ u,v \in V. \tag{5.3}$$

For simplicity, we assume in the following that the subdivision Λ of $[0,T]$ is uniform, i.e., $\lambda_{i+1} - \lambda_i = \lambda_{j+1} - \lambda_j$ for all $(i,j) \in \{0,1,2,\dots,n\}^2$.

Theorem 5.1 below shows that the solution u_Λ of (3.10) converges weakly in $\mathrm{MR}(V,H)$ and so the limit u, which is the solution of (1.4), belongs to the maximal regularity space $\mathrm{MR}(V,H)$. This gives another proof of Theorem 5.1 in [3] with $\mathfrak{a}$ symmetric and $B = \mathrm{Id}$.

Theorem 5.1. *Assume that $\mathfrak{a}$ is symmetric and Lipschitz continuous. Let $(f,u_0) \in L^2(0,T;H) \times V$. Then u_Λ, the solution of* (3.10), *converges weakly in $\mathrm{MR}(V,H)$ as $|\Lambda| \to 0$ and $u := \lim\limits_{|\Lambda|\to 0} u_\Lambda$ is the unique solution of* (1.4). *Moreover,*

$$\|u\|_{\mathrm{MR}(V,H)} \leqslant c[\|u_0\|_V + \|f\|_{L^2(0,T;H)}], \tag{5.4}$$

where the constant c depends merely on α, c_H, M and L.

Proof. Let $(f,u_0) \in L^2(0,T;H) \times V$. Let $u_\Lambda \in \mathrm{MR}(V,H)$ be the solution of (3.10). According to the proof of Theorem 3.2, it remains to prove that u_Λ is bounded in $\mathrm{MR}(V,H)$. We estimate first the derivative $\dot{u}_\Lambda$. Using (2.8) and (5.2) we obtain

$$\begin{aligned}\int_0^T \|\dot u_\Lambda(t)\|^2\,\mathrm{d}t &= \int_0^T \mathrm{Re}(-\mathcal{A}_\Lambda(t)u_\Lambda(t)\mid \dot u_\Lambda(t))\,\mathrm{d}t + \int_0^T \mathrm{Re}(f(t)\mid \dot u_\Lambda(t))\,\mathrm{d}t\\ &= \sum_{k=0}^{n-1}\int_{\lambda_k}^{\lambda_{k+1}} \mathrm{Re}(-\mathcal{A}_\Lambda(t)u_\Lambda(t)\mid \dot u_\Lambda(t))\,\mathrm{d}t + \int_0^T \mathrm{Re}(f(t)\mid \dot u_\Lambda(t))\,\mathrm{d}t\\ &= \sum_{k=0}^{n-1}\int_{\lambda_k}^{\lambda_{k+1}} \mathrm{Re}(-\mathcal{A}_k u_\Lambda(t)\mid \dot u_\Lambda(t))\,\mathrm{d}t + \int_0^T \mathrm{Re}(f(t)\mid \dot u_\Lambda(t))\,\mathrm{d}t\\ &= -\sum_{k=0}^{n-1}\int_{\lambda_k}^{\lambda_{k+1}} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{a}_k(u_\Lambda(t))\,\mathrm{d}t + \int_0^T \mathrm{Re}(f(t)\mid \dot u_\Lambda(t))\,\mathrm{d}t.\end{aligned}$$

For the first term on the right-hand side of the above equality we have

$$\begin{aligned}-\sum_{k=0}^{n-1}\int_{\lambda_k}^{\lambda_{k+1}} \frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{a}_k(u_\Lambda(t))\,\mathrm{d}t &= -\sum_{k=0}^{n-1}(\mathfrak{a}_k(u_\Lambda(\lambda_{k+1})) - \mathfrak{a}_k(u_\Lambda(\lambda_k)))\\ &= -\Big(\sum_{k=0}^{n-1}\mathfrak{a}_k(u_\Lambda(\lambda_{k+1})) - \sum_{k=-1}^{n-2}\mathfrak{a}_{k+1}(u_\Lambda(\lambda_{k+1}))\Big)\\ &= -\sum_{k=0}^{n-2}(\mathfrak{a}_k(u_\Lambda(\lambda_{k+1})) - \mathfrak{a}_{k+1}(u_\Lambda(\lambda_{k+1}))) - \mathfrak{a}_{n-1}(u_\Lambda(\lambda_n)) + \mathfrak{a}_0(u_\Lambda(0))\\ &\leqslant -\sum_{k=0}^{n-2}(\mathfrak{a}_k(u_\Lambda(\lambda_{k+1})) - \mathfrak{a}_{k+1}(u_\Lambda(\lambda_{k+1}))) + M\|u_\Lambda(0)\|_V^2.\end{aligned}$$

Now, using integration by substitution and Lipschitz continuity of $\mathfrak{a}$ we obtain

$$|\mathfrak{a}_k(u_\Lambda(\lambda_{k+1})) - \mathfrak{a}_{k+1}(u_\Lambda(\lambda_{k+1}))| \leqslant L(\lambda_{k+1}-\lambda_k)\|u_\Lambda(\lambda_{k+1}))\|_V^2 \quad \text{for every } k = 0,1,\ldots,n-2. \tag{5.5}$$

Let $k = 0,1,2,\ldots,n-2$ and $t_k \in [\lambda_k, \lambda_{k+1}[$ be arbitrary. Then $u_{\Lambda|_{[t_k,\lambda_{k+1}[}}$ belongs to $\mathrm{MR}(t_k, \lambda_{k+1}; D(A_k), H)$ and

$$\|u_\Lambda(\lambda_{k+1})\|_V^2 \leqslant c[\|u_\Lambda(t_k)\|_V^2 + \|f\|_{L^2(t_k,\lambda_{k+1};H)}^2] \tag{5.6}$$

where the constant c depends only on M, ω, α, c_H and T (see Lemma 2.2). Inserting (5.6) into (5.5) we obtain then for every $k = 0,1,\ldots,n-2$

$$\begin{aligned}&|\mathfrak{a}_k(u_\Lambda(\lambda_{k+1})) - \mathfrak{a}_{k+1}(u_\Lambda(\lambda_{k+1}))|\\ &\qquad\leqslant c(\lambda_{k+1}-\lambda_k)\|u_\Lambda(t_k)\|_V^2 + c(\lambda_{k+1}-\lambda_k)\|f\|_{L^2(0,T;H)}^2\\ &\qquad\leqslant c\int_{\lambda_k}^{\lambda_{k+1}} \|u_\Lambda(s)\|_V^2\,\mathrm{d}s + c(\lambda_{k+1}-\lambda_k)\|f\|_{L^2(0,T;H)}^2.\end{aligned}$$

For the last inequality, t_k is chosen such that

$$(\lambda_{k+1} - \lambda_k)\|u_\Lambda(t_k)\|_V^2 = \int_{\lambda_k}^{\lambda_{k+1}} \|u_\Lambda(s)\|_V^2 \, \mathrm{d}s$$

using the mean value theorem and the fact that $t_k \in [\lambda_k, \lambda_{k+1}[$ is arbitrary. Thus

$$\sum_{k=0}^{n-2} |\mathfrak{a}_k(u_\Lambda(\lambda_{k+1})) - a_{k+1}(u_\Lambda(\lambda_{k+1}))| \leqslant c[\|u_\Lambda\|^2_{L^2(0,T;V)} + \|f\|^2_{L^2(0,T;H)}] \tag{5.7}$$

for some $c = c(M, \omega, \alpha, c_H, T, L)$ (possibly different from the previous one). It follows that

$$\int_0^T \|\dot{u}_\Lambda(t)\|^2 \, \mathrm{d}t \leqslant c\big[\|u_\Lambda\|^2_{L^2(0,T;V)} + \|f\|^2_{L^2(0,T;H)}\big] + \int_0^T \mathrm{Re}(f(t) \mid \dot{u}_\Lambda(t)) \, \mathrm{d}t + M\|u_0\|_V^2.$$

Finally, from this inequality, the estimate (3.12) in Lemma 3.1, the Cauchy-Schwarz and the Young inequalities applied to the third term on the right-hand side, it follows that there is a constant $c = c(M, \omega, \alpha, c_H, T, L)$ such that

$$\int_0^T \|\dot{u}_\Lambda(t)\|^2 \, \mathrm{d}t + \int_0^T \|u_\Lambda(t)\|_V^2 \, \mathrm{d}t \leqslant c(\|u_0\|_V^2 + \|f\|^2_{L^2(0,T;H)}).$$

This completes the proof. $\square$

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Authors' addresses: Ahmed Sani, Department of Mathematics, University Ibn Zohr, Faculty of Sciences, BP 32/S, 80000 Agadir, Morocco, e-mail: `ahmedsani82@gmail.com`; Hafida Laasri, Fachbereich C-Mathematik und Naturwissenschaften, University of Wuppertal, Gaußstraße 20, 42097 Wuppertal, Germany, e-mail: `laasri@uni-wuppertal.de`.