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NEW CHARACTERIZATIONS FOR WEIGHTED COMPOSITION OPERATOR FROM ZYGMUND TYPE SPACES TO BLOCH TYPE SPACES

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Abstract. Let u be a holomorphic function and φ a holomorphic self-map of the open unit disk $\mathbb{D}$ in the complex plane. We provide new characterizations for the boundedness of the weighted composition operators uC_φ from Zygmund type spaces to Bloch type spaces in $\mathbb{D}$ in terms of u, φ, their derivatives, and φ^n, the n-th power of φ. Moreover, we obtain some similar estimates for the essential norms of the operators uC_φ, from which sufficient and necessary conditions of compactness of uC_φ follows immediately.

Keywords: weighted composition operator; Zygmund type space; Bloch type space; essential norm

MSC 2010: 47B38, 26A24, 30H30, 47B33

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ denote the space of all functions analytic on $\mathbb{D}$ and $S(\mathbb{D})$ the collection of all holomorphic self-maps of $\mathbb{D}$. The composition operator C_φ induced by $\varphi \in S(\mathbb{D})$ is defined on $H(\mathbb{D})$ by $C_\varphi(f) = f \circ \varphi$ for any $f \in H(\mathbb{D})$. This operator has been well studied for several decades, the two books [7], [21] and recent papers [1], [2], [3], [4], [5], [27], [29], [32] are good sources for information on the developments in the theory of composition operators.

For $u \in H(\mathbb{D})$, we define the weighted composition operator

$$uC_\varphi f(z) = u(z)f(\varphi(z)), \quad \text{for } f \in H(\mathbb{D}).$$

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As a combination of composition operators and pointwise multiplication operators, weighted composition operators arise naturally. Boundedness, compactness and estimates of the essential norms of weighted composition operators between weighted Banach spaces of analytic functions and Bloch type spaces have been studied by several authors, see e.g. [11], [14], [18], [20], [22], [23], [24], [30], [31] and the related references therein.

In this paper we give new characterizations for bounded and compact weighted composition operators acting from Zygmund type spaces to Bloch type spaces, and give new estimates of essential norms of such operators.

Let us first recall the definition of the weighted Banach space of analytic functions, which is defined by

$$H_\nu^\infty = \Big\{f \in H(\mathbb{D})\colon \ \|f\|_\nu := \sup_{\mathbb{D}} \nu(z)|f(z)| < \infty\Big\}$$

equipped with the norm $\|\cdot\|_\nu$, where the weight $\nu\colon \mathbb{D} \to \mathbb{R}_+$ is a continuous strictly positive and bounded function. The weight ν is called radial, if $\nu(z) = \nu(|z|)$ for all $z \in \mathbb{D}$. And for a weight ν, the associated weight $\tilde{\nu}$ is given by

$$\tilde{\nu}(z) := (\sup\{|f(z)|\colon \ f \in H_\nu^\infty, \ \|f\|_\nu \leqslant 1\})^{-1}, \quad z \in \mathbb{D}.$$

Besides the standard weights $\nu_\alpha = (1 - |z|^2)^\alpha$, $0 < \alpha < \infty$, we also consider the logarithmic weight

$$\nu_{\log}(z) := \Big(\log \frac{\mathrm{e}}{1 - |z|^2}\Big)^{-1}, \quad z \in \mathbb{D}.$$

It is not difficult to see that $\tilde{\nu}_\alpha = \nu_\alpha$ and $\tilde{\nu}_{\log} = \nu_{\log}$. What is more, the Banach space of bounded analytic functions on $\mathbb{D}$ is denoted by H^∞. In the following, let $\|f\|_{\nu_\alpha}$ and $\|f\|_{\nu_{\log}}$ denote the norms defined on the weighted Banach spaces $H_{\nu_\alpha}^\infty$ and $H_{\nu_{\log}}^\infty$, respectively.

The Bloch type space $\mathcal{B}^\alpha$ $(0 < \alpha < \infty)$ on the unit disk consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_\alpha := \sup_{\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

Furthermore, $\mathcal{B}^\alpha$ is a Banach space when endowed with the norm

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_\alpha < \infty.$$

As we all know, $\mathcal{B}^\alpha$ is a subspace of H^∞ for $0 < \alpha < 1$.

For $0 < \beta < \infty$, we denote by $\mathcal{Z}_\beta$ the Zygmund type space of the functions $f \in H(\mathbb{D})$ such that

$$\sup_{\mathbb{D}} (1 - |z|^2)^\beta |f''(z)| < \infty$$

equipped with the norm

$$\|f\|_{\mathcal{Z}_\beta} := |f(0)| + |f'(0)| + \sup_{\mathbb{D}}(1-|z|^2)^\beta |f''(z)|.$$

For $\beta = 1$ we obtain the classical Zygmund space $\mathcal{Z}$. For more information on function spaces, we can refer to two books [33], [35].

The essential norm of a continuous linear operator T is defined as the distance from T to the compact operators K, that is $\|T\|_{\mathrm{e}} = \inf\{\|T-K\|: K \text{ is compact}\}$. Notice that $\|T\|_{\mathrm{e}} = 0$ if and only if T is compact, so estimates on $\|T\|_{\mathrm{e}}$ lead to conditions for T to be compact.

Recently, there has been an increase interest in characterizing the boundedness and compactness of composition operators or integral-type composition operators acting on Bloch type spaces in terms of the n-th power φ^n of φ, see e.g. [6], [9], [10], [12], [13], [15], [16], [25], [28]. The natural question to ask is whether the essential norm formula for composition operators between Bloch-types paces $\mathcal{B}_\alpha$ can be generalized to weighted composition operators. In 2012, Manhas and Zhao [19] showed that the question has an affirmative answer when $\alpha \neq 1$; for the case $\alpha = 1$, the problem was solved by Hyvärinen and Lindström in [13]. In 2013, Esmaeili and Lindström [8] gave similar characterizations for the weighted composition operators acting between Zygmund type spaces. The characterization on the classical Zygmund spaces was presented by Hu and Ye in [26]. Very recently, Liang and Zhou [17] used an approach due to Hyvärinen and Lindström in [13] and Esmaeili and Lindström in [8] to obtain new characterizations for bounded weighted composition operators from Bloch type spaces to Zygmund type spaces, and to give similar estimates of the essential norms of such operators.

In this paper, our goal is to estimate the essential norm of uC_φ from Zygmund type spaces to Bloch type spaces in terms of u, φ, their derivatives and the n-th power φ^n of φ, which is inspired by the above papers. The results of this paper make the characterizations for the boundedness and essential norms of weighted composition operators between Bloch type and Zygmund type spaces more perfect.

Throughout this paper, C denotes a positive constant, the exact value of which may be different. The notation $A \preceq B$, $A \succeq B$ and $A \asymp B$ means that there may be different positive constants C such that $A \leqslant CB$, $A \geqslant CB$ and $B/C \leqslant A \leqslant CB$ hold.

2. Some lemmas

In this section, we present some lemmas which will be used in the proofs of our main results in the next sections. The following two lemmas are crucial to the new characterizations of boundedness and essential norm of a weighted composition operator.

Lemma 2.1 ([20], Theorem 2.1, or [12], Theorem 2.4). *Let ν and w be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Then*

(i) *the weighted composition operator uC_φ maps H_ν^∞ into H_w^∞ if and only if*

$$\sup_{n\geqslant 0}\frac{\|u\varphi^n\|_w}{\|z^n\|_\nu}\asymp\sup_{\mathbb{D}}\frac{w(z)}{\tilde{\nu}(\varphi(z))}|u(z)|<\infty,$$

with norm comparable to the above supremum.

(ii)

$$\begin{aligned}\|uC_\varphi\|_{\mathrm{e},H_\nu^\infty\to H_w^\infty} &= \limsup_{n\to\infty}\|u\varphi^n\|_w/\|z^n\|_\nu\\ &= \limsup_{|\varphi(z)|\to 1} w(z)|u(z)|/\tilde{\nu}(\varphi(z))(1-|z|^2)^\alpha|f''_{\varphi(w)}(z)|.\end{aligned}$$

Lemma 2.2 ([13], Lemma 2.1). *For $0<\alpha<\infty$ we have*

(i) $\lim\limits_{n\to\infty}(n+1)^\alpha\|z^n\|_{\nu_\alpha}=(2\alpha/\mathrm{e})^\alpha$,

(ii) $\lim\limits_{n\to\infty}(\log n)\|z^n\|_{\nu_{\log}}=1$.

The next two lemmas are well-known characterizations for the Zygmund type spaces and Bloch type spaces on the unit disc.

Lemma 2.3 ([8], Lemma 1.1). *For every $f\in\mathcal{Z}_\alpha$, $\alpha>0$ we have:*

(i) $|f'(z)|\preceq\|f\|_{\mathcal{Z}_\alpha}$ *and* $|f(z)|\preceq\|f\|_{\mathcal{Z}_\alpha}$ *for every* $0<\alpha<1$,

(ii) $|f'(z)|\preceq\|f\|_{\mathcal{Z}}\log(\mathrm{e}/(1-|z|^2))$ *and* $|f(z)|\preceq\|f\|_{\mathcal{Z}}$ *for* $\alpha=1$,

(iii) $|f'(z)|\preceq\|f\|_{\mathcal{Z}_\alpha}/(1-|z|^2)^{\alpha-1}$ *for every* $\alpha>1$,

(iv) $|f(z)|\preceq\|f\|_{\mathcal{Z}_\alpha}$ *for every* $1<\alpha<2$,

(v) $|f(z)|\preceq\|f\|_{\mathcal{Z}_2}\log(\mathrm{e}/(1-|z|^2))$ *for* $\alpha=2$,

(vi) $|f(z)|\preceq\|f\|_{\mathcal{Z}_\alpha}/(1-|z|^2)^{\alpha-2}$, *for every* $\alpha>2$.

Lemma 2.4 ([34]). *Let $f\in\mathcal{B}^\alpha, m\in\mathbb{D}$ and $\alpha>0$. Then $f(z)\in\mathcal{B}^\alpha$ if and only if*

$$\sup_{\mathbb{D}}(1-|z|^2)^{\alpha+m-1}|f^{(m)}(z)|<\infty$$

The following criterion for compactness follows from an easy modification of Proposition 3.11 of [7]. Therefore, we omit the details.

Lemma 2.5. *Suppose X and Y are two Banach spaces, the weighted composition operator $uC_\varphi\colon X \to Y$ is compact. If $\{f_k\}$ is bounded in X and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $uC_\varphi f_k \to 0$ in Y as $k \to \infty$.*

The above lemma is critical for the estimate of the lower bound of the essential norm of a weighted composition operator. The next two lemmas are the key to the estimate of the upper bound.

Lemma 2.6 ([17], Lemma 4.2)**.** *Let $0 < \varphi < \infty$, let the weighted composition operator $uC_\varphi\colon \mathcal{B}^\alpha \to H^\infty_{\nu_\beta}$ be bounded.*
(i) *If $0 < \alpha < 1$, then $uC_\varphi\colon \mathcal{B}^\alpha \to H^\infty_{\nu_\beta}$ is compact.*
(ii) *If $\alpha = 1$, then*

$$\|uC_\varphi\|_{\mathrm{e},\mathcal{B}\to H^\infty_{\nu_\beta}} \asymp \limsup_{n\to\infty} (\log n)\|u\varphi^n\|_{\nu_\beta}. \tag{2.1}$$

(iii) *If $\alpha > 1$, then*

$$\|uC_\varphi\|_{\mathrm{e},\mathcal{B}^\alpha\to H^\infty_{\nu_\beta}} \asymp \limsup_{n\to\infty} (n+1)^{\alpha-1}\|u\varphi^n\|_{\nu_\beta}. \tag{2.2}$$

Lemma 2.7 ([8], Theorem 3.3)**.** *Let $0 < \varphi < \infty$, let the weighted composition operator $uC_\varphi\colon \mathcal{Z}_\alpha \to H^\infty_\nu$ be bounded.*
(i) *If $0 < \alpha < 2$, then uC_φ is compact.*
(ii) *If $\alpha = 2$, then*

$$\|uC_\varphi\|_{\mathrm{e},\mathcal{Z}_2\to H^\infty_\nu} \asymp \limsup_{n\to\infty} (\log n)\|u\varphi^n\|_\nu. \tag{2.3}$$

(iii) *If $\alpha > 2$, then*

$$\|uC_\varphi\|_{\mathrm{e},\mathcal{Z}_\alpha\to H^\infty_\nu} \asymp \limsup_{n\to\infty} (n+1)^{\alpha-2}\|u\varphi^n\|_\nu. \tag{2.4}$$

On the other hand, the following lemma is useful for us to get an estimate of the essential norm of $uC_\varphi\colon \mathcal{Z}_\alpha \to \mathcal{B}^\beta$ which can be proved in a way similar to [8], Lemma 3.1. Denote $\widetilde{\mathcal{Z}}_\alpha = \{f \in \mathcal{Z}_\alpha\colon f(0) = f'(0) = 0\}$.

Lemma 2.8. *If $0 < \alpha < \infty$ and $uC_\varphi\colon \mathcal{Z}_\alpha \to \mathcal{B}^\beta$ is a bounded weighted composition operator, then*

$$\|uC_\varphi\|_{\mathrm{e},\widetilde{\mathcal{Z}}_\alpha\to\mathcal{B}^\beta} = \|uC_\varphi\|_{\mathrm{e},\mathcal{Z}_\alpha\to\mathcal{B}^\beta}. \tag{2.5}$$

3. Boundedness

In this section, we will give new characterizations for the boundedness of $uC_\varphi\colon \mathcal{Z}_\alpha \to \mathcal{B}^\beta$ in five cases according to Lemma 2.3.

Theorem 3.1. *If $0 < \alpha < 1$, then uC_φ maps $\mathcal{Z}_\alpha$ boundedly into $\mathcal{B}^\beta$ if and only if $u \in \mathcal{B}^\beta$ and $u\varphi' \in H^\infty_{\nu_\beta}$.*

Proof. On the one hand, if uC_φ maps $\mathcal{Z}_\alpha$ boundedly into $\mathcal{B}^\beta$, then we can easily get $u \in \mathcal{B}^\beta$ and $u\varphi' \in H^\infty_{\nu_\beta}$ by taking the test functions $f = 1$ and $f = z$.

On the other hand, if $u \in \mathcal{B}^\beta$ and $u\varphi' \in H^\infty_{\nu_\beta}$, by Lemma 2.3 (i) we have that

$$\begin{aligned}(1-|z|^2)^\beta|(uC_\varphi f)'(z)| &\leqslant (1-|z|^2)^\beta|u'(z)f(\varphi(z))| \\ &\quad + (1-|z|^2)^\beta|u(z)\varphi'(z)f'(\varphi(z))| \\ &\preceq \|u\|_{\mathcal{B}^\beta}\|f\|_{\mathcal{Z}_\alpha} + \|u\varphi'\|_{H^\infty_{\nu_\beta}}\|f\|_{\mathcal{Z}_\alpha} \\ &\preceq C\|f\|_{\mathcal{Z}_\alpha}\end{aligned}$$

for any $f \in \mathcal{Z}_\alpha$. The proof is complete. □

Theorem 3.2. *If $\alpha = 1$, then uC_φ maps $\mathcal{Z}$ boundedly into $\mathcal{B}^\beta$ if and only if $u \in \mathcal{B}^\beta$ and*

$$\sup_{\mathbb{D}}(1-|z|^2)^\beta|u(z)\varphi'(z)|\log\frac{\mathrm{e}}{1-|\varphi(z)|^2} \asymp \sup_{n\geqslant 0}(\log n)\|u\varphi'\varphi^n\|_{\nu_\beta} < \infty. \tag{3.1}$$

Proof. If $u \in \mathcal{B}^\beta$ and (3.1) holds then by Lemma 2.3 (ii) we can prove the boundedness of uC_φ easily as in Theorem 3.1. Here we omit the details.

For the other direction, assume that uC_φ is bounded. For any $w \in \mathbb{D}$, introduce a test function $f_w(z) = \int_0^z \log(\mathrm{e}/(1-\overline{w}\zeta))\,\mathrm{d}\zeta$. Note that $f_w'(z) = \log(\mathrm{e}/(1-\overline{w}z))$ and $f_w''(z) = \overline{w}/(1-\overline{w}z)$, hence we have

$$(1-|z|^2)|f_w''(z)| = (1-|z|^2)\frac{\overline{|w|}}{|1-\overline{\varphi(w)}z|} \leqslant \frac{2(1-|z|)}{1-|z|} = 2.$$

Combining the above with Lemma 2.3 (ii), we obtain $\|f_w\|_{\mathcal{Z}} \leqslant C$ and $f_w \in \mathcal{Z}$, and

$$C\|f_{\varphi(w)}\|_{\mathcal{Z}} \geqslant \|uC_\varphi f_{\varphi(w)}\|_{\mathcal{B}^\beta} \geqslant (1-|w|^2)^\beta |u(w)\varphi'(w)f'_{\varphi(w)}(\varphi(w))| - (1-|w|^2)^\beta |u'(w)f_{\varphi(w)}(\varphi(w))|.$$

Thus

$$(1-|w|^2)^\beta |u(w)\varphi'(z)f'_{\varphi(w)}(\varphi(w))| \preceq C\|f_{\varphi(w)}\|_{\mathcal{Z}} + \|u\|_{\mathcal{B}^\beta}\|f_{\varphi(w)}\|_{\mathcal{Z}} \leqslant C.$$

Then by Lemmas 2.1 and 2.2, (3.1) holds. This completes the proof of the theorem. □

The sufficiency of the next three theorems can be proved in the same way as in Theorems 3.1 and 3.2, so we omit this part. The only differences are that we use (iii), (iv) of Lemma 2.3 in Theorem 3.3; (iii), (v) of Lemma 2.3 in Theorem 3.4; (iii), (vi) of Lemma 2.3 in Theorem 3.5.

Theorem 3.3. *If $1 < \alpha < 2$, then uC_φ maps $\mathcal{Z}_\alpha$ boundedly into $\mathcal{B}^\beta$ if and only if $u \in \mathcal{B}^\beta$ and*

$$\sup_{\mathbb{D}} \frac{(1-|z|^2)^\beta |u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}} \asymp \sup_{n\geqslant 0}(n+1)^{\alpha-1}\|u\varphi'\varphi^n\|_{\nu_\beta} < \infty. \tag{3.2}$$

P r o o f. Necessity. If uC_φ is bounded, we choose the test function defined by

$$f_{\varphi(w)}(z) = \frac{1}{(1-\overline{\varphi(w)}z)^{\alpha-2}} - \frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{\alpha-1}} \in \mathcal{Z}_\alpha$$

where $w \in \mathbb{D}$. Here it is easy to verify $f_{\varphi(w)}(\varphi(w)) = 0$, $\|f_{\varphi(w)}\|_{\mathcal{Z}_\alpha} \leqslant C$ and $f_{\varphi(w)} \in \mathcal{Z}_\alpha$ by

$$\begin{aligned}
&(1-|z|^2)^\alpha |f''_{\varphi(w)}(z)| \\
&= (1-|z|^2)^\alpha \Big|\frac{\overline{\varphi(w)}^2(\alpha-2)(\alpha-1)}{(1-\overline{\varphi(w)}z)^\alpha} - \frac{\overline{\varphi(w)}^2\alpha(\alpha-1)(1-|\varphi(w)|^2)}{(1-\overline{\varphi(w)}z)^{\alpha+1}}\Big| \\
&\leqslant (1-|z|^2)^\alpha \Big|\frac{(\alpha-2)(\alpha-1)}{(1-\overline{\varphi(w)}z)^\alpha}\Big| + (1-|z|^2)^\alpha\Big|\frac{\alpha(\alpha-1)(1-|\varphi(w)|^2)}{(1-\overline{\varphi(w)}z)^{\alpha+1}}\Big| \\
&\leqslant (\alpha-1)(\alpha+2)2^\alpha.
\end{aligned}$$

Then

$$\begin{aligned}
C\|f_{\varphi(w)}\|_{\mathcal{Z}_\alpha} \geqslant \|uC_\varphi f_{\varphi(w)}\|_{\mathcal{B}^\beta} &\geqslant (1-|w|^2)^\beta |u(w)\varphi'(w)f'_{\varphi(w)}(\varphi(w))| \\
&\quad - (1-|w|^2)^\beta |u'(w)f_{\varphi(w)}(\varphi(w))| \\
&= (1-|w|^2)^\beta |u(w)\varphi'(w)f'_{\varphi(w)}(\varphi(w))|.
\end{aligned}$$

Combining the above with Lemmas 2.1 and 2.2, it follows that (3.2) holds. This ends the proof of the theorem. □

Theorem 3.4. *If* $\alpha = 2$, *then* uC_φ *maps* $\mathcal{Z}_2$ *boundedly into* $\mathcal{B}^\beta$ *if and only if*

$$\sup_{\mathbb{D}}(1-|z|^2)^\beta|u'(z)|\log\frac{\mathrm{e}}{1-|\varphi(z)|^2} \asymp \sup_{n\geqslant 0}\log n\|u'\varphi^n\|_{\nu_\beta} < \infty \tag{3.3}$$

and

$$\sup_{\mathbb{D}}(1-|z|^2)^\beta\frac{|u(z)\varphi'(z)|}{1-|\varphi(z)|^2} \asymp \sup_{n\geqslant 0}(n+1)\|u\varphi'\varphi^n\|_{\nu_\beta} < \infty. \tag{3.4}$$

Proof. Necessity. If uC_φ is bounded, we can show that (3.4) holds as in the proof of (3.2) in Theorem 3.3. Thus we only need to give the proof of (3.3).

Let the test function be

$$f_{\varphi(w)}(z) = \log\frac{\mathrm{e}}{1-\overline{\varphi(w)}z},$$

where $\|f_{\varphi(w)}\|_{\mathcal{Z}_2} \leqslant C$ and $f_{\varphi(w)} \in \mathcal{Z}_2$ which can be verified as in Theorems 3.2 and 3.3. Therefore,

$$\begin{aligned} C\|f_{\varphi(w)}\|_{\mathcal{Z}_2} \geqslant \|uC_\varphi f_{\varphi(w)}\|_{\mathcal{B}^\beta} \geqslant (1-|w|^2)^\beta|u'(w)f_{\varphi(w)}(\varphi(w))| \\ - (1-|w|^2)^\beta|u(w)\varphi'(w)f'_{\varphi(w)}(\varphi(w))|. \end{aligned}$$

Then, by using (3.4), we have

$$\begin{aligned} (1-|w|^2)^\beta|u'(w)f_{\varphi(w)}(\varphi(w))| &= (1-|w|^2)^\beta|u'(w)|\log\frac{\mathrm{e}}{1-|\varphi(w)|^2} \\ &\leqslant C\|f_{\varphi(w)}\|_{\mathcal{Z}_2} + (1-|w|^2)^\beta|u(w)\varphi'(w)f'_{\varphi(w)}(\varphi(w))| \\ &\preceq C\|f_{\varphi(w)}\|_{\mathcal{Z}_2} + (1-|w|^2)^\beta\frac{|u(w)\varphi'(w)|}{1-|\varphi(w)|^2} \leqslant C. \end{aligned}$$

By Lemmas 2.1 and 2.2, it follows that

$$\sup_{\mathbb{D}}(1-|w|^2)^\beta|u'(w)|\log\frac{\mathrm{e}}{1-|\varphi(w)|^2} \asymp \sup_{n\geqslant 0}\log n\|u'\varphi^n\|_{\nu_\beta} < \infty,$$

that is, (3.3) holds. We completed the proof. □

Theorem 3.5. *If $\alpha > 2$, then uC_φ maps $\mathcal{Z}_\alpha$ boundedly into $\mathcal{B}^\beta$ if and only if*

$$\sup_{\mathbb{D}}(1-|z|^2)^\beta \frac{|u'(z)|}{(1-|\varphi(z)|^2)^{\alpha-2}} \asymp \sup_{n\geqslant 0}(n+1)^{\alpha-2}\|u'\varphi^n\|_{\nu_\beta} < \infty \tag{3.5}$$

and

$$\sup_{\mathbb{D}}(1-|z|^2)^\beta \frac{|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}} \asymp \sup_{n\geqslant 0}(n+1)^{\alpha-1}\|u\varphi'\varphi^n\|_{\nu_\beta} < \infty. \tag{3.6}$$

P r o o f. Necessity. Let uC_φ map $\mathcal{Z}_\alpha$ boundedly into $\mathcal{B}^\beta$. Like in the proof of (3.2), (3.6) holds as well. Take the test function to be

$$f_{\varphi(w)}(z) = \frac{1}{(1-\overline{\varphi(w)}z)^{\alpha-2}}.$$

It is easy to show that $\|f_{\varphi(w)}\|_{\mathcal{Z}_\alpha} \leqslant C$ and $f_{\varphi(w)} \in \mathcal{Z}_\alpha$. Thus

$$\begin{aligned} C\|f_{\varphi(w)}\|_{\mathcal{Z}_\alpha} \geqslant \|uC_\varphi f_{\varphi(w)}\|_{\mathcal{B}^\beta} \geqslant (1-|w|^2)^\beta |u'(w) f_{\varphi(w)}(\varphi(w))| \\ - (1-|w|^2)^\beta |u(w)\varphi'(w) f'_{\varphi(w)}(\varphi(w))|. \end{aligned}$$

Therefore, by Lemmas 2.1 and 2.2, we obtain that

$$\begin{aligned} &\sup_{\mathbb{D}}(1-|w|^2)^\beta |u'(w) f_{\varphi(w)}(\varphi(w))| \\ &\quad = \sup_{\mathbb{D}} |(1-|w|^2)^\beta \frac{|u'(w)|}{(1-|\varphi(w)|^2)^{\alpha-2}} \\ &\quad \asymp \sup_{n\geqslant 0}(n+1)^{\alpha-2}\|u'\varphi^n\|_{\nu_\beta} \\ &\quad \preceq C\|f_{\varphi(w)}\|_{\mathcal{Z}_\alpha} + \sup_{\mathbb{D}}(1-|w|^2)^\beta |u(w)\varphi'(w) f'_{\varphi(w)}(\varphi(w))| \\ &\quad \preceq C\|f_{\varphi(w)}\|_{\mathcal{Z}_\alpha} + \sup_{\mathbb{D}}(1-|w|^2)^\beta \frac{|u(w)\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha-1}} < C, \end{aligned}$$

that is, (3.5) holds. This completes the proof. □

4. Essential norms

In this section, we estimate the essential norms of $uC_\varphi\colon \mathcal{Z}_\alpha \to \mathcal{B}^\beta$ in terms of u, φ, their derivatives and φ^n. Denote $\widetilde{\mathcal{B}}^\alpha = \{f \in \mathcal{B}^\alpha\colon f(0) = 0\}$.

Since $uC_\varphi\colon \mathcal{Z}_\alpha \to \mathcal{B}^\beta$ is bounded, by Lemma 2.3 we know that $u'C_\varphi$ maps $\mathcal{Z}_\alpha$ boundedly into $H^\infty_{\nu_\beta}$ from $u \in \mathcal{Z}_\beta$ for $0 < \alpha < 2$, (3.3) for $\alpha = 2$ and (3.5) for $\alpha > 2$. And by Lemma 2.4., we have that $u\varphi' C_\varphi$ maps $\mathcal{B}^\alpha$ boundedly into $H^\infty_{\nu_\beta}$ from $u\varphi' \in H^\infty_{\nu_\beta}$ for $0 < \alpha < 1$, (3.1) for $\alpha = 1$, (3.2) for $1 < \alpha < 2$, (3.4) for $\alpha = 2$ and, (3.6) for $\alpha > 2$. Then we can make use of the essential norms of the two operators by Lemmas 2.6 and 2.7.

Moreover, we have other two results which are similar to Lemma 2.8:

$$\|u'C_\varphi\|_{\mathrm{e},\widetilde{\mathcal{Z}}_\alpha \to H^\infty_{\nu_\beta}} = \|u'C_\varphi\|_{\mathrm{e},\mathcal{Z}_\alpha \to H^\infty_{\nu_\beta}} \tag{4.1}$$

and

$$\|u\varphi'C_\varphi\|_{\mathrm{e},\widetilde{\mathcal{B}}^\alpha \to H^\infty_{\nu_\beta}} = \|u\varphi'C_\varphi\|_{\mathrm{e},\mathcal{B}^\alpha \to H^\infty_{\nu_\beta}}. \tag{4.2}$$

Because $(uC_\varphi f)' = u'C_\varphi f + u\varphi' C_\varphi f'$, $\mathcal{D}_\alpha\colon \mathcal{Z}_\alpha \to \mathcal{B}_\alpha$ and $\mathcal{S}_\alpha\colon \mathcal{B}^\alpha \to H_{\nu^\infty_\alpha}$ are linear isometries on $\widetilde{\mathcal{Z}}_\alpha$ and $\widetilde{\mathcal{B}}^\alpha$, it follows that

$$\|uC_\varphi\|_{\mathrm{e},\widetilde{\mathcal{Z}}_\alpha \to \mathcal{Z}^\beta} \preceq \|u'C_\varphi\|_{\mathrm{e},\widetilde{\mathcal{Z}}_\alpha \to H^\infty_{\nu_\beta}} + \|u\varphi'C_\varphi\|_{\mathrm{e},\widetilde{\mathcal{B}}^\alpha \to H^\infty_{\nu_\beta}}. \tag{4.3}$$

Therefore, combining Lemmas 2.6 and 2.7 with (2.5) and (4.1)–(4.3), we can directly get the upper bound of a weighted composition operator from

$$\|uC_\varphi\|_{\mathrm{e},\mathcal{Z}_\alpha \to \mathcal{B}^\beta} \preceq \|u'C_\varphi\|_{\mathrm{e},\mathcal{Z}_\alpha \to H^\infty_{\nu_\beta}} + \|u\varphi'C_\varphi\|_{\mathrm{e},\mathcal{B}^\alpha \to H^\infty_{\nu_\beta}}.$$

So the critical parts in the following theorems are to estimate the lower bound.

Besides, we can immediately get sufficient and necessary conditions of the compactness of the operators uC_φ from the estimates of the essential norms.

Theorem 4.1. *Let $0 < \alpha < 1$. Suppose the weighted composition operator $uC_\varphi\colon \mathcal{Z}_\alpha \to \mathcal{B}^\beta$ is bounded. Then $\|uC_\varphi\|_{\mathrm{e},\mathcal{Z}_\alpha \to \mathcal{B}^\beta} = 0$.*

Proof. By Lemma 2.6 (i) and Lemma 2.7 (i), we can directly obtain the result from

$$\|uC_\varphi\|_{\mathrm{e},\mathcal{Z}_\alpha \to \mathcal{Z}^\beta} \preceq \|u'C_\varphi\|_{\mathrm{e},\mathcal{Z}_\alpha \to H^\infty_{\nu_\beta}} + \|u\varphi'C_\varphi\|_{\mathrm{e},\mathcal{B}^\alpha \to H^\infty_{\nu_\beta}} = 0.$$

That is, when $0 < \alpha < 1$, the weighted composition operator $uC_\varphi\colon \mathcal{Z}_\alpha \to \mathcal{B}^\beta$ is bounded if and only if uC_φ is compact. □

Theorem 4.2. *Let $\alpha = 1$. Suppose the weighted composition operator $uC_\varphi\colon \mathcal{Z} \to \mathcal{B}^\beta$ is bounded. Then*

$$(4.4)\qquad \|uC_\varphi\|_{e,\mathcal{Z}_\alpha\to\mathcal{B}^\beta} \asymp \limsup_{n\to\infty} \log n \|u\varphi'\varphi^n\|_{\nu_\beta} \asymp \limsup_{|\varphi(z)|\to 1} (1-|z|^2)^\beta |u(z)\varphi'(z)| \log\frac{e}{1-|\varphi(z)|^2}.$$

Proof. When $\alpha = 1$, the boundedness of uC_φ guarantees that $u'C_\varphi$ maps $\mathcal{Z}$ boundedly and compactly into $H^\infty_{\nu_\beta}$ and $u\varphi'C_\varphi$ maps $\mathcal{B}$ boundedly into $H^\infty_{\nu_\beta}$.

Therefore, by Lemma 2.1 (ii), 2.2 (ii), 2.7 (i) and (2.1) we have

$$(4.5)\qquad \begin{aligned}\|uC_\varphi\|_{e,\mathcal{Z}\to\mathcal{B}^\beta} &\preceq \|u'C_\varphi\|_{e,\mathcal{Z}\to H^\infty_{\nu_\beta}} + \|u\varphi'C_\varphi\|_{e,\mathcal{B}\to H^\infty_{\nu_\beta}} \asymp \|u\varphi'C_\varphi\|_{e,\mathcal{B}\to H^\infty_{\nu_\beta}}\\ &\asymp \limsup_{|\varphi(z)|\to 1} (1-|z|^2)^\beta |u(z)\varphi'(z)| \log\frac{e}{1-|\varphi(z)|^2}\\ &\asymp \limsup_{n\to\infty} \log n \|u\varphi'\varphi^n\|_{\nu_\beta}.\end{aligned}$$

In order to search for the lower bound, let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| > 1/2$ and $|\varphi(z_k)| \to 1$ as $k \to \infty$. Define

$$f_k(z) = \int_0^z \frac{(\log(e/(1-\overline{\varphi(z_k)}w)))^2}{\log(e/(1-|\varphi(z_k)|^2))}\,dw.$$

It is obvious that $\{f_k\}$ is a bounded sequence in $\mathcal{Z}$ and converges to zero uniformly on compact subsets of $\mathbb{D}$. Then for every compact operator $T\colon \mathcal{Z} \to \mathcal{B}^\beta$ we have $\|Tf_k\|_{\mathcal{B}^\beta} \to 0$ as $k \to \infty$. Thus

$$\begin{aligned}\|uC_\varphi - T\|_{\mathcal{Z}\to\mathcal{B}^\beta} &\succeq \limsup_{k\to\infty} \|uC_\varphi f_k\|_{\mathcal{B}^\beta} - \limsup_{k\to\infty} \|T(f_k)\|_{\mathcal{B}^\beta}\\ &\succeq \limsup_{k\to\infty} (1-|z_k|^2)^\beta |u(z_k)\varphi'(z_k)| \log\frac{e}{1-|\varphi(z_k)|^2}\\ &\quad - \limsup_{k\to\infty} (1-|z_k|^2)^\beta |u'(z_k) f_k(\varphi(z_k))|.\end{aligned}$$

Since $u'C_\varphi$ is compact, it follows that

$$\inf_{T\in\mathcal{K}(\mathcal{Z},\mathcal{B}^\beta)} \|uC_\varphi - T\|_{\mathcal{Z}\to\mathcal{B}^\beta} \succeq \limsup_{k\to\infty} (1-|z_k|^2)^\beta |u(z_k)\varphi'(z_k)| \log\frac{e}{1-|\varphi(z_k)|^2},$$

which implies that

$$(4.6)\qquad \begin{aligned}\|uC_\varphi\|_{e,\mathcal{Z}_\alpha\to\mathcal{B}^\beta} &\succeq \limsup_{|\varphi(z)|\to 1} (1-|z|^2)^\beta |u(z)\varphi'(z)| \log\frac{e}{1-|\varphi(z)|^2}\\ &\asymp \limsup_{n\to\infty} \log n \|u\varphi'\varphi^n\|_{\nu_\beta}.\end{aligned}$$

Combining (4.5) with (4.6), we conclude that (4.4) holds. This completes the proof. □

In the following three theorems, we can use a technique similar to that in the above theorem to get the upper bound of essential norms. Besides, in the proof of other cases, we have the results by the corresponding conditions in Lemmas 2.1 and 2.2.

Theorem 4.3. *Let $1 < \alpha < 2$. Suppose the weighted composition operator $uC_\varphi\colon \mathcal{Z}_\alpha \to \mathcal{B}^\beta$ is bounded. Then*

$$\begin{aligned} \|uC_\varphi\|_{\mathrm{e},\mathcal{Z}_\alpha\to\mathcal{B}^\beta} &\asymp \limsup_{n\to\infty}(n+1)^{\alpha-1}\|u\varphi'\varphi^n\|_{\nu_\beta} \\ &\asymp \limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\beta\frac{|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}}. \end{aligned} \tag{4.7}$$

P r o o f. Here we only prove

$$\|uC_\varphi\|_{\mathrm{e},\mathcal{Z}_\alpha\to\mathcal{B}^\beta} \succeq \limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\beta\frac{|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}}.$$

We choose the test function as

$$f_k(z) = \frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}z)^{\alpha-1}},$$

where z_k are chosen like in Theorem 4.2 and $\{f_k\}$ is a bounded sequences in $\mathcal{Z}_\alpha$ and converges to zero uniformly on compact subsets of $\mathbb{D}$. Then for every compact operator $T\colon \mathcal{Z}_\alpha \to \mathcal{B}^\beta$ we have $\|Tf_k\|_{\mathcal{B}^\beta} \to 0$ as $k\to\infty$. Thus

$$\begin{aligned} \|uC_\varphi - T\|_{\mathcal{Z}_\alpha\to\mathcal{B}^\beta} &\succeq \limsup_{k\to\infty}\|uC_\varphi f_k\|_{\mathcal{B}^\beta} - \limsup_{k\to\infty}\|T(f_k)\|_{\mathcal{B}^\beta} \\ &\succeq \limsup_{k\to\infty}(1-|z_k|^2)^\beta\frac{|u(z_k)\varphi'(z_k)|}{1-|\varphi(z_k)|^2} \\ &\quad - \limsup_{k\to\infty}(1-|z_k|^2)^\beta|u'(z_k)f_k(\varphi(z_k))| \\ &= \limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\beta\frac{|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}} \\ &\asymp \limsup_{n\to\infty}(n+1)^{\alpha-1}\|u\varphi'\varphi^n\|_{\nu_\beta}. \end{aligned}$$

Therefore,

$$\inf_{\mathcal{K}(\mathcal{Z}_\alpha\to\mathcal{B}^\beta)}\|uC_\varphi - T\|_{\mathcal{Z}_\alpha\to\mathcal{B}^\beta} \succeq \limsup_{n\to\infty}(n+1)^{\alpha-1}\|u\varphi'\varphi^n\|_{\nu_\beta}.$$

We completed the proof of the theorem. □

Theorem 4.4. *Let $\alpha = 2$. Suppose the weighted composition operator $uC_\varphi\colon \mathcal{Z}_2 \to \mathcal{B}^\beta$ is bounded. Then*

$$(4.8)\qquad \|uC_\varphi\|_{e,\mathcal{Z}_2\to\mathcal{B}^\beta} \asymp \max\Big\{\limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\beta|u'(z)|\log\frac{e}{1-|\varphi(z)|^2},\ \limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\beta\frac{|u(z)\varphi'(z)|}{1-|\varphi(z)|^2}\Big\}$$
$$\asymp \max\Big\{\limsup_{n\to\infty}\log n\|u'\varphi^n\|_{\nu_\beta},\ \limsup_{n\to\infty}(n+1)\|u\varphi'\varphi^n\|_{\nu_\beta}\Big\}.$$

Proof. The most important task for us is to choose the test functions separately; here let

$$f_k(z) = \frac{1-|\varphi(z_k)|^2}{1-\overline{\varphi(z_k)}z} - \frac{(1-|\varphi(z_k)|^2)^2}{(1-\overline{\varphi(z_k)}z)^2}$$

and

$$g_k(z) = \frac{(\log(e/(1-\overline{\varphi(z_k)}z)))^2}{2\log(e/(1-|\varphi(z_k)|^2))} - \frac{(\log(e/(1-\overline{\varphi(z_k)}z)))^3}{3(\log(e/(1-|\varphi(z_k)|^2)))^2}$$

where z_k are chosen as in Theorem 4.2, $f_k(z)$ and $g_k(z)$ are bounded sequences in $\mathcal{Z}_2$ and converge to zero uniformly on compact subsets of $\mathbb{D}$. Then like in the last theorem, we have that

$$\|uC_\varphi\|_{e,\mathcal{Z}_2\to\mathcal{B}^\beta} \succeq \limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\beta\frac{|u(z)\varphi'(z)|}{1-|\varphi(z)|^2} \asymp \limsup_{n\to\infty}(n+1)\|u\varphi'\varphi^n\|_{\nu_\beta},$$
$$\|uC_\varphi\|_{e,\mathcal{Z}_2\to\mathcal{B}^\beta} \succeq \limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\beta|u'(z)|\log\frac{e}{1-|\varphi(z)|^2} \asymp \limsup_{n\to\infty}\log n\|u'\varphi^n\|_{\nu_\beta}.$$

Therefore,

$$\|uC_\varphi\|_{e,\mathcal{Z}_2\to\mathcal{B}^\beta}$$
$$\succeq \max\Big\{\limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\beta\frac{|u(z)\varphi'(z)|}{1-|\varphi(z)|^2},\ \limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\beta|u'(z)|\log\frac{e}{1-|\varphi(z)|^2}\Big\}$$
$$\asymp \max\Big\{\limsup_{n\to\infty}\log n\|u'\varphi^n\|_{\nu_\beta},\ \limsup_{n\to\infty}(n+1)\|u\varphi'\varphi^n\|_{\nu_\beta}\Big\}.$$

We completed the proof. □

Theorem 4.5. *Let $\alpha > 2$. Suppose the weighted composition operator $uC_\varphi\colon \mathcal{Z}_\alpha \to \mathcal{B}^\beta$ is bounded. Then*

$$\begin{aligned}
&\|uC_\varphi\|_{\mathrm{e},\mathcal{Z}_\alpha\to\mathcal{B}^\beta}\\
&\asymp \max\Big\{\limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\beta\frac{|u'(z)|}{(1-|\varphi(z)|^2)^{\alpha-2}}, \limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\beta\frac{|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}}\Big\}\\
&\asymp \max\Big\{\limsup_{n\to\infty}(n+1)^{\alpha-2}\|u'\varphi^n\|_{\nu_\beta},\ \limsup_{n\to\infty}(n+1)^{\alpha-1}\|u\varphi'\varphi^n\|_{\nu_\beta}\Big\}.
\end{aligned}$$

Proof. The test functions are chosen as

$$f_k(z) = \frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}z)^{\alpha-1}} - \frac{(1-|\varphi(z_k)|^2)^2}{(1-\overline{\varphi(z_k)}z)^{\alpha}}$$

and

$$g_k(z) = \frac{1-|\varphi(z_k)|^2}{(\alpha-1)(1-\overline{\varphi(z_k)}z)^{\alpha-1}} - \frac{(1-|\varphi(z_k)|^2)^2}{\alpha(1-\overline{\varphi(z_k)}z)^{\alpha}}.$$

Therefore, as in the above theorems, we have

$$\begin{aligned}
&\|uC_\varphi\|_{\mathrm{e},\mathcal{Z}_\alpha\to\mathcal{B}^\beta}\\
&\succeq \max\Big\{\limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\beta\frac{|u'(z)|}{(1-|\varphi(z)|^2)^{\alpha-2}}, \limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\beta\frac{|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}}\Big\}\\
&\asymp \max\Big\{\limsup_{n\to\infty}(n+1)^{\alpha-2}\|u'\varphi^n\|_{\nu_\beta},\ \limsup_{n\to\infty}(n+1)^{\alpha-1}\|u\varphi'\varphi^n\|_{\nu_\beta}\Big\}.
\end{aligned}$$

This completes the proof of the theorem. □

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