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NON SUPERCYCLIC SUBSETS OF LINEAR ISOMETRIES ON BANACH SPACES OF ANALYTIC FUNCTIONS

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Abstract. Let X be a Banach space of analytic functions on the open unit disk and Γ a subset of linear isometries on X. Sufficient conditions are given for non-supercyclicity of Γ. In particular, we show that the semigroup of linear isometries on the spaces S^p $(p > 1)$, the little Bloch space, and the group of surjective linear isometries on the big Bloch space are not supercyclic. Also, we observe that the groups of all surjective linear isometries on the Hardy space H^p or the Bergman space L^p_a $(1 < p < \infty,\ p \neq 2)$ are not supercyclic.

Keywords: supercyclicity; hypercyclic operator; semigroup; isometry

MSC 2010: 47A16, 47B33, 47B38

1. Introduction and preliminary results

Let X be a Banach space over the field of complex numbers and $\dim X > 1$. By $B(X)$ we mean the set of bounded linear operators on X. A set $\Gamma \subseteq B(X)$ is hypercyclic (supercyclic) if there exists a vector $x \in X$ such that $O(x, \Gamma) = \{Tx\colon T \in \Gamma\}$ $(\mathbb{C} \cdot O(x, \Gamma) = \{\lambda Tx\colon T \in \Gamma,\ \lambda \in \mathbb{C}\})$ is a dense subset of X. An operator $T \in B(X)$ is called hypercyclic (supercyclic) if the semigroup $\Gamma = \{T^n\colon n \in \mathbb{N}_0\}$ $(\mathbb{C} \cdot \Gamma = \{\lambda T^n\colon n \in \mathbb{N}_0,\ \lambda \in \mathbb{C}\})$ is hypercyclic (supercyclic). Here $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.

In 1969, Rolewicz [19] showed that any scalar multiple λB of the unilateral backward shift B is hypercyclic on ℓ_p $(1 \leqslant p < \infty)$ whenever $|\lambda| > 1$. Some researchers believe that the linear dynamics is a branch of functional analysis, which was born in 1982 with the Toronto Ph.D. thesis of Kitai [15]. For decades, hypercyclic and supercyclic operators have been the focus of much works in linear dynamics. The

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reader can see [2] to get more information about hypercyclic and supercyclic operators. It was proved by Ansari [1] that if T is hypercyclic then so is T^n for any $n \geqslant 2$, with the same hypercyclic vectors. Later, this fact was proved in a general form by Bourdon and Feldman [5] who say that the T-orbits of each $x \in X$ is either dense or nowhere-dense in X.

León-Saavedra and Müller [16] proved that if the semigroup $\Gamma = \{\lambda T^n \colon n \in \mathbb{N}_0,\ |\lambda| = 1\}$ is hypercyclic then so is the operator T. In 2007, Conejero, Müller and Peris [6] showed that if the C_0-semigroup $\Gamma = \{(T_t)_t \colon t \geqslant 0\}$ is hypercyclic then each individual operator T_{t_0} $(t_0 > 0)$ is also hypercyclic. Recall that a one-parameter family of operators $(T_t)_{t\geqslant 0}$ is a C_0-semigroup (or strongly continuous semigroup) if $T_0 = I$, $T_{t+s} = T_t T_s$ for any $t, s \geqslant 0$ and $\lim\limits_{t\to s} T_t(x) = T_s(x)$ for each $s \geqslant 0$ and all $x \in X$.

The set of all unitary operators on a Hilbert space is supercyclic, see [9], page 183. Also, there are examples of supercyclic group of isometries on Banach spaces. The group of all isometries on $L^p(\mu)$ $(1 \leqslant p < \infty)$ where μ is a homogeneous measure is supercyclic, see [12]. Moreover, for $1 \leqslant p < \infty$, the group of all isometries on the Banach space $L^p(X, \mu)$ is also supercyclic, where X is the disjoint union of an uncountable family of copies of the interval $[0, 1]$ and μ is a certain measure on X, see [3]. In this paper, we study the supercyclicity behavior of sets of linear isometries on some spaces of analytic functions on the unit disc $\mathbb{D}$. In Section 2, we show that the semigroup of linear isometries on the spaces S^p $(p > 1)$ and the little Bloch space $\mathcal{B}_0$ is not supercyclic and neither is the group of surjective linear isometries on the big Bloch space $\mathcal{B}$. In Section 3, we prove that the groups of all surjective linear isometries on the Hardy space H^p and the Bergman space L^p_a $(1 < p < \infty,\ p \neq 2)$ are not supercyclic.

2. Banach spaces of analytic functions

For each Banach space X with the norm $\|\cdot\|$ there is a norm $|||\cdot|||$ on X equivalent to $\|\cdot\|$ such that the set of all surjective linear isometries on $(X, |||\cdot|||)$ is $\{\lambda I \colon |\lambda| = 1\}$ where I is the identity operator on X, see [14]. Thus, there is a Banach space on which the group of surjective linear isometries is not supercyclic. On the other hand, the set of all unitary operators on a Hilbert space is supercyclic. Therefore, if H is a Hilbert space with the norm $\|\cdot\|$ induced by the inner product of H, then there is a norm $|||\cdot|||$ on H equivalent to $\|\cdot\|$ such that $(H, |||\cdot|||)$ is not a Hilbert space.

By a Banach space of analytic functions on $\mathbb{D}$, we mean the Banach space X consisting of functions that are analytic on $\mathbb{D}$ such that for each $z \in \mathbb{D}$ the functional $e_z \colon X \to \mathbb{C}$ of evaluation at z given by $e_z(f) = f(z)$ is bounded and there exists a sequence of unit vectors in X converging pointwise to zero.

Some classical examples of such spaces run as follows:

1. The Hardy space H^p, $1 \leqslant p < \infty$, consists of those functions f analytic in $\mathbb{D}$ for which $\|f\|_{H^p}^p = \sup\limits_{0<r<1} \int_{\mathbb{T}} |f(rz)|^p \,\mathrm{d}m(z) < \infty$ where $\mathrm{d}m$ is the normalized arc length measure on $\mathbb{T}$, where $\mathbb{T} = \{z \in \mathbb{C}\colon\ |z| = 1\}$.

2. The space of all bounded analytic functions on $\mathbb{D}$, H^∞, with the norm $\|f\|_\infty = \sup\limits_{\mathbb{D}} |f(z)|$.

3. The weighted Bergman space A^p_α $(1 \leqslant p < \infty,\ \alpha > -1)$ is defined as the space of all f in $H(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} = \left(\int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha (\alpha + 1) \,\mathrm{d}A(z)\right)^{1/p} < \infty,$$

where $\mathrm{d}A$ is the normalized Lebesgue area measure on $\mathbb{D}$.

4. The weighted Dirichlet-type space $\mathcal{D}^p_\alpha$ $(1 \leqslant p < \infty,\ -1 < \alpha < \infty)$ is the space of all $f \in H(\mathbb{D})$ such that $f' \in A^p_\alpha$, equipped with the norm

$$\|f\|_{\mathcal{D}^p_\alpha} = |f(0)| + \|f'\|_{A^p_\alpha} < \infty.$$

5. The big Bloch space $\mathcal{B}$ which is the set of all analytic functions f on $\mathbb{D}$ such that $f(0) = 0$ and

$$\|f\| = \sup\{|f'(z)|(1 - |z^2|)\colon\ z \in \mathbb{D}\} < \infty,$$

and the little Bloch space $\mathcal{B}_0$, the subspace of $\mathcal{B}$ spanned by the polynomials.

6. The space $\mathcal{D}$, the set of analytic functions f on $\mathbb{D}$ such that $\|f\| = \sup\{|f(z)|(1-|z|^2)\colon\ z \in \mathbb{D}\} < \infty$, and the subspace $\mathcal{D}_0$ of $\mathcal{D}$, which is $\mathcal{D}_0 = \{f'\colon\ f \in \mathcal{B}_0\}$.

7. The space S^p, $1 \leqslant p < \infty$, consisting of all analytic functions f on $\mathbb{D}$ for which

$$\|f\| = \|f\|_\infty + \|f'\|_{H^p} < \infty.$$

8. The space $H^\infty_{\nu_p}(\mathbb{D})$, $(p > 0)$ with the standard weights $\nu_p(z) = (1 - |z|^2)^p$, consisting of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{\nu_p} = \sup_{z \in \mathbb{D}} \nu_p(z)|f(z)| < \infty.$$

9. The analytic Besov space B^p, $1 < p < \infty$, is defined as the set of all analytic functions in the disk such that

$$\|f\|_{B^p}^p = |f(0)|^p + (p - 1) \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{(p-2)} \,\mathrm{d}A(z) < \infty.$$

We note that when $X = H^p$ $(1 \leqslant p \leqslant \infty)$, $\|z^n\| = 1$ for $n \geqslant 0$. Also if X is the Bloch space, $\|z^n\|_X = (2n/(n+1)) \cdot ((n-1)/(n+1))^{(n-1)/2}$ which converges to $2/\mathrm{e}$ as $n \to \infty$. Moreover, $\|z^n\|_{\mathcal{D}} = 2/(\mathrm{e}^2(n+2))$, $\|z^n\|_{S^p} = 1 + n^p$ and $\|z^n\|_{A^p_\alpha} = (\alpha + 1)\beta(np/2+1, \alpha+1)$, where $\beta(x,y)$ is the beta function given by $\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t$ for $x > 0$ and $y > 0$. Furthermore, $\|z^n\|_{\mathcal{D}^p_\alpha} = n^p(\alpha+1)\beta((n-1)p/2+1, \alpha+1)$, $\|z^n\|^p_{B^p} = n^p(p-1)\beta((n-1)p/2+1, p-1)$ and $\|z^n\|_{\nu_p} = (1-n/(n+2p))^p(n/(n+2p))^{n/2}$. Since $\beta(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ where $\Gamma(\cdot)$ is the gamma function and limit asymptotic approximation is $\lim\limits_{\nu\to\infty} \Gamma(\nu+\alpha)/(\Gamma(\nu+\beta)\nu^{\alpha-\beta}) = 1$ (see [8], page 62), it is easy to verify that if X is any of the above spaces then

$$\lim_{n\to\infty} \frac{|z|^n}{\|z^n\|_X} = 0, \quad z \in \mathbb{D}.$$

We remark that if X is a separable reflexive Banach space consisting of functions that are analytic on $\mathbb{D}$ such that for each $z \in \mathbb{D}$ the functional e_z is bounded then there exists a sequence of unit vectors in X weakly converging to zero. Indeed, X^* is separable and $\mathrm{ball}(X)$ is weakly compact, so it is weakly metrizable. On the other hand, the weak closure of the unit sphere of X is $\mathrm{ball}(X)$; thus there exists a sequence $(x_n)_n$ of unit vectors which weakly converges to zero. Hence such X is a Banach space of analytic functions.

Definition 2.1. Let X be a Banach space of analytic functions on $\mathbb{D}$. A sequence $(T_n)_n$ in $B(X)$ is said to converge pointwise evaluation to $T \in B(X)$ if $\lim\limits_{n\to\infty} (T_n f)(z) = (Tf)(z)$ for all $f \in X$ and all $z \in \mathbb{D}$.

Theorem 2.2. *Let X be a Banach space of analytic functions on $\mathbb{D}$ and let $\Gamma \subset B(X)$ be bounded away from 0 and ∞ at each point of X, i.e., for each $f \in X$ there exist two positive constants c_f and d_f such that $c_f\|f\| \leqslant \|Tf\| \leqslant d_f\|f\|$ for all $T \in \Gamma$. If each sequence $(T_n)_n$ in Γ has a subsequence which converges pointwise evaluation to some $T \in \Gamma$ then Γ is not supercyclic.*

Proof. Assume, on the contrary that $f \in X$ is a unit supercyclic vector for Γ. So, for any $g \in X$ there is a sequence $(\alpha_n)_n$ in $\mathbb{C}$ and a sequence $(T_n)_n$ in Γ such that $\lim\limits_{n\to\infty} \alpha_n T_n(f) = g$. Thus, $|\alpha_n| \leqslant (\|g\|+1)/c_f$ for sufficiently large n, which implies that $\lim\limits_k \alpha_{n_k} = \alpha$ exists for some subsequence $(\alpha_{n_k})_k$. Now, $(T_{n_k})_k$ has a subsequence which converges pointwise evaluation to some $T \in \Gamma$ and, without loss of generality, we may assume that this subsequence is $(T_{n_k})_k$. So $g(z) = \lim\limits_k \alpha_{n_k}(T_{n_k}f)(z) = \alpha(Tf)(z)$, for all z in $\mathbb{D}$, which implies that $X = \{\alpha Tf\colon \alpha \in \mathbb{C},\ T \in \Gamma\}$. Let $\{f_n\}_n$ be the sequence of unit vectors converging pointwise to zero. Hence for each n there is an operator $S_n \in \Gamma$ and $\lambda_n \in \mathbb{C}$ such that

$$\lambda_n S_n(f) = f_n.$$

It follows that there exist subsequences $(\lambda_{n_k})_k$ and $(S_{n_k})_k$ such that $\lambda_{n_k} \to \lambda$, for some nonzero constant λ and $S_{n_k} \to S \in \Gamma$ pointwise evaluation. Hence $\lambda(Sf)(z) = \lim_k \lambda_{n_k}(S_{n_k} f)(z) = \lim_k f_{n_k}(z) = 0$ for all z in $\mathbb{D}$. Thus, $\|f\| = 0$, which is impossible. $\square$

In the sequel, we need the following lemmas.

Lemma 2.3. *For $r \geqslant 0$ let $\mathbb{D}_r = \{z\colon |z| \leqslant r\}$. Suppose that $|\lambda_n| = 1$ and $a_n \in \mathbb{D}_r$ for all n, where $0 \leqslant r < 1$. If $\varphi_n(z) = \lambda_n(z - a_n)/(1 - \overline{a}_n z)$ converges pointwise to some function φ on $\mathbb{D}_r$, then $\varphi(z) = \lambda(z - a)/(1 - \overline{a}z)$ where $\lambda = \lim\limits_{k\to\infty} \lambda_{n_k}$ for some subsequence $(\lambda_{n_k})_k$, $a = -\varphi(0)/\lambda$ and $|a| \leqslant r$. In particular, when φ_n, $n = 1, 2, 3, \ldots$ are rotations then φ is also a rotation.*

P r o o f. Since $\lim\limits_{n\to\infty}(-\lambda_n a_n) = \lim\limits_{n\to\infty} \varphi_n(0) = \varphi(0)$, $|\varphi(0)| \leqslant r$, the sequence $(a_n)_n$ has a subsequence $(a_{n_k})_k$ which converges to some point a. On the other hand, the sequence $(\lambda_{n_k})_k$ has also a convergent subsequence which is denoted, without loss of generality, again by $(\lambda_{n_k})_k$. So we have $-\lambda_{n_k} a_{n_k} \to -a\lambda$ for some λ in $\mathbb{T}$. Thus, $-a\lambda = \varphi(0)$ and clearly $|a| \leqslant r$. Hence $\varphi(z) = \lim\limits_{k\to\infty} \varphi_{n_k}(z) = \lim\limits_{k\to\infty} \lambda_{n_k}(z - a_{n_k})/(1 - \overline{a}_{n_k} z) = \lambda(z - a)/(1 - \overline{a}z)$. $\square$

Recall that $H(\mathbb{D})$ is the space of all analytic functions on $\mathbb{D}$ for which $f_n \to f$ in $H(\mathbb{D})$ if and only if f_n converges to f uniformly on all compact subsets of $\mathbb{D}$. Moreover, a family $\mathcal{F} \subseteq H(\mathbb{D})$ is normal if each sequence in $\mathcal{F}$ has a subsequence which converges to a function f in $H(\mathbb{D})$.

Lemma 2.4. *The set of all automorphisms of the unit disk is normal in $H(\mathbb{D})$.*

P r o o f. Denote the set of all automorphisms of $\mathbb{D}$ by $\mathrm{Aut}(\mathbb{D})$. Let $z_0 \in \mathbb{D}$. Since the set $\{\varphi(z_0)\colon \varphi \in \mathrm{Aut}(\mathbb{D})\}$ is bounded, it has a compact closure in $\mathbb{C}$. We show that $\mathrm{Aut}(\mathbb{D})$ is equicontinuous at z_0. To see this, let $\varepsilon > 0$ and let $\varphi(z) = \lambda(z - a)/(1 - \overline{a}z)$ be an arbitrary element in $\mathrm{Aut}(\mathbb{D})$. Let $|z - z_0| < 1 - |z_0|$. Then

$$\begin{aligned}
|\varphi(z) - \varphi(z_0)| &= \left|\frac{(z-a)(1-\overline{a}z_0) - (z_0 - a)(1 - \overline{a}z)}{(1-\overline{a}z)(1-\overline{a}z_0)}\right| \\
&= \left|\frac{(z - z_0)(1 - |a|^2)}{(1-\overline{a}z)(1-\overline{a}z_0)}\right| \leqslant \frac{|z - z_0|}{(1 - |a||z|)(1 - |a||z_0|)} \\
&\leqslant \frac{|z - z_0|}{(1 - |z|)(1 - |z_0|)} \leqslant \frac{|z - z_0|}{(1 - |z - z_0| - |z_0|)(1 - |z_0|)}.
\end{aligned}$$

It is easy to see that

$$\frac{|z - z_0|}{(1 - |z - z_0| - |z_0|)(1 - |z_0|)} < \varepsilon$$

if and only if

$$|z - z_0| < \frac{\varepsilon(1 - |z_0|)^2}{1 + \varepsilon(1 - |z_0|)}.$$

Since $\varepsilon(1 - |z_0|)^2/(1 + \varepsilon(1 - |z_0|)) < 1 - |z_0|$ it is enough to consider $|z - z_0| < \delta = \varepsilon(1 - |z_0|)^2/(1 + \varepsilon(1 - |z_0|))$. Thus, $\mathrm{Aut}(\mathbb{D})$ is normal, by applying the Arzelà-Ascoli Theorem [7]. $\square$

For $\varphi \in \mathrm{Aut}(\mathbb{D})$ we define a bounded linear functional Z_φ on X by $Z_\varphi f = f(\varphi(0))$ and a composition operator C_φ on X by $C_\varphi f = f \circ \varphi$ for each function f in X.

Theorem 2.5. *Let X be a Banach space of analytic functions on $\mathbb{D}$. Suppose that $\Gamma = \{\lambda((\varphi')^\alpha C_\varphi + \beta Z_\varphi)\colon \varphi \in \mathrm{Aut}(\mathbb{D}),\ |\varphi(0)| \leqslant r,\ |\lambda| = 1\}$ for constants $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}$ and $0 \leqslant r < 1$. If Γ is a subset of linear isometries on X, then Γ is not supercyclic.*

Proof. Suppose that $(\lambda_n(({\varphi_n}')^\alpha C_{\varphi_n} + \beta Z_{\varphi_n}))_n$ is a sequence in Γ. Without loss of generality we assume that the sequence $(\lambda_n)_n$ converges to a unimodular constant λ. By Lemmas 2.3 and 2.4 there is a subsequence $(\varphi_{n_k})_k$ and an automorphism φ with $|\varphi(0)| \leqslant r$ such that $\varphi_{n_k} \to \varphi$ in $H(\mathbb{D})$. If $f \in X$ then $\lim\limits_{k\to\infty} \lambda_{n_k}(({\varphi_{n_k}}')^\alpha C_{\varphi_{n_k}} f + \beta Z_{\varphi_{n_k}} f)(z) = \lambda((\varphi')^\alpha C_\varphi f + \beta Z_\varphi f)(z)$ for all $z \in \mathbb{D}$. Hence $\lambda_{n_k}(({\varphi_{n_k}}')^\alpha C_{\varphi_{n_k}} + \beta Z_{\varphi_{n_k}})$ converges pointwise evaluation to $\lambda((\varphi')^\alpha C_\varphi + \beta Z_\varphi) \in \Gamma$. Now the result follows from Theorem 2.2. $\square$

Take $\alpha = \beta = r = 0$ in Theorem 2.5. Since the set of all isometries on S^p is of the form $\Gamma = \{\lambda C_\varphi\colon \varphi$ is a rotation, $|\lambda| = 1\}$, see [18], we have

Corollary 2.6. *The semigroup of linear isometries on the space S^p, $p > 1$ is not supercyclic.*

Corollary 2.7. *The semigroup of linear isometries on the little Bloch space $\mathcal{B}_0$ and the group of surjective linear isometries on the big Bloch space $\mathcal{B}$ are not supercyclic.*

Proof. Let $\alpha = 0$, $\beta = -1$ and $r = 0$ (note that every function in the Bloch space vanishes at zero) in Theorem 2.5. Since the set of all linear isometries on $\mathcal{B}_0$ and the set of all surjective linear isometries on $\mathcal{B}$ has the form $\Gamma = \{\lambda(C_\varphi - Z_\varphi)\colon$ φ is a rotation, $|\lambda| = 1\}$, see [10], the result follows. $\square$

Corollary 2.8. *The semigroup* $\Gamma = \{C_\varphi \colon \varphi \in \operatorname{Aut}(\mathbb{D})$ *and* C_φ *is an isometry*$\}$ *is not supercyclic on the weighted Bergman space* A^p_α $(1 \leqslant p < \infty)$, *the analytic Besov space* B^p $(2 < p < \infty)$, *the weighted Dirichlet-type space* $\mathcal{D}^p_\alpha$ $(1 \leqslant p < \infty, -1 < \alpha < \infty)$ *and the space* $H^\infty_{\nu_p}(\mathbb{D})$ $(p > 0)$.

Proof. For all of these spaces $\Gamma = \{C_\varphi \colon \varphi \text{ is a rotation}\}$. The relevant references are respectively, see [17], Theorems 1.3 and 1.4, [11], Corollary 2.3, and [4], Corollary 12. Thus, the result follows from Theorem 2.5. $\square$

Corollary 2.9. *Let* X *be the space* H^1 *or* L^1_a *and let* Γ *be the set of all surjective linear isometries on* X *such that* $0 < c < |(T1)(0)|$ *for all* $T \in \Gamma$ *and some constant* c. *Then* Γ *is not supercyclic.*

Proof. Regarding Theorem 2.5, put $\alpha = 1$, $\beta = 0$ for H^1 and $\alpha = 2$, $\beta = 0$ for L^1_a, see [10], page 88, and [13]. We see that $|(T1)(0)| = (1 - |\varphi(0)|^2)^\alpha > c$; thus, $|\varphi(0)| < \sqrt{1 - c^{1/\alpha}}$. $\square$

Corollary 2.10. *Let* Γ *be the set of all surjective linear isometries on* H^∞ *such that* $|T(\mathrm{id})(0)| \leqslant c < 1$ *for all* $T \in \Gamma$ *and some constant* c *where* $\mathrm{id}(z) = z$ *for all* $z \in \mathbb{D}$. *Then* Γ *is not supercyclic.*

Proof. If $T \in \Gamma$ then $T = \lambda C_\varphi$ for some $|\lambda| = 1$ and some automorphism φ, see [10], page 80. Thus, $|\varphi(0)| = |T(\mathrm{id})(0)| \leqslant c$. Now apply Theorem 2.5. $\square$

3. Reflexive Banach spaces

For the algebra $B(X)$ of all bounded operators on a Banach space X, the weak operator topology (WOT) is the one in which a net T_α converges to T if and only if $T_\alpha(x) \to T(x)$ weakly for all $x \in X$. By the notation $x_\alpha \xrightarrow{w} x$ we mean x_α converges to x weakly.

Theorem 3.1. *Let* X *be an infinite dimensional reflexive Banach space and* Γ *a subset of* $B(X)$ *that is bounded and bounded away from* 0 *at each point of* X. *If* Γ *is* WOT*-closed in* $B(X)$ *then* Γ *is not supercyclic.*

Proof. Suppose that $x_0 \in X$ is a unit supercyclic vector for Γ. So for any $x \in X$ there is a sequence $(\alpha_n)_n$ in $\mathbb{C}$ and a sequence $(T_n)_n$ in Γ such that $\lim\limits_{n\to\infty} \alpha_n T_n(x_0) = x$. Since Γ is bounded away from 0 there is a positive constant c_x such that $\|Tx\| \geqslant c_x\|x\|$ for all $T \in \Gamma$. Therefore, $|\alpha_n| \leqslant (\|x\| + 1)/c_{x_0}$ for sufficiently large n, which implies that (by passing to a subsequence) $\lim\limits_n \alpha_n = \alpha$

exists. Since the unit ball of $B(X)$ is WOT-compact, $(T_n)_n$ has a WOT-convergent subnet $(T_{n_j})_j$ with limit $T \in \Gamma$; i.e., $T_{n_j}(x) \overset{w}{\to} T(x)$ for all $x \in X$. In particular, $\alpha_{n_j} T_{n_j}(x_0) \overset{w}{\to} \alpha T(x_0)$. On the other hand, $\alpha_{n_j} T_{n_j}(x_0) \overset{w}{\to} x$, which implies that $X = \{\alpha T(x_0)\colon\ \alpha \in \mathbb{C},\ T \in \Gamma\}$. Since the weak closure of the unit sphere of X is ball(X), there exists a net $(x_\alpha)_\alpha$ of unit vectors weakly converging to zero. Therefore, for each α there is an operator $S_\alpha \in \Gamma$ and $\lambda_\alpha \in \mathbb{C}$ such that $\lambda_\alpha S_\alpha(x_0) = x_\alpha$. It follows that there exist subnets $(\lambda_{\alpha_i})_{\alpha_i}$ and $(S_{\alpha_i})_{\alpha_i}$ such that $\lambda_{\alpha_i} \to \lambda$ for some nonzero constant λ, and $S_{\alpha_i} \overset{\text{WOT}}{\longrightarrow} S \in \Gamma$. Hence, in the weak topology, $\lambda S(x_0) = \lim\limits_i \lambda_{\alpha_i} S_{\alpha_i}(x_0) = \lim\limits_i x_{\alpha_i} = 0$. Thus, $\|x_0\| = 0$, which is impossible. $\square$

Proposition 3.2. *Let X be the Hardy space H^p or the Bergman space L^p_a ($1 < p < \infty$, $p \neq 2$). If Γ is the group of all surjective linear isometries on X then the* WOT*-closure of Γ is* $\Gamma \cup \{0\}$.

P r o o f. If $T_n \in \Gamma$ then $T_n = \mu_n(\varphi_n')^\alpha C_{\varphi_n}$ where $|\mu_n| = 1$, $\alpha = 1/p$ or $\alpha = 2/p$ and $\varphi_n(z) = \lambda_n(z - a_n)/(1 - \overline{a}_n z)$ (see [10], [13]). Suppose that $T_n \overset{\text{WOT}}{\longrightarrow} T$. Now we have two cases: if $|\varphi_n(0)| \leqslant r < 1$ for all n, then by applying Lemma 2.3 we see that $T = \mu(\varphi')^\alpha C_\varphi$ where $\varphi \in \mathrm{Aut}(\mathbb{D})$ and $|\mu| = 1$. Hence $T \in \Gamma$. Otherwise, by passing to a subsequence if necessary, we can assume that $(\varphi_n(0))_n$ converges to some point on the unit circle. An easy computation shows that $(\varphi_n)_n$ converges pointwise to some point with unimodular 1 and also $(\varphi_n')_n$ converges pointwise to zero. This, in turn, implies that $T_n(p(z)) \to 0$ for every polynomial p. Since the polynomials are dense in X we conclude that $T = 0$. $\square$

Proposition 3.3. *Let X be the Hardy space H^p or the Bergman space L^p_a* ($1 <$ *$p < \infty$, $p \neq 2$). If Γ is the group of all surjective linear isometries on X then Γ is not supercyclic.*

P r o o f. Assume, on the contrary, that $f \in X$ is a unit supercyclic vector for Γ. There are sequences $(\mu_n)_n$ in $\mathbb{C}$ and $(S_n)_n$ in Γ such that $\lim\limits_{n\to\infty} \mu_n S_n(f) = \mathrm{id}$. Thus, $\lim\limits_{n\to\infty} |\mu_n| = 1$, which implies that $\lim\limits_k \mu_{n_k} = \mu$ exists for some subsequence $(\mu_{n_k})_k$. Now, $(S_n)_n$ has a subsequence which converges (WOT) to some $S \in \Gamma$ and, without loss of generality, we denote this subsequence by $(S_{n_k})_k$. So $\mu_{n_k} S_{n_k} \overset{\text{WOT}}{\longrightarrow} \mu S$, which implies that $(\mu_{n_k} S_{n_k})(f) \overset{w}{\to} \mu S(f)$. Hence $\mu S(f) = \mathrm{id}$. This gives that $f = \mu^{-1} S^{-1}(\mathrm{id}) = \mu^{-1}(\varphi')^\alpha C_\varphi(\mathrm{id})$ where $\varphi(z) = \lambda(z - a)/(1 - \overline{a}z)$ and $\alpha = 1/p$ or $\alpha = 2/p$, which implies that $f(a) = 0$. In a similar way, for the constant function $1 \in X$, there is a scalar β and an operator $T \in \Gamma$ satisfying $\beta T f = 1$. Thus, $f = \beta^{-1}(\psi')^\alpha C_\psi(1) = \beta^{-1}(\psi')^\alpha = \beta^{-1}(\eta(1 - |b|^2)/(1 - \overline{b}z)^2)^\alpha$ where $\psi(z) = \eta(z - b)/(1 - \overline{b}z)$. This equation gives $f(a) \neq 0$, which is a contradiction. $\square$

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