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Czechoslovak Mathematical Journal, Vol. 65 (2015), No. 2, 301–316

Persistent URL: http://dml.cz/dmlcz/144270

Czechoslovak Mathematical Journal, 65 (140) (2015), 301–316

OSCILLATION OF THIRD ORDER DIFFERENTIAL EQUATION WITH DAMPING TERM

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(Received October 3, 2012)

Dedicated to the memory of Professor Marko Švec

Abstract. We study asymptotic and oscillatory properties of solutions to the third order differential equation with a damping term

$$x'''(t)+q(t)x'(t)+r(t)|x|^{\lambda}(t)\operatorname{sgn} x(t)=0, \quad t\geqslant 0.$$

We give conditions under which every solution of the equation above is either oscillatory or tends to zero. In case $\lambda\leqslant 1$ and if the corresponding second order differential equation $h''+q(t)h=0$ is oscillatory, we also study Kneser solutions vanishing at infinity and the existence of oscillatory solutions.

Keywords: third order nonlinear differential equation; vanishing at infinity solution; Kneser solution; oscillatory solution

MSC 2010: 34C10, 34C15

1. Introduction

The aim of this paper is to investigate oscillatory and asymptotic properties of solutions to the third order nonlinear differential equation with a damping term

$$x'''(t)+q(t)x'(t)+r(t)|x|^{\lambda}(t)\operatorname{sgn} x(t)=0, \quad t\geqslant 0 \tag{1}$$

where $\lambda>0$, $q\in C^2(\mathbb{R}_+)$ and $r\in C(\mathbb{R}_+)$ are positive functions on $\mathbb{R}_+$, $\mathbb{R}_+=[0,\infty)$.

Research is supported by the grant GAP 201/11/0768 of the Czech Grant Agency.

When $\lambda \leqslant 1$ we will also assume that the corresponding second order linear equation

$$h''(t) + q(t)h(t) = 0 \tag{2}$$

is oscillatory.

By a solution of (1) we mean a function x defined on $[T_x, \infty)$, $T_x \geqslant 0$, which is differentiable up to the third order and satisfies (1) on $[T_x, \infty)$. A solution x of (1) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$; otherwise it is said to be nonoscillatory. Observe that if $\lambda \geqslant 1$, according to [15], Theorem 11.5, all nontrivial solutions of (1) satisfy $\sup\{|x(t)|\colon\ t \geqslant T\} > 0$ for $T \geqslant T_x$, contrary to the case $\lambda < 1$ when eventually vanishing solutions can exist.

The asymptotic and oscillatory properties of solutions for equation (1) have been deeply investigated in literature. The pioneering work is due to M. Švec [20] and G. Villari [22] for the two-term linear differential equation

$$x'''(t) + r(t)x(t) = 0, \quad t \geqslant 0 \tag{L$_0$}$$

where $r(t) > 0$ for $t \geqslant 0$. If there exists $\mu > 0$ such that

$$\int^{\infty} t^{2-\mu} r(t)\,\mathrm{d}t = \infty,$$

then (L$_0$) has both oscillatory and nonoscillatory solutions and every its nonoscillatory solution x tends to zero as $t \to \infty$ and satisfies

$$x(t)x'(t) < 0, \quad x(t)x''(t) > 0 \quad \text{for large } t. \tag{3}$$

This property, sometimes called property A, has been extended to linear and nonlinear higher order equations in various directions, see e.g. the monographs [2], [12], [13], and [3], [6], [14], [16], [17], [19]. Special attention has been paid to the third order equations with quasi-derivatives, see e.g. [11], [10], with damping term [5], [9], [7] or with deviating argument [1] and the references therein.

If (2) is nonoscillatory and h is its positive solution, then (1) can be written as

$$\Big(h^2(t)\Big(\frac{1}{h(t)}x'(t)\Big)'\Big)' + h(t)r(t)|x|^{\lambda}(t)\operatorname{sgn} x(t) = 0,$$

which is an equation with quasi-derivatives. This approach has been used by many authors, see e.g. [5], [8], [9], [10], [21].

If (2) is nonoscillatory, then the result [3], Theorem 2.2 with $n = 3$ reads as follows.

Theorem A. *Assume* $\int^{\infty} tq(t)\,\mathrm{d}t < \infty$. *Then every nonoscillatory solution of equation* (1) *tends to zero as* $t \to \infty$ *if and only if*

$$\int^{\infty} t^2 r(t)\,\mathrm{d}t = \infty.$$

Our aim here is to investigate oscillation of (1) when (2) is oscillatory. In this case, a prototype of equation (1) is

$$x'''(t) + x'(t) + r(t)|x|^{\lambda}(t)\operatorname{sgn} x(t) = 0, \quad \lambda \neq 1, \tag{4}$$

which is a special case of higher-order differential equations with a damping term investigated in [14]. A result [14], Corollary 1.7 with $n = 3$, reads as follows.

Theorem B. *Assume* $\lambda \neq 1$ *and* $r(t) \geqslant r_0/t$ *for some* $r_0 > 0$ *and large* t. *Then every nonoscillatory solution of equation* (4) *satisfies* $\lim x^{(i)}(t) = 0$ *for* $i = 0, 1, 2$.

Observe that the statement of Theorem B fails to hold under a weaker condition

$$\int_0^{\infty} r(t)\,\mathrm{d}t = \infty, \tag{5}$$

as is shown in [14], Remark after Corollary 1.7, by an example of equation (4) having a nonoscillatory solution which does not tend to zero as $t \to \infty$.

Theorem B has been extended in [7], [6] to equation (1) where

$$q(t) \geqslant q_0 \quad \text{for some } q_0 > 0 \text{ and } \quad q'(t) \leqslant 0. \tag{6}$$

It has been proved in [6] that if $\int^{\infty} |q'(t)|\,\mathrm{d}t < \infty$ and $\int^{\infty} r(t)\,\mathrm{d}t < \infty$, then (1) has simultaneously oscillatory solutions and nonoscillatory solutions with oscillating first derivative (so called weakly oscillatory solutions).

The aim of our paper is to give conditions under which every solution of (1) is either oscillatory or tends to zero. All cases $q(t) \to c$ $(c > 0)$, $q(t) \to 0$ or $q(t) \to \infty$ as $t \to \infty$, will be treated. In case $\lambda \leqslant 1$ and if the corresponding second order differential equation (2) is oscillatory, we also study solutions satisfying (3) (so called *Kneser solution*) which are vanishing at infinity, and the existence of oscillatory solutions of (1). Our approach is based on the uniform estimates for positive solutions of quasilinear equations given in [3] and an energy function.

2. Preliminaries

In this section we establish the main tools needed in the next section.

Consider the third-order quasi-linear differential equation

$$\dddot{y}(s) + Q_2(s)\ddot{y}(s) + Q_1(s)\dot{y}(s) + R(s)|y|^{\lambda}(s)\operatorname{sgn} y(s) = 0, \quad \lambda > 1 \tag{7}$$

where Q_i and R are continuous functions on $[a,\infty)$, $a > 0$ and $\dot{} = \mathrm{d}/\mathrm{d}s$.

In [4], the following estimates for positive solutions of (7) have been proved.

Proposition 1 ([4], Theorem 3.4)**.** *Assume $\lambda > 1$. Let y be a positive solution of* (7) *defined on $[a,b]$ and*

$$R(s) \geqslant r_*, \quad |Q_1(s)| \leqslant Q^2, \quad |Q_2(s)| \leqslant Q \tag{8}$$

for some constants $r_* > 0$ and $Q > 0$. Then we have the estimate

$$y(s) \leqslant L r_*^{-1/(\lambda-1)} \delta^{-3/(\lambda-1)}(s) \quad \text{for } s \in (a,b),$$

where L is a suitable constant which depends on λ and

$$\delta = \min\{s-a, \alpha\}, \quad \alpha = \frac{2^{-11}}{Q}.$$

This result can be extended to the non-compact interval in the following way.

Lemma 1. *Assume that* (8) *holds on $[a,\infty)$. Then any positive solution y of* (7) *satisfies*

$$y(s) \leqslant L r_*^{-1/(\lambda-1)} \alpha^{-3/(\lambda-1)} \quad \text{for } s \in [a+\alpha,\infty), \tag{9}$$

where L and α are as in Proposition 1.

P r o o f. Let $\alpha = 2^{-11}/Q$ and $b \geqslant \alpha + a$. By Proposition 1, applied on $[a,b]$, we have $\delta = \delta(s) \equiv \alpha$ for $s \in [a+\alpha, b]$ and (9) holds for $s \in [a+\alpha, b]$. Letting $b \to \infty$, we get the conclusion. $\square$

Lemma 2. *Let $f \in C^2([a,\infty))$, where $f(t) > 0$ is such that*

$$\int_a^\infty \sqrt{f(\sigma)}\,\mathrm{d}\sigma = \infty,$$

and consider the transformation

$$s = \int_a^t \sqrt{f(\sigma)}\,\mathrm{d}\sigma, \quad x(t) = y(s), \quad \dot{} = \frac{\mathrm{d}}{\mathrm{d}s}.$$

Then x is a solution of equation (1) *on $[a,\infty)$ if and only if y is a solution of the equation*

$$\dddot{y}(s) + \frac{3f'(t(s))}{2f^{3/2}(t(s))}\ddot{y}(s) + \Big(\frac{f''(t(s))}{2f^2(t(s))} - \frac{(f'(t(s)))^2}{4f^3(t(s))} + \frac{q(t(s))}{f(t(s))}\Big)\dot{y}(s) + \frac{r(t(s))}{f^{3/2}(t(s))}|y(s)|^\lambda \operatorname{sgn} y(s) = 0$$

on $[0,\infty)$, where $t = t(s)$ is the inverse function to $s = s(t)$.

Proof. We have

$$x'(t) = \dot{y}(s)\sqrt{f(t)}, \quad x''(t) = \ddot{y}(s)f(t) + \frac{\dot{y}(s)f'(t)}{2\sqrt{f(t)}},$$
$$x'''(t) = \dddot{y}(s)f^{3/2}(t) + \frac{3}{2}\ddot{y}(s)f'(t) + \dot{y}(s)\Big(\frac{f''(t)}{2\sqrt{f(t)}} - \frac{(f'(t))^2}{4f^{3/2}(t)}\Big).$$

By substitution into (1) we get the conclusion. $\square$

3. Super-linear equation

In this section we study (1) in the case $\lambda > 1$. Our first result is the following statement.

Theorem 1. *Let $\lambda > 1$. Assume that there exist constants $M > 0$ and $k \leqslant 2$ such that*

$$0 < q(t) \leqslant \frac{M}{t^k} \tag{10}$$

for large t and

$$\lim_{t\to\infty} t^{3k/2} r(t) = \infty. \tag{11}$$

Then every solution x of (1) *is either oscillatory or satisfies*

$$\lim_{t\to\infty} x(t) = 0. \tag{12}$$

P r o o f. Let x be a positive solution of (1) on $[a,\infty) \subset \mathbb{R}_+$, $a > 0$ and let (10) hold on $[a,\infty)$. Consider the transformation from Lemma 2 with $f(t) = t^{-k}$, i.e.,

$$\begin{aligned} s &= \Big(1-\frac{k}{2}\Big)^{-1}(t^{1-k/2} - a^{1-k/2}) \quad \text{if } k < 2,\\ s &= \ln\frac{t}{a} \quad \text{if } k = 2. \end{aligned}$$

Using Lemma 2, we transform equation (1) into

$$\begin{aligned} \dddot{y}(s) - \frac{3}{2}kt^{-1+k/2}\ddot{y}(s) + \Big(\frac{k}{4}(k+2)t^{k-2} + q(t)t^k\Big)\dot{y}(s)&\\ + r(t)t^{3k/2}|y(s)|^\lambda \operatorname{sgn} y(s) = 0.& \end{aligned} \tag{13}$$

This equation is of the form (7) with

$$Q_1(s) = \frac{k}{4}(k+2)t^{k-2}(s)+q(t)t^k, \quad Q_2(s) = -\frac{3}{2}kt^{-1+k/2}(s), \quad R(s) = r(t(s))t^{3k/2}(s)$$

where $t = (\frac{1}{2}(2-k)s + a^{1-k/2})^{2/(2-k)}$ for $k < 2$ and $t = a\exp s$ for $k = 2$. Obviously, there exists $Q > 0$ such that $|Q_1(s)| \leqslant Q^2$ and $|Q_2(s)| \leqslant Q$ for $s \in [a,\infty)$.

Let $n \in \mathbb{N}$ and let $\{a_n\}_{n=1}^\infty$ be an increasing sequence such that $a_1 \geqslant \max\{1,a\}$, $\lim\limits_{n\to\infty} a_n = \infty$ and

$$\varrho_n = \min_{t\in[a_n,\infty)} r(t)t^{3k/2} = r(a_n)a_n^{3k/2}, \quad n \in \mathbb{N}. \tag{14}$$

Put

$$s_n = \int_a^{a_n} t^{-k/2}\,\mathrm{d}t, \quad n \in \mathbb{N}.$$

Then $a_n = t(s_n)$ and $R(s) \geqslant \varrho_n$ for $s \in [s_n,\infty)$. Thus the assumption (8) is satisfied with $r_* = \varrho_n$. Now applying Lemma 1 to equation (13) on $[s_n,\infty)$ we get

$$y(s) \leqslant L\varrho_n^{-1/(\lambda-1)}\alpha^{-3/(\lambda-1)}, \quad s \in [s_n+\alpha,\infty),$$

where $\alpha = 2^{-11}/Q$ and L is a constant given by Proposition 1.

Using (11) and (14) we have

$$\lim_{n\to\infty} \varrho_n = \lim_{n\to\infty} r(a_n)a_n^{3k/2} = \infty,$$

thus

$$0 \leqslant \lim_{t\to\infty} x(t) = \lim_{s\to\infty} y(s) \leqslant L\alpha^{-3/(\lambda-1)} \lim_{n\to\infty} \varrho_n^{-1/(\lambda-1)} = 0,$$

i.e. (12) holds. □

Remark 1. The statement of the Theorem 1 holds if $\lambda > 1$ and one of the following conditions hold:

(i) there exists $q_0 > 0$ such that

$$q(t) \sim q_0, \quad \lim_{t\to\infty} r(t) = \infty;$$

(ii) there exist positive constants q_0, r_0 such that

$$q(t) \sim q_0 t^m, \quad r(t) \sim r_0 t^i, \quad \text{where } 0 < 3m < 2i,$$

where the symbol $f \sim g$ means $\lim\limits_{t\to\infty} f(t)/g(t) = 1$.

Our next result proves the statement of Theorem 1 under different conditions.

Theorem 2. *Let* $\lambda > 1$,

$$\lim_{t\to\infty} \frac{r(t)}{q^{3/2}(t)} = \infty, \tag{15}$$

and let there exist constants c_1 *and* c_2 *such that for* $t \in \mathbb{R}_+$

$$\frac{|q'(t)|}{q^{3/2}(t)} \leqslant c_1, \quad \frac{|q''(t)|}{q^2(t)} \leqslant c_2. \tag{16}$$

Then the conclusion of Theorem 1 holds.

P r o o f. Let x be a positive solution of (1) on $[a,\infty)$ $(a \geqslant 0)$. Consider the transformation from Lemma 2 with $f = q$, i.e.,

$$s = \int_a^t \sqrt{q(\sigma)}\,\mathrm{d}\sigma, \quad x(t) = y(s), \quad {}^{\cdot} = \frac{\mathrm{d}}{\mathrm{d}s}.$$

Using Lemma 2 we have

$$x'(t) = \dot{y}(s)\sqrt{q(t)}, \qquad x''(t) = \ddot{y}(s)q(t) + \frac{\dot{y}(s)q'(t)}{2\sqrt{q(t)}},$$
$$x'''(t) = \dddot{y}(s)q^{3/2}(t) + \frac{3}{2}\ddot{y}(s)q'(t) + \dot{y}(s)\Big(\frac{q''(t)}{2\sqrt{q(t)}} - \frac{(q'(t))^2}{4q^{3/2}(t)}\Big).$$

By substitution into (1) we get

$$\dddot{y}(s) + \frac{3q'(t)}{2q^{3/2}(t)}\ddot{y}(s) + \Big(\frac{q''(t)}{2q^2(t)} - \frac{(q'(t))^2}{4q^3(t)} + 1\Big)\dot{y}(s) + \frac{r(t)}{q^{3/2}(t)}|y(s)|^{\lambda}\operatorname{sgn} y(s) = 0, \tag{17}$$

where $t = t(s)$ is the inverse function to $s = s(t)$. This equation is of the form (7) with

$$Q_1(s) = \frac{q''(t)}{2q^2(t)} - \frac{(q'(t))^2}{4q^3(t)} + 1, \quad Q_2(s) = \frac{3q'(t)}{2q^{3/2}(t)}, \quad R(s) = \frac{r(t)}{q^{3/2}(t)},$$

where $t = t(s)$.

Let $n \in \mathbb{N}$ and let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence such that $a_n \in [a, \infty)$, $\lim_{n\to\infty} a_n = \infty$ and

$$\frac{r(a_n)}{q^{3/2}(a_n)} = \min_{t\in[a_n,\infty)} \frac{r(t)}{q^{3/2}(t)}. \tag{18}$$

Put

$$s_n = \int_a^{a_n} \sqrt{q(\sigma)}\,\mathrm{d}\sigma, \quad \varrho_n = \frac{r(a_n)}{q^{3/2}(a_n)}, \quad Q = \max\Big\{\frac{3}{2}c_1, \Big(\frac{c_2}{2} + \frac{c_1^2}{4} + 1\Big)^{1/2}\Big\}.$$

Then by (16) we have $|Q_1(s)| \leqslant Q^2$, $|Q_2(s)| \leqslant Q$ for $s \in [a, \infty)$, $a_n = t(s_n)$ and $R(s) \geqslant \varrho_n$ for $s \in [s_n, \infty)$. Thus the assumption (8) is satisfied with $r_* = \varrho_n$. Applying Lemma 1 to equation (17) on $[s_n, \infty)$, we get

$$y(s) \leqslant L\varrho_n^{-1/(\lambda-1)}\alpha^{-3/(\lambda-1)}, \quad s \in [s_n + \alpha, \infty),$$

where $\alpha = 2^{-11}/Q$ and L is given by Proposition 1.

Moreover, using (15) and (18) we have

$$\lim_{n\to\infty} \varrho_n = \lim_{n\to\infty} \frac{r(a_n)}{q^{3/2}(a_n)} = \infty. \tag{19}$$

Using the same argument as in the proof of Theorem 1, we get $\lim_{t\to\infty} x(t) = 0$. $\square$

Remark 2. If $q(t) = q_0 t^{-k}$ ($k \leqslant 2$), then Theorem 1 follows from Theorem 2. However, in general, conditions (10) and (16) are independent.

Theorems 1, 2 do not require that equation (2) be oscillatory. Both the theorems cover the limiting case for oscillation of equation (2), for example when (2) is the Euler equation. In this case the assumptions of Theorem A are not satisfied, thus Theorems 1, 2 extend Theorem A.

Theorems 1, 2 also complete [7], Theorems 4.1, 5.1, where it is proved that if (5) and (6) hold, then every nonoscillatory solution x of (1) satisfies $\limsup_{t\to\infty} |x^{(i)}(t)| < \infty$, $i = 0, 1, 2$.

4. Sub-linear and linear equations

Now we study (1) in the case $\lambda \leqslant 1$.

Definition. We say that a solution x of (1) is a *Kneser solution* if (3) holds.

Obviously, if x is a Kneser solution of (1), then $\lim_{t\to\infty} x'(t) = 0$.
Theorems 1 and 2 can be extended to $\lambda \leqslant 1$ in the following way.

Theorem 3. *Let* $\lambda \leqslant 1$ *and let either* (10), (11) *or* (15), (16) *hold.*
Then every nonoscillatory solution of equation (1) *is either unbounded or tends to zero as* $t \to \infty$.

Proof. Let x be a bounded positive solution of (1) on $[t_0, \infty)$. Then $v = x$ is a solution of the equation

$$v'''(t) + q(t)v'(t) + R(t)v^3(t) = 0, \tag{20}$$

where

$$R(t) = \frac{r(t)}{(x(t))^{3-\lambda}} \geqslant Kr(t) \quad \text{for } t \geqslant t_0,$$

where $K > 0$ is a suitable constant. We can apply Theorem 1 or 2 to (20) and get the conclusion. □

Remark 3. Theorem 3 extends [9], Theorem 4, where conditions ensuring that every Kneser solution of (1) tends to zero as $t \to \infty$ are given.

In the sequel, we assume that equation (2) is oscillatory. Theorem 3 can be completed in the following way.

Theorem 4. *Let* (2) *be oscillatory and let the assumptions of Theorem* 3 *hold. In addition, assume either*

$$q'(t) \leqslant 0 \quad \textit{for large } t \tag{21}$$

or

$$\limsup_{t\to\infty} \frac{q'(t)}{r(t)} = 0. \tag{22}$$

Then every nonoscillatory solution of (1) *which tends to zero as* $t \to \infty$ *is a Kneser solution.*

Our main result in this section is the following statement.

Theorem 5. *Let* (2) *be oscillatory. Let* $\lambda \leqslant 1$, $q'(t) \leqslant 0$ *for* $t \geqslant 0$ *and let either* (10), (11) *or* (15), (16) *hold. In addition, when* $\lambda < 1$, *assume that there exists* $M > 0$ *such that*

$$r(t) \geqslant M q^{1/2}(t) \quad \text{for large } t. \tag{23}$$

Then every solution of (1) *which has a zero is oscillatory, and every nonoscillatory solution is a Kneser solution and tends to zero as* $t \to \infty$.

To prove Theorems 4, 5, the following lemmas will be needed.

Lemma 3 ([8], Proposition 2). *Let* (2) *be oscillatory. Then any nonoscillatory solution* x *of* (1) *is the one of the following types:*

(a) $x(t)x'(t) \leqslant 0$ *for large* t;
(b) x' *changes its sign for large* t.

Lemma 4. *Assume* $\lambda \leqslant 1$. *Let* x *be a nonoscillatory solution of* (1) *and*

$$F(t) = -x(t)x''(t) + \frac{1}{2}x'^2(t) - \frac{1}{2}q(t)x^2(t). \tag{24}$$

(i) *If either* (21) *or* (22) *holds and* x *is a bounded solution, then the function* F *is nonpositive and nondecreasing for large* t.
(ii) *If* $q'(t) \leqslant 0$ *for* $t \geqslant 0$, *then the function* F *is nonpositive and nondecreasing for* $t \geqslant 0$.

P r o o f. First we prove monotonicity of F. We have for $t \geqslant 0$

$$F'(t) = r(t)|x(t)|^{\lambda+1} - \frac{1}{2}q'(t)x^2(t) = x^2(t)\Big(\frac{r(t)}{|x(t)|^{1-\lambda}} - \frac{q'(t)}{2}\Big). \tag{25}$$

If $q'(t) \leqslant 0$ for $t \geqslant 0$ (for t large), then $F'(t) \geqslant 0$ for $t \in \mathbb{R}_+$ (for t large). Suppose (22). Then for any nonoscillatory solution x of (1) there exist t_0 and M such that $0 < x(t) \leqslant M$ and

$$\frac{r(t)}{x^{1-\lambda}(t)} - \frac{q'(t)}{2} \geqslant M^{\lambda-1}r(t) - \frac{q'(t)}{2} \geqslant \frac{r(t)}{2}\Big(2M^{\lambda-1} - \frac{q'(t)}{r(t)}\Big) \geqslant \frac{M^{\lambda-1}}{2}r(t)$$

for $t \geqslant t_0$. Hence, (25) implies that F is nondecreasing for large t.

Now we prove that F is nonpositive for large t or for $t \geqslant 0$ in claim (i) or (ii), respectively. By Lemma 3 any eventually positive solution of (1) is of type (a) or (b). First note that any solution of type (a) satisfies either $x''(t) > 0$ or x'' is oscillatory, i.e., x'' has infinitely many zeros. Indeed, if $x''(t) < 0$ for large t, then x' is negative decreasing for large t and x becomes negative.

Let x be a positive solution of type (a) and $x''(t) > 0$ for large t. Then x' is increasing and negative and $\lim\limits_{t\to\infty} x'(t) = 0$, otherwise x becomes negative for large t. Hence, $\lim\limits_{t\to\infty} F(t) \leqslant 0$ and F is nonpositive.

Let x be a positive solution of type (a) and let x'' be oscillatory. Let $\{t_k\}$ be an increasing sequence tending to infinity such that x' has local maxima at t_k, $k \in \mathbb{N}$. Since $x'(t) \leqslant 0$, we get

$$\lim_{k\to\infty} x'(t_k) = 0,$$

otherwise x becomes negative for large t. Since $x''(t_k) = 0$, we have from (24) that

$$\lim_{k\to\infty} F(t_k) \leqslant 0.$$

Thus, in view of the monotonicity, $\lim\limits_{t\to\infty} F(t) \leqslant 0$ and F is nonpositive.

Let x be a positive solution of type (b). Then x' changes its sign and x'' has infinitely many zeros. Let $\{t_k\}$ and $\{\tau_k\}$ be increasing sequences tending to infinity such that $t_k < \tau_k$, x' has local minima at t_k,

$$x'(t_k) < 0, \quad x'(\tau_k) = 0, \quad x''(t) > 0, \quad t \in (t_k, \tau_k).$$

This implies

$$x'(t) < 0, \quad t \in [t_k, \tau_k), \quad x''(t_k) = 0,$$

and there exists a sequence $\{\xi_k\}$ such that $t_k < \xi_k < \tau_k$ and

$$-x(\xi_k)x''(\xi_k) + \frac{1}{2}(x'(\xi_k))^2 = 0, \quad k \in \mathbb{N}.$$

From this equation and (24) we have $F(\xi_k) < 0$, $k \in \mathbb{N}$ and by the monotonicity of F we get the conclusion. $\square$

Proof of Theorem 4. Let x be a nonoscillatory solution of (1) which tends to zero as $t \to \infty$. We prove that it is a Kneser solution. First, suppose that $x'''(t) \neq 0$ for large t. Then by Lemma 3 we have $x'(t) < 0$ and $x''(t) > 0$ for large t.

Assume that x''' is oscillatory. If $x''(t) > 0$ for large t, then the conclusion holds. Let x'' be oscillatory. Consider a function F defined by (24). By Lemma 4 (i) there exists $t_0 \geqslant 0$ such that F is nonpositive and nondecreasing on $[t_0, \infty)$. Let $\{t_k\}_{k=1}^{\infty}$ be an increasing sequence such that $t_0 \leqslant t_1$, x'' has local minima at t_k, $k = 1, 2, \ldots$, and $x''(t_k) \leqslant 0$. Hence, $x'''(t_k) = 0$ and from equation (1) we have

$$-x'(t_k) = \frac{r(t_k)}{q(t_k)} x^{\lambda}(t_k), \quad k \in \mathbb{N}. \tag{26}$$

Moreover, (24) and (26) imply

$$0 \geqslant 2F(t_k) \geqslant x'^2(t_k) - q(t_k)x^2(t_k) = \Big(\frac{r(t_k)}{q(t_k)}\Big)^2 x^{2\lambda}(t_k) - q(t_k)x^2(t_k)$$

for $k \in \mathbb{N}$. This and (15) yield

$$x(t_k) \geqslant \Big(\frac{r(t_k)}{q^{3/2}(t_k)}\Big)^{1/(1-\lambda)}.$$

Letting $k \to \infty$, we get a contradiction with the boundedness of x. $\square$

Proof of Theorem 5. Let x be a solution of (1) and let F be defined by (24). If $x(t_0) = 0$ for some $t_0 \in \mathbb{R}_+$, then $F(t_0) \geqslant 0$. If x is nonoscillatory, then by Lemma 4 (ii) we get $F(t) < 0$ for $t \geqslant t_0$, a contradiction. Hence x must be oscillatory.

Now we prove that if x is nonoscillatory, then it is a Kneser solution and tends to zero as $t \to \infty$. If x is a bounded solution, then by Theorem 3 we have $\lim\limits_{t\to\infty} x(t) = 0$ and by Theorem 4 x is a Kneser solution.

Assume that x is an unbounded solution. According to Lemma 3, such solutions are of type (b). Let $\{t_k\}_{k=1}^{\infty}$ be an increasing sequence such that x has local maxima at t_k and

$$\lim_{k\to\infty} x(t_k) = \infty.$$

Then $x'(t_k) = 0$ and $x''(t_k) \leqslant 0$. Moreover, equation (1) gives $x'''(t_k) < 0$. Let $\{\tau_k\}_{k=1}^{\infty}$ be the sequence such that $t_k < \tau_k$ and τ_k is the first zero of x''' lying to the right of t_k, $k = 1, 2, \ldots$. Denote

$$\Delta_k = (t_k, \tau_k), \quad k \in \mathbb{N}.$$

Then x' and x'' are negative decreasing, and x is positive decreasing on Δ_k. Consider the function F defined by (24). Then by Lemma 4, $F(t) \leqslant 0$ for $t \in \Delta_k$ and

$$|x''(t)| \leqslant \frac{q(t)}{2} x(t), \quad |x'(t)| \leqslant \sqrt{q(t)}\, x(t), \quad t \in \Delta_k. \tag{27}$$

Moreover, from (25) we have

$$\int_a^{\infty} r(t) x^{\lambda+1}(t)\,\mathrm{d}t \leqslant F(\infty) - F(a) < \infty \tag{28}$$

for any $a > 0$, and from (1)

$$|x'(\tau_k)| = \frac{r(\tau_k)}{q(\tau_k)} x^{\lambda}(\tau_k). \tag{29}$$

Hence, (27) and (29) imply

$$\frac{r(\tau_k)}{q(\tau_k)}x^{\lambda}(\tau_k) = |x'(\tau_k)| \leqslant q^{1/2}(\tau_k)x(\tau_k).$$

Now, if $\lambda = 1$, then

$$1 \geqslant \frac{r(\tau_k)}{q^{3/2}(\tau_k)}$$

and letting $k \to \infty$ we get a contradiction with (15) or (10), (11). Hence, (1) has no unbounded nonoscillatory solution.

Let $\lambda < 1$. Then

$$x(\tau_k) \geqslant \Big(\frac{r(\tau_k)}{q^{3/2}(\tau_k)}\Big)^{1/(1-\lambda)}, \tag{30}$$

so from (27), (29)

$$\begin{aligned}\frac{r(\tau_k)}{q(\tau_k)}x^{\lambda}(\tau_k) = |x'(\tau_k)| &= \int_{\Delta_k} |x''(s)|\,\mathrm{d}s \leqslant |x''(\tau_k)|(\tau_k - t_k)\\ &\leqslant \frac{q(\tau_k)}{2}x(\tau_k)(\tau_k - t_k)\end{aligned}$$

and this implies

$$\Delta_k = \tau_k - t_k \geqslant 2\frac{r(\tau_k)}{q^2(\tau_k)}x^{\lambda-1}(\tau_k). \tag{31}$$

Put $\xi_k = \min\limits_{t\in\Delta_k} r(t)$. From (30), (31) we get

$$\begin{aligned}\int_{\Delta_k} r(t)x^{\lambda+1}(t)\,\mathrm{d}t \geqslant r(\xi_k)x^{\lambda+1}(\tau_k)\Delta_k &\geqslant 2r(\xi_k)\frac{r(\tau_k)}{q^2(\tau_k)}x^{2\lambda}(\tau_k)\\ &\geqslant 2r(\xi_k)\frac{r(\tau_k)}{q^2(\tau_k)}\Big(\frac{r(\tau_k)}{q^{3/2}(\tau_k)}\Big)^{2\lambda/(1-\lambda)}.\end{aligned}$$

The function $r(t)/q^{3/2}(t)$ is bounded away from zero, and so, in view of (23),

$$\begin{aligned}\int_a^{\infty} r(t)x^{\lambda+1}(t)\,\mathrm{d}t &\geqslant \sum_{k=1}^{\infty}\int_{\Delta_k} r(t)x^{\lambda+1}(t)\,\mathrm{d}t\\ &\geqslant 2\sum_{k=1}^{\infty} r(\xi_k)\frac{r(\tau_k)}{q^2(\tau_k)}\Big(\frac{r(\tau_k)}{q^{3/2}(\tau_k)}\Big)^{2\lambda/(1-\lambda)}\\ &\geqslant 2\min_{s\geqslant t_1}\Big(\frac{r(s)}{q^{3/2}(s)}\Big)^{2\lambda/(1-\lambda)+1}\sum_{k=1}^{\infty}\frac{r(t_k)}{q^{1/2}(t_k)}\\ &\geqslant 2M\min_{s\geqslant t_1}\Big(\frac{r(s)}{q^{3/2}(s)}\Big)^{(\lambda+1)/(1-\lambda)}\sum_{k=1}^{\infty}1 = \infty,\end{aligned}$$

which contradicts (28). Thus (1) has no unbounded solutions. □

Remark 4. Theorem 5 extends [7], Theorem 5.3, where it is proved that if $\lambda \leqslant 1$, (6), and $r(t) \geqslant M > 0$ for large t, then any nonoscillatory solution x of (1) satisfies $\lim_{t\to\infty} x^{(i)}(t) = 0$, $i = 0, 1, 2$.

We conclude this section with an application of our results to the linear equation

$$x'''(t) + q(t)x'(t) + r(t)x(t) = 0, \quad t \geqslant 0. \tag{L}$$

This equation has a nonoscillatory solution if $q'(t) \leqslant 0$ for $t \geqslant 0$, see [13], Theorem 1.13. Thus, applying Theorem 5 to (L) we get the existence of a Kneser solution for (L).

Corollary 1. *Let $q'(t) \leqslant 0$ for $t \geqslant 0$ and let either* (10), (11) *or* (15), (16) *hold. Then* (L) *has both oscillatory and nonoscillatory solutions and every nonoscillatory solution tends to zero as $t \to \infty$ and satisfies* (3) *for all $t \geqslant 0$.*

Remark 5. Corollary 1 extends [9], Theorem 2, where the existence of a Kneser solution vanishing at infinity was studied.

The following examples illustrate our results.

Example 1. Consider the equation

$$x'''(t) + \frac{1}{t^2}x'(t) + 7t^{\lambda-4}|x|^{\lambda}(t)\operatorname{sgn} x(t) = 0, \quad t \geqslant 1, \tag{32}$$

where $\lambda \geqslant 1$. If $\lambda > 1$, applying Theorem 1 with $k = 2$ we get that every nonoscillatory solution of (32) is vanishing at infinity. If $\lambda = 1$, then by Theorem 5 every solution with a zero is oscillatory and every nonoscillatory solution is a Kneser solution vanishing at infinity. One can check that $x(t) = 1/t$ is a solution of (32) with $\lambda = 1$.

Example 2. Consider the equation

$$x'''(t) + \frac{1}{t}x'(t) + \frac{1}{\sqrt{t}}|x|^{\lambda}(t)\operatorname{sgn} x(t) = 0, \quad t \geqslant 1, \tag{33}$$

where $\lambda \leqslant 1$. By Theorem 5, every solution of (33) with a zero is oscillatory and every nonoscillatory solution is a Kneser solution vanishing at infinity.

Example 3. Consider the equation

$$x'''(t) + e^t x'(t) + e^{2t}|x|^{\lambda}(t)\operatorname{sgn} x(t) = 0, \tag{34}$$

where $\lambda > 1$. By Theorem 2, every nonoscillatory solution of (34) is vanishing at infinity. Observe that Theorem 1 is not applicable to this equation.

Concluding remarks. (1) Sometimes the assumption $q \in C^2$ can be weakened. In particular, Theorem 1 holds under a weaker assumption $q \in C$ and Theorems 3, 4 with assumptions (10), (11) hold under $q \in C$ and Theorem 5 under $q \in C^1$.

(2) Sansone (see [13]) constructed a function q such that $q(t) > 0$ and the linear equation (L) with $r(t) = q'(t)$ has all solutions oscillatory. It is an interesting problem to find conditions for all solutions of (1) to be oscillatory. In view of Theorems 1, 5 this problem reduces to the nonexistence of nonoscillatory solutions tending to zero as $t \to \infty$.

(3) After this paper was written, the comprehensive monograh [18] concerning oscillatory and asymptotic properties of solutions for various types of third order differential equations has been published. Theorems 1–5 extend results of Sections 2.5 and 4.1 of [18]. In particular, Theorem 5 extends [18] , Theorems 4.1.14–4.1.20, where it is assumed that $q(t) \leqslant 0$, and Corollary 1 extends a result of S. Padhi [18], Theorem 2.5.12, where it is assumed that $r(t) \geqslant M > 0$ for large t.

Acknowledgement. The authors would like to thank the referee for helpful suggestions to the paper.

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