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EXISTENCE AND CONTROLLABILITY FOR NONDENSELY DEFINED PARTIAL NEUTRAL FUNCTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. We give sufficient conditions for the existence of integral solutions for a class of neutral functional differential inclusions. The assumptions on the generator are reduced by considering nondensely defined Hille-Yosida operators. Existence and controllability results are established by combining the theory of addmissible multivalued contractions and Frigon's fixed point theorem. These results are applied to a neutral partial differential inclusion with diffusion.

Keywords: nondensely operator; neutral differential inclusion; multivalued map; fixed point; controllability; C_0-semigroup

MSC 2010: 34A60, 34K35, 93B05

1. Introduction

The aim of this paper is to establish several results on the existence of integral solutions of the partial neutral functional differential inclusion

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}[y(t)-f(t,y_t)] - A[y(t)-f(t,y_t)] \in F(t,y_t) \quad \text{for a.e. } t \in J=[0,a], \\ y_0 = \varphi \in C = C([-r,0];E), \end{cases} \tag{1.1}$$

and on the controllability of the partial neutral functional differential inclusion

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}[y(t)-f(t,y_t)] - A[y(t)-f(t,y_t)] \in F(t,y_t) + Bu(t) \quad \text{for a.e. } t \in J, \\ y_0 = \varphi \in C, \end{cases} \tag{1.2}$$

where A is a nondensely defined linear operator on a Banach space E, C is the space of continuous functions from $[-r,0]$ to E endowed with the uniform norm topology.

For $t \geqslant 0$, as usual, the function $y_t \in C$ is defined by

$$y_t(\theta) = y(t+\theta) \quad \text{for } \theta \in [-r, 0].$$

$F\colon J \times C \to \mathcal{P}(E)$ is a multivalued map with compact values, $f\colon J \times C \to E$ is a continuous function, B is a bounded linear operator defined from a Banach space $\mathcal{U}$ into E and $u(\cdot) \in L^2(J,\mathcal{U})$.

In the literature, there has been much current interest in studying neutral partial functional differential equations either if A satisfies all the conditions of the Hille-Yosida Theorem or A is not necessarily densely defined. When A is the infinitesimal generator of a strongly continuous semigroup on X, we refer for instance to [20], [21], [19], [13], [17], [16], [18] while when A is a Hille-Yosida operator, we refer for instance to [3] and [4].

For semilinear functional differential inclusions, Benchohra and Ouahab [9] used Frigon's fixed point theorem [12] to study the controllability of the neutral functional differential inclusion

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}[y(t) - f(t, y_t)] - Ay(t) \in F(t, y_t) + Bu(t) & \text{for a.e. } t \in [0, \infty), \\ y_0 = \varphi \in C, \end{cases} \tag{1.3}$$

where A is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ of bounded linear operators on E and F, u, B are as in (1.2) with $J = [0, \infty)$.

We note that the existence and controllability results obtained in [9] rely on some assumptions on the semigroup $(T(t))_{t\geqslant 0}$ and its generator A which imply that A is a bounded operator and E is a finite dimensional space (see [15]).

For partial functional differential inclusions with nondensely defined operators, we refer to the work of Henderson and Ouahab [14] in which the authors studied the existence of integral solutions for the semilinear functional differential inclusion

$$\begin{cases} y'(t) - Ay(t) \in F(t, y_t) & \text{for a.e. } t \in [0, \infty), \\ y_0 = \varphi \in C, \end{cases}$$

and discussed the existence of integral solutions of the problem

$$\begin{cases} y'(t) - Ay(t) \in F(t, y_t) + Bu(t) & \text{for a.e. } t \in [0, \infty), \\ y_0 = \varphi. \end{cases}$$

More recently, it has been shown that the density condition is not necessary for dealing with the existence of integral solutions and the controllability for many classes of both the functional differential equations (see [7], [5], [8]) and the semilinear functional differential inclusions (see [1], [2], [6]).

The purpose of this work is to show that the boundedness of the operator A and the density of its domain are not needed to get results on the existence of integral solutions and controllability even if we work with neutral and multivalued partial functional differential inclusions.

It should be pointed out that in [14], the authors assumed that the operator B takes values in $\overline{D(A)}$, and then the linear operator W defined on $L^2([0,n];\mathcal{U})$ $(n > 0)$ by

$$Wu = \int_0^n T_0(n-s)Bu(s)\,\mathrm{d}s$$

is forced to take values in $\overline{D(A)}$. Without assuming those conditions on B and W, we give a generalization to partial neutral functional differential inclusions.

This work is organized as follows. In Section 2, we recall some preliminary results on multivalued analysis. In Section 3, we extend the existence result obtained in [14] to partial neutral functional differential inclusions with Hille-Yosida operators of the form (1.1). In Section 4, we study the controllability of (1.2). The last section is devoted to the study of some reaction-diffusion inclusions.

2. Preliminary results on multivalued mappings

In this section, we recall some results on multivalued functions and on the non-linear alternative for multivalued admissible contractions in Fréchet spaces due to Frigon [12].

Given a space X, a directed set Λ, and a metrics d_α, $\alpha \in \Lambda$ on X, define

$$\begin{aligned}
\mathcal{P}(X) &= \{Y \subset X\colon\ Y \neq \emptyset\},\\
\mathcal{P}_{\rm cl}(X) &= \{Y \in \mathcal{P}(X)\colon\ Y \text{ closed}\},\\
\mathcal{P}_{\rm cp}(X) &= \{Y \in \mathcal{P}(X)\colon\ Y \text{ compact}\},
\end{aligned}$$

and denote by $D_\alpha, \alpha \in \Lambda$ the Hausdorff pseudometric induced by d_α:

$$\begin{aligned}
D_\alpha(A,B) = \inf\{\varepsilon > 0\colon\ &\text{for all } x \in A,\ y \in B,\ \text{there exist } \overline{x} \in A,\ \overline{y} \in B\\
&\text{such that } d_\alpha(x,\overline{y}) \leqslant \varepsilon,\ d_\alpha(\overline{x},y) \leqslant \varepsilon\}
\end{aligned}$$

with $\inf \emptyset = 1$.

Definition 2.1. A multivalued map $F\colon X \to \mathcal{P}(E)$ is called an admissible contraction with constants $\{k_\alpha\}_{\alpha\in\Lambda}$ if for each $\alpha \in \Lambda$ there exists $k_\alpha \in (0,1)$ such that

(i) $D_\alpha(F(x), F(y)) \leqslant k_\alpha d_\alpha(x, y)$ for all $x, y \in X$,
(ii) for every $x \in X$ and every $\varepsilon \in (0, \infty)^\Lambda$ there exists $y \in F(x)$ such that

$$d_\alpha(x, y) \leqslant d_\alpha(x, F(x)) + \varepsilon_\alpha \quad \text{for all } \alpha \in \Lambda.$$

The following result gives sufficient conditions for the existence of a fixed point for admissible multivalued contractions.

Theorem 2.2 ([12]). *Let X be a Fréchet space and V an open neighborhood with its origin in X and let $N\colon \overline{V} \to P(X)$ be an admissible multivalued contraction. Assume that N is bounded, then one of the following statements holds:*
(C_1) *N has a fixed point,*
(C_2) *there exists $\lambda \in [0, 1)$ and $x \in \partial V$ such that $x \in \lambda N(x)$.*

To apply Theorem 2.2, we consider $H_d\colon \mathcal{P}(E) \times \mathcal{P}(E) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\Big(\sup_{a\in A} d(a, B),\ \sup_{b\in B} d(A, b)\Big),$$

where $d(A, b) = \inf_{a\in A} d(a, b)$ and $d(a, B) = \inf_{b\in B} d(a, b)$.

Then the space $(\mathcal{P}_{\rm cl}(X), H_d)$ is a generalized metric space.

For compact valued measurable multifunctions, we have the following result.

Proposition 2.3 ([10]). *If Γ_1 and Γ_2 are compact valued measurable multifunctions then the multifunction $t \to \Gamma_1(t) \cap \Gamma_2(t)$ is measurable.*

Theorem 2.4 ([10]). *Let X be a separable metric space, $(T, \mathcal{T})$ a mesurable space, Γ a multifunction from T to complete nonempty subsets of X. If for each open set V in X, $\Gamma^-(V) = \{t\colon \Gamma(t) \cap V \neq \emptyset\}$ belongs to $\mathcal{T}$, then Γ admits a measurable selection.*

3. Integral solutions

In this section, we establish sufficient conditions for the existence of integral solutions for problem (1.1). We assume that A satisfies the following hypothesis:

(H_1) A is a Hille-Yosida operator, namely, there exist $M_0 \geqslant 0$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \varrho(A)$ and

$$\|(\lambda - A)^{-n}\| \leqslant \frac{M_0}{(\lambda-\omega)^n} \quad \text{for } n \in \mathbb{N} \text{ and } \lambda > \omega,$$

where $\varrho(A)$ is the resolvent set of A.

In the sequel, we introduce the part A_0 of A in $\overline{D(A)}$ defined as

$$D(A_0) = \{x \in D(A),\ Ax \in \overline{D(A)}\},$$
$$A_0 x = Ax \quad \text{for } x \in D(A_0).$$

It is well known that A_0 generates a strongly continuous semigroup $(T_0(t))_{t\geqslant 0}$ on $\overline{D(A)}$.

We define now integral solutions of (1.1).

Definition 3.1. A continuous function $y\colon [-r,a] \to E$ is called an integral solution of (1.1) if there exists a function $g \in S_{F,y} = \{g \in L^1(J,E)\colon\ g(t) \in F(t,y_t)$ for a.e. $t \in J\}$ such that

(i) $\int_0^t (y(s) - f(s,y_s))\,\mathrm{d}s \in D(A)$,

(ii) $y(t) = f(t,y_t) + (\varphi(0) - f(0,\varphi)) + A\int_0^t (y(s) - f(s,y_s))\,\mathrm{d}s + \int_0^t g(s)\,\mathrm{d}s$ for $0 \leqslant t \leqslant a$,

(iii) $y_0 = \varphi$.

Remark 3.2. One can observe that if y is an integral solution of (1.1) then for all $t \in [0,a]$, $y(t) - f(t,y_t) \in \overline{D(A)}$. In fact, $t^{-1}\int_0^t (y(s) - f(s,y_s))\,\mathrm{d}s \in D(A)$ and $t^{-1}\int_0^t (y(s) - f(s,y_s))\,\mathrm{d}s$ goes to $y(t) - f(t,y_t)$ as t goes to 0. In particular, we get $\varphi(0) - f(0,\varphi) \in \overline{D(A)}$.

Under additional conditions, we will show that $\varphi(0) - f(0,\varphi) \in \overline{D(A)}$ is also sufficient for obtaining the existence of at least one integral solution of (1.1).

If an integral solution of (1.1) exists, then it is given as in [4] by

$$\begin{aligned} y(t) = f(t,y_t) &+ T_0(t)(\varphi(0) - f(0,\varphi)) \\ &+ \lim_{\lambda\to\infty}\int_0^t T_0(t-s)A_\lambda f(s,y_s)\,\mathrm{d}s \\ &+ \lim_{\lambda\to\infty}\int_0^t T_0(t-s)A_\lambda g(s)\,\mathrm{d}s, \end{aligned}$$

where $A_\lambda = \lambda(\lambda - A)^{-1}$.

In the sequel, we assume that the function $F\colon\ J \times C \to \mathcal{P}(E)$ is a Carathéodory function, namely,

(i) $t \to F(t,\varphi)$ is measurable for each $\varphi \in C$,

(ii) $\varphi \to F(t,\varphi)$ is continuous for almost all $t \in [0,a]$,

(iii) for each $q > 0$, there exists $h_q \in L^1([0,a];\mathbb{R}_+)$ such that

$$\|F(t,\varphi)\| = \sup\{\|g\|,\ g \in F(t,\varphi)\} \leqslant h_q(t) \quad \text{for all } \|\varphi\| < q \text{ and for almost all } t \in [0,a].$$

To study the existence of integral solutions, we add the following assumptions:
(H_2) There exists $K_1 < 1$ such that

$$\|f(t,\varphi_1) - f(t,\varphi_2)\| \leqslant K_1\|\varphi_1 - \varphi_2\| \quad \text{for } t \in [0,a] \text{ and } \varphi_1, \varphi_2 \in C.$$

(H_3) There exist $\psi\colon [0,\infty) \to (0,\infty)$ continuous and nondecreasing and $p \in L^1([0,a];\mathbb{R}_+)$ such that

$$\|F(t,\varphi)\| \leqslant p(t)\psi(\|\varphi\|) \quad \text{for a.e. } t \in [0,a] \text{ and } \varphi \in C \text{ with } \int_1^\infty \frac{\mathrm{d}s}{s+\psi(s)} = \infty.$$

(H_4) There exists $l_a \in L^1([0,a];\mathbb{R}_+)$ such that

$$H_d(F(t,\varphi_1),F(t,\varphi_2)) \leqslant l_a(t)\|\varphi_1 - \varphi_2\| \quad \text{for } t \in [0,a] \text{ and } \varphi_1,\varphi_2 \in C,$$

and

$$d(0,F(t,0)) \leqslant l_a(t) \quad \text{for a.e. } t \in [0,a].$$

Remark 3.3. As an immediate consequence of assumption (H_2), we have the estimate

$$\|f(t,\varphi)\| \leqslant K_1\|\varphi\| + K_2 \quad \text{for all } t \in [0,a] \text{ and } \varphi \in C, \tag{3.1}$$

where $K_2 = \sup\limits_{t\in[0,a]} \|f(t,0)\|$.

We give now our main existence result.

Theorem 3.4. *Assume that* (H_1)–(H_4) *are satisfied and let* φ *be such that* $\varphi(0) - f(0,\varphi) \in \overline{D(A)}$. *Then problem* (1.1) *has at least one integral solution on* $[0,a]$.

Proof. To prove Theorem 3.4, we consider the operator $N\colon C([-r,a];E) \to \mathcal{P}(C([-r,a];E))$ defined as

$$N(y)(t) = \begin{cases} \varphi(t) \quad \text{for } t \in [-r,0] \\ f(t,y_t) + T_0(t)(\varphi(0) - f(0,\varphi)) + \lim\limits_{\lambda\to\infty} \displaystyle\int_0^t T_0(t-s)A_\lambda f(s,y_s)\,\mathrm{d}s \\ \qquad + \lim\limits_{\lambda\to\infty} \displaystyle\int_0^t T_0(t-s)A_\lambda g(s)\,\mathrm{d}s \quad \text{for } t \in [0,a], \end{cases} \tag{3.2}$$

where $g \in S_{F,y} = \{g \in L^1(J,E)\colon\ g(t) \in F(t,y_t) \quad \text{for a.e. } t \in J\}$.

We will show that N has a fixed point which is then an integral solution of (1.1).

Claim 1: Let y be a solution of (1.1). Then there exists $c_1 > 0$ such that $\|y\| \leqslant c_1$. Indeed, there exists $g \in S_{F,y}$ such that for each $t \in [0,a]$,

$$
\begin{aligned}
y(t) = {} & f(t, y_t) + T_0(t)(\varphi(0) - f(0,\varphi)) \\
& + \lim_{\lambda\to\infty}\int_0^t T_0(t-s)A_\lambda f(s,y_s)\,\mathrm{d}s \\
& + \lim_{\lambda\to\infty}\int_0^t T_0(t-s)A_\lambda g(s)\,\mathrm{d}s.
\end{aligned} \tag{3.3}
$$

Without loss of generality, we assume that $M_0 = 1$ and $\omega > 0$. Then

$$
\|T_0(t)\| \leqslant \mathrm{e}^{\omega t} \quad \text{for } t \geqslant 0. \tag{3.4}
$$

By using (3.1), (3.4) and assumption (H_3), we get

$$
\begin{aligned}
\|y(t)\| \leqslant {} & (K_1\|y_t\| + K_2) + \mathrm{e}^{\omega t}((1+K_1)\|\varphi\| + K_2) \\
& + \mathrm{e}^{\omega t}\int_0^t \mathrm{e}^{-\omega s}(K_1\|y_s\| + K_2)\,\mathrm{d}s \\
& + \mathrm{e}^{\omega t}\int_0^t \mathrm{e}^{-\omega s}p(s)\psi(\|y_s\|)\,\mathrm{d}s.
\end{aligned} \tag{3.5}
$$

Hence, for all $t \in [0,a]$, we have

$$
\begin{aligned}
(1-K_1)\|y(t)\| \leqslant K_2 + \mathrm{e}^{\omega t}\Big\{ & (1+K_1)\|\varphi\| + K_2 + K_1\int_0^t \mathrm{e}^{-\omega s}\|y_s\|\,\mathrm{d}s + aK_2 \\
& + \int_0^t \mathrm{e}^{-\omega s}p(s)\psi(\|y_s\|)\,\mathrm{d}s\Big\}.
\end{aligned}
$$

The last inequality along with the fact that

$$
\|y(s)\| \leqslant \|\varphi\| \quad \text{for all } s \in [-r, 0]
$$

implies that

$$
\sup_{-r\leqslant s\leqslant t}\|y(s)\| \leqslant \mathrm{e}^{\omega t}v(t),
$$

where v is defined by

$$
\begin{aligned}
v(t) = \frac{1}{1-K_1}\Big\{ & K_2\mathrm{e}^{-\omega t} + (1+K_1)\|\varphi\| + K_2(1+a) \\
& + K_1\int_0^t \mathrm{e}^{-\omega s}\|y_s\|\,\mathrm{d}s + \int_0^t \mathrm{e}^{-\omega s}p(s)\psi(\|y_s\|)\,\mathrm{d}s\Big\} \quad \text{for } t \in [0,a].
\end{aligned} \tag{3.6}
$$

Furthermore, $v(0) = 1/(1-K_1)\{(1+K_1)\|\varphi\| + K_2(2+a)\}$,

$$(\mathrm{e}^{\omega t}v(t))' \leqslant \omega \mathrm{e}^{\omega t}v(t) + \frac{K_1}{1-K_1}\mathrm{e}^{\omega t}v(t) + \frac{p(t)}{1-K_1}\psi(\mathrm{e}^{\omega t}v(t))$$
$$\leqslant m(t)(\mathrm{e}^{\omega t}v(t) + \psi(\mathrm{e}^{\omega t}v(t))),$$

where $m(t) = \max\{\omega + K_1/(1-K_1), p(t)/(1-K_1)\}$.

Hence,

$$\int_{v(0)}^{\mathrm{e}^{\omega t}v(t)} \frac{\mathrm{d}s}{s+\psi(s)} = \int_0^t \frac{(\mathrm{e}^{\omega s}v(s))'}{\mathrm{e}^{\omega s}v(s) + \psi(\mathrm{e}^{\omega s}v(s)} \,\mathrm{d}s \leqslant \int_0^t m(s)\,\mathrm{d}s < \infty.$$

By (H_3), the last inequality implies that $(\mathrm{e}^{\omega t}v(t))$ is bounded uniformly with respect to v and we deduce that there exists a constant $\widetilde{c}_1$ such that $\mathrm{e}^{\omega t}v(t) \leqslant \widetilde{c}_1$ for $t \in [0,a]$.

Let $c_1 = \max\{\|\varphi\|, \widetilde{c}_1\}$. Then we have

$$\sup_{-r \leqslant s \leqslant t} \|y(s)\| \leqslant c_1 \quad \text{for all } t \in [0,a].$$

This implies that

$$\|y\| \leqslant c_1.$$

Set

$$U_1 = \Big\{y \in C([-r,a];E)\colon \sup_{t\in[0,a]} \|y(t)\| < c_1 + 1\Big\}.$$

We can see that N is bounded. We have to show that the operator $N\colon \overline{U}_1 \to \mathcal{P}(C([-r,a];E))$ is an admissible contraction. Let us introduce on $C([-r,a];E)$ a new norm $\|\cdot\|_a$ by

$$\|y\|_a = \sup_{t\in[0,a]} \mathrm{e}^{-(\omega t + \tau \overline{L}(t))}\|y(t)\|,$$

where τ will be chosen sufficiently large, and the functions $\overline{L}$ and $\overline{l}$ are given by

$$\overline{L}(t) = \int_0^t \overline{l}(s)\,\mathrm{d}s,$$

and

$$\overline{l}(t) = \max\{K_1,\ l_a(t)\}.$$

Claim 2: N is a contraction, which means that there exists $\delta < 1$ such that

$$H_d(N(y), N(\overline{y})) \leqslant \delta\|y - \overline{y}\|_a \quad \text{for } y,\ \overline{y} \in C([-r,a];E).$$

Let $y, \overline{y} \in C([-r,a];E)$. Then for each $t \in [0,a]$ and $h \in N(y)$, there exists $g \in S_{F,y}$ such that

$$\begin{aligned} h(t) &= f(t,y_t) + T_0(t)(\varphi(0) - f(0,\varphi)) \\ &\quad + \lim_{\lambda\to\infty} \int_0^t T_0(t-s)A_\lambda f(s,y_s)\,\mathrm{d}s \\ &\quad + \lim_{\lambda\to\infty} \int_0^t T_0(t-s)A_\lambda g(s)\,\mathrm{d}s. \end{aligned}$$

Assumption (H_4) implies that

$$H_d(F(t,y(t)), F(t,\overline{y}(t))) \leqslant l_a(t)\|y_t - \overline{y}_t\|.$$

Hence, there exists $x \in F(t,\overline{y}_t)$ such that

$$\|g(t) - x\| \leqslant l_a(t)\|y_t - \overline{y}_t\| \quad \text{for } t \in [0,a].$$

Let $U_*\colon [0,a] \to \mathcal{P}(E)$ be given by

$$U_*(t) = \{x \in E\colon \|g(t) - x\| \leqslant l_a(t)\|y_t - \overline{y}_t\|\}.$$

It follows from Proposition 2.3 that the multivalued operator $V_*(t) = U_*(t) \cap F(t,\overline{y}_t)$ is measurable. Then V_* admits a measurable selection $\overline{g}$.

Hence, $\overline{g}(t) \in F(t,\overline{y}_t)$ and

$$\|g(t) - \overline{g}(t)\| \leqslant l_a(t)\|y_t - \overline{y}_t\| \quad \text{for } t \in [0,a].$$

Let $\overline{h} \in N(\overline{y})$ be defined for $t \in [0,a]$ by

$$\begin{aligned} \overline{h}(t) &= f(t,\overline{y}_t) + T_0(t)(\varphi(0) - f(0,\varphi)) \\ &\quad + \lim_{\lambda\to\infty} \int_0^t T_0(t-s)A_\lambda f(s,\overline{y}_s)\,\mathrm{d}s \\ &\quad + \lim_{\lambda\to\infty} \int_0^t T_0(t-s)A_\lambda \overline{g}(s)\,\mathrm{d}s. \end{aligned}$$

Then

$$\begin{aligned} \|h(t) - \overline{h}(t)\| &\leqslant \|f(t,y_t) - f(t,\overline{y}_t)\| \\ &\quad + \left\| \lim_{\lambda\to\infty} \int_0^t T_0(t-s)A_\lambda(f(s,y_s) - f(s,\overline{y}_s))\,\mathrm{d}s \right\| \\ &\quad + \left\| \lim_{\lambda\to\infty} \int_0^t T_0(t-s)A_\lambda(g(s) - \overline{g}(s))\,\mathrm{d}s \right\| \\ &\leqslant \Big(K_1 + \frac{2}{\tau}\Big)\mathrm{e}^{\omega t + \tau\overline{L}(t)}\|y - \overline{y}\|_a. \end{aligned} \tag{3.7}$$

In fact, assumption (H_2) yields that

$$\begin{aligned}\|f(t,y_t)-f(t,\overline{y}_t)\| &\leqslant K_1\|y_t-\overline{y}_t\| \\ &\leqslant K_1\mathrm{e}^{\omega t+\tau\overline{L}(t)}\mathrm{e}^{-(\omega t+\tau\overline{L}(t))}\|y-\overline{y}\| \\ &\leqslant K_1\mathrm{e}^{\omega t+\tau\overline{L}(t)}\|y-\overline{y}\|_a.\end{aligned} \tag{3.8}$$

On the other hand, by virtue of (H_1), (H_2), we get

$$\begin{aligned}&\left\|\lim_{\lambda\to\infty}\int_0^t T_0(t-s)A_\lambda(f(s,y_s)-f(s,\overline{y}_s))\,\mathrm{d}s\right\| \\ &\quad\leqslant \mathrm{e}^{\omega t}\int_0^t \mathrm{e}^{-\omega s}\|f(s,y_s)-f(s,\overline{y}_s)\|\,\mathrm{d}s \\ &\quad\leqslant \mathrm{e}^{\omega t}\int_0^t \mathrm{e}^{-\omega s}K_1\|y_s-\overline{y}_s\|\,\mathrm{d}s \\ &\quad\leqslant \mathrm{e}^{\omega t}\int_0^t \overline{l}(s)\mathrm{e}^{\tau\overline{L}(s)}\mathrm{e}^{-\tau\overline{L}(s)}\mathrm{e}^{-\omega s}\|y-\overline{y}\|\,\mathrm{d}s \\ &\quad= \mathrm{e}^{\omega t}\int_0^t \overline{l}(s)\mathrm{e}^{\tau\overline{L}(s)}\|y-\overline{y}\|_a\,\mathrm{d}s \\ &\quad= \frac{\mathrm{e}^{\omega t}}{\tau}\int_0^t (\tau\overline{L}(s))'\mathrm{e}^{\tau\overline{L}(s)}\,\mathrm{d}s\|y-\overline{y}\|_a \\ &\quad= \frac{1}{\tau}\mathrm{e}^{\omega t+\tau\overline{L}(t)}\|y-\overline{y}\|_a.\end{aligned} \tag{3.9}$$

Similarly, by using (H_1), (H_4), we obtain

$$\begin{aligned}\left\|\lim_{\lambda\to\infty}\int_0^t T_0(t-s)A_\lambda(g(s)-\overline{g}(s))\,\mathrm{d}s\right\| &\leqslant \mathrm{e}^{\omega t}\int_0^t \mathrm{e}^{-\omega s}\|g(s)-\overline{g}(s)\|\,\mathrm{d}s \\ &\leqslant \mathrm{e}^{\omega t}\int_0^t \mathrm{e}^{-\omega s}l_a(s)\|y_s-\overline{y}_s\|\,\mathrm{d}s \\ &\leqslant \mathrm{e}^{\omega t}\int_0^t \overline{l}(s)\mathrm{e}^{\tau\overline{L}(s)}\|y-\overline{y}\|_a\,\mathrm{d}s \\ &= \frac{1}{\tau}\mathrm{e}^{\omega t+\tau\overline{L}(t)}\|y-\overline{y}\|_a.\end{aligned} \tag{3.10}$$

Therefore, inequality (3.7) holds and consequently, we get

$$\|h-\overline{h}\|_a \leqslant \Big(K_1+\frac{2}{\tau}\Big)\|y-\overline{y}\|_a,$$

and we deduce by interchanging the roles of y and $\overline{y}$ that

$$H_d(N(y),N(\overline{y})) \leqslant \Big(K_1+\frac{2}{\tau}\Big)\|y-\overline{y}\|_a.$$

By choosing τ large enough such that $K_1 + 2/\tau < 1$, we deduce that N is a contraction.

Claim 3: N is an admissible multivalued map.

Let $y \in C([-r,a];E)$ and define $N\colon\ C([-r,a];E) \to \mathcal{P}_{\mathrm{cl}}(C([-r,a];E))$ by

$$N(y)(t) = \begin{cases} \varphi(t) \quad \text{for } t \in [-r,0], \\ f(t,y_t) + T_0(t)(\varphi(0) - f(0,\varphi)) + \lim\limits_{\lambda\to\infty} \displaystyle\int_0^t T_0(t-s)A_\lambda f(s,y_s)\,\mathrm{d}s, \\ \qquad\qquad + \lim\limits_{\lambda\to\infty} \displaystyle\int_0^t T_0(t-s)A_\lambda g(s)\,\mathrm{d}s \quad \text{for } t \in [0,a], \end{cases}$$

where $g \in S_{F,y}$.

Since F is a multivalued map with compact values, we can prove as in [9], [14] that for every $y \in C([-r,a];E)$ we have $N(y) \in \mathcal{P}_{\mathrm{cp}}(C([-r,a];E))$ and there exists $y_* \in C([-r,a];E)$ such that $y_* \in N(y_*)$.

Let $\overline{y} \in \overline{U_1}$, $\varepsilon > 0$.

If $y_* \in N(\overline{y})$, then $\|y_* - N(\overline{y})\| = 0$ and we have

$$\|\overline{y} - y_*\| \leqslant \|\overline{y} - N\overline{y}\| + \|y_* - h\|.$$

Let $h \in C([-r,a];E)$ be such that $\|h - y_*\|_a \leqslant \varepsilon$, then

$$\|\overline{y} - y_*\|_a \leqslant \|\overline{y} - N\overline{y}\|_a + \|y_* - h\|_a \leqslant \|\overline{y} - N\overline{y}\|_a + \varepsilon.$$

If $y_* \notin N(\overline{y})$ then $\|y_* - N(\overline{y})\| \neq 0$. Since $N(\overline{y})$ is compact, there exists $\overline{x} \in N(\overline{y})$ such that $\|y_* - N(\overline{y})\| = \|y_* - \overline{x}\|$.

Let $h \in C([-r,a];E)$ be such that $\|\overline{x} - h\|_a \leqslant \varepsilon$. Since $\overline{x} \in N(\overline{y})$, we get

$$\|\overline{y} - \overline{x}\| \leqslant \|\overline{y} - N\overline{y}\| + \|\overline{x} - h\|,$$

which leads to

$$\|\overline{y} - \overline{x}\|_a \leqslant \|\overline{y} - N\overline{y}\|_a + \varepsilon.$$

Hence, N is an admissible multifunction. Moreover, due to the choice of U_1, there is no $y \in \partial U_1$ such that $y \in \lambda N(y)$ for some $\lambda \in [0,1)$. We deduce from Theorem 2.2 that N has at least one fixed point which is an integral solution of (1.1). □

In this section, we are concerned with the controllability of problem (1.2). We start by introducing the following definitions.

Definition 4.1. A continuous function $y\colon [-r,a] \to E$ is called an integral solution of (1.2) if there exists $g \in S_{F,y}$ such that

(i) $\int_0^t (y(s) - f(s,y_s))\,\mathrm{d}s \in D(A)$,

(ii) $y(t) = f(t,y_t) + (\varphi(0) - f(0,\varphi)) + A\int_0^t (y(s) - f(s,y_s))\,\mathrm{d}s + \int_0^t g(s)\,\mathrm{d}s + \int_0^t Bu(s)\,\mathrm{d}s$ for $t \in [0,a]$,

(iii) $y_0 = \varphi$.

If y is an integral solution of (1.2), then it is given as in [4] by the formula

$$
\begin{aligned}
y(t) = f(t,y_t) + T_0(t)(\varphi(0) - f(0,\varphi)) &+ \lim_{\lambda\to\infty}\int_0^t T_0(t-s)A_\lambda f(s,y_s)\,\mathrm{d}s \\
+ \lim_{\lambda\to\infty}\int_0^t T_0(t-s)A_\lambda g(s)\,\mathrm{d}s &+ \lim_{\lambda\to\infty}\int_0^t T_0(t-s)A_\lambda Bu(s)\,\mathrm{d}s,
\end{aligned}
\tag{4.1}
$$

where $A_\lambda = \lambda(\lambda - A)^{-1}$.

Definition 4.2. We say that problem (1.2) is controllable on $[0,a]$ if for any continuous function φ on $[-r,0]$ satisfying $\varphi(0) - f(0,\varphi) \in \overline{D(A)}$ and for any $x_1 \in E$ there exists a control $u \in L^2([0,a];\mathcal{U})$ such that the integral solution y of (1.2) satisfies $y(a) = x_1$.

In addition to (H_1)–(H_4), we assume the following assumption:

(H_5) The operator $W\colon\ L^2([0,a];\mathcal{U}) \to E$ defined by

$$
Wu = \lim_{\lambda\to\infty}\int_0^a T_0(a-s)A_\lambda Bu(s)\,\mathrm{d}s
$$

induces a bounded inverse W^{-1} defined on $L^2([0,a];\mathcal{U}) \setminus \operatorname{Ker} W$.

Let M_1, M_2 be positive constants such that

$$
\|B\| \leqslant M_1 \text{ and } \|W^{-1}\| \leqslant M_2.
$$

We are now in position to state our controllability result.

Theorem 4.3. *Assume that* (H_1)–(H_5) *are verified and let* φ *be such that* $\varphi(0)-f(0,\varphi)\in\overline{D(A)}$. *If* $(1+aM_1M_2\mathrm{e}^{\omega a})K_1<1$ *then problem* (1.2) *is controllable on* $[0,a]$.

Proof. According to (H_5), we define for each $y(\cdot)$ and $g\in S_{F,y}$ the control

$$
(4.2)\quad u_y(t)=W^{-1}\Big[y(a)-f(a,y_a)-T_0(a)(\varphi(0)-f(0,\varphi)) \\
-\lim_{\lambda\to\infty}\int_0^a T_0(a-s)A_\lambda f(s,y_s)\,\mathrm{d}s-\lim_{\lambda\to\infty}\int_0^a T_0(a-s)A_\lambda g(s)\,\mathrm{d}s\Big].
$$

Define an operator $N\colon\ C([-r,a];E)\to\mathcal{P}(C([-r,a];E))$ by

$$
N(y)(t)=\begin{cases}\varphi(t)\quad\text{for } t\in[-r,0],\\ f(t,y_t)+T_0(t)(\varphi(0)-f(0,\varphi))\\ \quad+\lim\limits_{\lambda\to\infty}\int_0^t T_0(t-s)A_\lambda f(s,y_s)\,\mathrm{d}s+\lim\limits_{\lambda\to\infty}\int_0^t T_0(t-s)A_\lambda g(s)\,\mathrm{d}s\\ \quad+\lim\limits_{\lambda\to\infty}\int_0^t T_0(t-s)A_\lambda(Bu_y)(s)\,\mathrm{d}s\quad\text{for } t\in[0,a].\end{cases}
$$

Clearly, the fixed points of N are integral solutions of (1.2).

Claim 1: Let y be a solution of (1.2). Then there exists $c_2>0$ such that $\|y\|\leqslant c_2$.

Indeed, there exists $g\in S_{F,y}$ such that for each $t\in[0,a]$,

$$
y(t)=f(t,y_t)+T_0(t)(\varphi(0)-f(0,\varphi))+\lim_{\lambda\to\infty}\int_0^t T_0(t-s)A_\lambda f(s,y_s)\,\mathrm{d}s \\
+\lim_{\lambda\to\infty}\int_0^t T_0(t-s)A_\lambda g(s)\,\mathrm{d}s+\lim_{\lambda\to\infty}\int_0^t T_0(t-s)A_\lambda(Bu_y)(s)\,\mathrm{d}s.
$$

We can see from (3.1), (3.4), and assumption (H_3) that

$$
(4.3)\quad \Big\|f(t,y_t)+T_0(t)(\varphi(0)-f(0,\varphi)) \\
+\lim_{\lambda\to\infty}\int_0^t T_0(t-s)A_\lambda f(s,y_s)\,\mathrm{d}s+\lim_{\lambda\to\infty}\int_0^t T_0(t-s)A_\lambda g(s)\,\mathrm{d}s\Big\| \\
\leqslant(K_1\|y_t\|+K_2)+\mathrm{e}^{\omega t}((1+K_1)\|\varphi\|+K_2) \\
+\mathrm{e}^{\omega t}\int_0^t\mathrm{e}^{-\omega s}(K_1\|y_s\|+K_2)\,\mathrm{d}s+\mathrm{e}^{\omega t}\int_0^t\mathrm{e}^{-\omega s}p(s)\psi(\|y_s\|)\,\mathrm{d}s.
$$

Moreover, by using (4.2), the same argument as in (4.3) allows us to get the estimate

$$\begin{aligned}(4.4)\qquad &\Big\|\lim_{\lambda\to\infty}\int_0^t T_0(t-s)A_\lambda(Bu_y)(s)\,\mathrm{d}s\Big\| \\ &\leqslant aM_1M_2\mathrm{e}^{\omega t}\Big\{\|x_1\|+K_1\|y_a\|+K_2+\mathrm{e}^{\omega a}((1+K_1)\|\varphi\|+K_2) \\ &\quad+\mathrm{e}^{\omega a}\int_0^a \mathrm{e}^{-\omega s}(K_1\|y_s\|+K_2)\,\mathrm{d}s+\mathrm{e}^{\omega a}\int_0^a \mathrm{e}^{-\omega s}p(s)\psi(\|y_s\|)\,\mathrm{d}s\Big\}.\end{aligned}$$

By virtue of (4.3) and (4.4), we get for $t\in[0,a]$

$$\begin{aligned}(1-K_1)\|y(t)\| &\leqslant K_2+\mathrm{e}^{\omega t}\Big\{(1+K_1)\|\varphi\|+(1+a)K_2 \\ &\quad+K_1\int_0^a \mathrm{e}^{-\omega s}\|y_s\|\,\mathrm{d}s+\int_0^t \mathrm{e}^{-\omega s}p(s)\psi(\|y_s\|)\,\mathrm{d}s\Big\} \\ &\quad+aM_1M_2\mathrm{e}^{\omega t}\Big\{\|x_1\|+K_1\|y_a\|+K_2+\mathrm{e}^{\omega a}((1+K_1)\|\varphi\|+K_2) \\ &\quad+aK_2\mathrm{e}^{\omega a}+K_1\mathrm{e}^{\omega a}\int_0^a \mathrm{e}^{-\omega s}\|y_s\|\,\mathrm{d}s+\mathrm{e}^{\omega a}\int_0^a \mathrm{e}^{-\omega s}p(s)\psi(\|y_s\|)\,\mathrm{d}s\Big\}\end{aligned}$$

which implies that

$$\sup_{-r\leqslant s\leqslant t}\|y(s)\|\leqslant \mathrm{e}^{\omega t}\delta(t),$$

where δ is defined by

$$\begin{aligned}\delta(t)=\frac{1}{1-K_1}\Big\{&K_2\mathrm{e}^{-\omega t}+\int_0^t \mathrm{e}^{-\omega s}p(s)\psi(\|y_s\|)\,\mathrm{d}s+(1+K_1)\|\varphi\|+(1+a)K_2 \\ &+K_1\int_0^a \mathrm{e}^{-\omega s}\|y_s\|\,\mathrm{d}s+aM_1M_2\Big[\|x_1\|+K_1\|y_a\|+K_2+\mathrm{e}^{\omega a}\Big((1+K_1)\|\varphi\| \\ &\quad+(1+a)K_2+K_1\int_0^a \mathrm{e}^{-\omega s}\|y_s\|\,\mathrm{d}s+\int_0^a \mathrm{e}^{-\omega s}p(s)\psi(\|y_s\|)\,\mathrm{d}s\Big)\Big]\Big\}\end{aligned}$$

with

$$\begin{aligned}\delta(0)=\frac{1}{1-K_1}\Big\{&K_2+(1+K_1)\|\varphi\|+(1+a)K_2 \\ &+K_1\int_0^a \mathrm{e}^{-\omega s}\|y_s\|\,\mathrm{d}s+aM_1M_2\Big[\|x_1\|+K_1\|y_a\|+K_2 \\ &+\mathrm{e}^{\omega a}\Big((1+K_1)\|\varphi\|+(1+a)K_2 \\ &+K_1\int_0^a \mathrm{e}^{-\omega s}\|y_s\|\,\mathrm{d}s+\int_0^a \mathrm{e}^{-\omega s}p(s)\psi(\|y_s\|)\,\mathrm{d}s\Big)\Big]\Big\}\end{aligned}$$

and

$$\delta'(t) = \frac{1}{1-K_1}(-\omega K_2 e^{-\omega t} + e^{-\omega t} p(t)\psi(\|y_t\|)).$$

Since ψ is increasing and $\omega > 0$, we deduce that

$$\delta'(t) \leqslant \frac{1}{1-K_1} e^{-\omega t} p(t)\psi(e^{\omega t}\delta(t)),$$

$$(e^{\omega t}\delta(t))' \leqslant \omega e^{\omega t}\delta(t) + \frac{1}{1-K_1} p(t)\psi(e^{\omega t}\delta(t)) \leqslant q(t)(e^{\omega t}\delta(t) + \psi(e^{\omega t}\delta(t))),$$

where $q(t) = \max\{\omega, 1/(1-K_1)p(t)\}$.

By (H_3) and since

$$\int_{\delta(0)}^{e^{\omega t}\delta(t)} \frac{ds}{s+\psi(s)} = \int_0^t \frac{(e^{\omega s}\delta(s))'}{e^{\omega s}\delta(s) + \psi(e^{\omega s}\delta(s))}\, ds \leqslant \int_0^t q(s)\, ds < \infty,$$

we deduce that there exists a constant $\widetilde{c}_2$ such that

$$e^{\omega t}\delta(t) \leqslant \widetilde{c}_2 \quad \text{for } t \in [0,a].$$

This implies that

$$\|y\| \leqslant \widetilde{c}_2 \quad \text{for } t \in [0,a].$$

Let $c_2 = \max\{\|\varphi\|, \widetilde{c}_2\}$. Then

$$\|y\| \leqslant c_2.$$

Set

$$U_2 = \Big\{ y \in C([-r,a];E)\colon \sup_{t\in[0,a]} \|y(t)\| < c_2 + 1 \Big\}.$$

The space $C([-r,a];E)$ is now endowed with the new norm

$$\|y\|_a = \sup_{t\in[0,a]} e^{-(\omega t + \tau L(t))}\|y(t)\|,$$

where the functions L and $\hat{l}$ are defined as

$$L(t) = \int_0^t \hat{l}(s)\, ds,$$

and

$$\hat{l}(t) = \max\{K_1,\ l_a(t),\ aM_1M_2e^{\omega a}K_1,\ aM_1M_2e^{\omega a}\|l_a\|\}. \tag{4.5}$$

Claim 2: The operator $N\colon \overline{U}_2 \to \mathcal{P}(C([-r,a];E))$ is an admissible contraction.

Let us first show that N is a contraction.

Let $y, \overline{y} \in C([-r,a];E)$. Then for each $t \in [0,a]$ and $h \in N(y)$, there exists $g(t) \in F(t, y_t)$ such that

$$\begin{aligned} h(t) = f(t,y_t) + T_0(t)(\varphi(0) - f(0,\varphi)) &+ \lim_{\lambda\to\infty} \int_0^t T_0(t-s)A_\lambda f(s,y_s)\,\mathrm{d}s \\ + \lim_{\lambda\to\infty} \int_0^t T_0(t-s)A_\lambda g(s)\,\mathrm{d}s &+ \lim_{\lambda\to\infty} \int_0^t T_0(t-s)A_\lambda (Bu_y)(s)\,\mathrm{d}s. \end{aligned}$$

The proof of Theorem 3.4 yields that there exists a function $\overline{g}$ such that $\overline{g}(t) \in F(t, \overline{y}_t)$ and

$$\|g(t) - \overline{g}(t)\| \leqslant l_a(t)\|y_t - \overline{y}_t\| \quad \text{for } t \in [0,a].$$

Let $\overline{h} \in N(\overline{y})$ be defined for each $t \in [0,a]$ by

$$\begin{aligned} \overline{h}(t) = f(t,\overline{y}_t) + T_0(t)(\varphi(0) - f(0,\varphi)) &+ \lim_{\lambda\to\infty} \int_0^t T_0(t-s)A_\lambda f(s,\overline{y}_s)\,\mathrm{d}s \\ + \lim_{\lambda\to\infty} \int_0^t T_0(t-s)A_\lambda \overline{g}(s)\,\mathrm{d}s &+ \lim_{\lambda\to\infty} \int_0^t T_0(t-s)A_\lambda (Bu_{\overline{y}})(s)\,\mathrm{d}s. \end{aligned}$$

Then

$$\|h(t) - \overline{h}(t)\| \leqslant \Big((1 + aM_1M_2\mathrm{e}^{\omega a})K_1 + \frac{4}{\tau}\Big)\mathrm{e}^{\omega t + \tau L(t)}\|y - \overline{y}\|_a. \tag{4.6}$$

In fact, by using the same argument as in inequalities (3.8)–(3.10) we get

$$\begin{aligned} \Big\| f(t,y_t) - f(t,\overline{y}_t) &+ \lim_{\lambda\to\infty} \int_0^t T_0(t-s)A_\lambda (f(s,y_s) - f(s,\overline{y}_s))\,\mathrm{d}s \\ &+ \lim_{\lambda\to\infty} \int_0^t T_0(t-s)A_\lambda (g(s) - \overline{g}(s))\,\mathrm{d}s \Big\| \\ &\leqslant \Big(K_1 + \frac{2}{\tau}\Big)\mathrm{e}^{\omega t + \tau L(t)}\|y - \overline{y}\|_a. \end{aligned}$$

On the other hand, using (4.2) and (4.5), we have the estimate

$$
\begin{aligned}
(4.7)\qquad &\left\| \lim_{\lambda\to\infty} \int_0^t T_0(t-s)A_\lambda B(u_y(s)-u_{\overline{y}}(s))\,\mathrm{d}s \right\| \\
&\leqslant aM_1M_2\mathrm{e}^{\omega a}K_1\|y-\overline{y}\| + aM_1M_2\mathrm{e}^{\omega a}\mathrm{e}^{\omega t}\int_0^t \mathrm{e}^{-\omega s}K_1\|y-\overline{y}\|\,\mathrm{d}s \\
&\quad + aM_1M_2\mathrm{e}^{\omega a}\mathrm{e}^{\omega t}\int_0^t \mathrm{e}^{-\omega s}\|l_a\|\|y-\overline{y}\|\,\mathrm{d}s \\
&\leqslant aM_1M_2\mathrm{e}^{\omega a}K_1\mathrm{e}^{\omega t+\tau L(t)}\|y-\overline{y}\|_a + \mathrm{e}^{\omega t}\int_0^t \hat{l}(s)\mathrm{e}^{\tau L(s)}\|y-\overline{y}\|_a\,\mathrm{d}s \\
&\quad + \mathrm{e}^{\omega t}\int_0^t \hat{l}(s)\mathrm{e}^{\tau L(s)}\|y-\overline{y}\|_a\,\mathrm{d}s \\
&\leqslant \Big(aM_1M_2\mathrm{e}^{\omega a}K_1\mathrm{e}^{\omega t+\tau L(t)} + \frac{2}{\tau}\mathrm{e}^{\omega t+\tau L(t)}\Big)\|y-\overline{y}\|_a.
\end{aligned}
$$

Hence, inequality (4.6) is verified and we have

$$\|h-\overline{h}\|_a \leqslant \Big((1+aM_1M_2\mathrm{e}^{\omega a})K_1 + \frac{4}{\tau}\Big)\|y-\overline{y}\|_a.$$

Consequently, we get

$$H_d(N(y),N(\overline{y})) \leqslant \Big((1+aM_1M_2\mathrm{e}^{\omega a})K_1 + \frac{4}{\tau}\Big)\|y-\overline{y}\|_a.$$

By choosing τ large enough such that $(1+aM_1M_2\mathrm{e}^{\omega a})K_1+4/\tau<1$, we deduce that N is a contraction. Using the same reasoning as in Theorem 3.4, we can show that N is an admissible multivalued map.

Claim 3: Problem (1.2) is controllable.

By applying Theorem 2.2 and since there is no $y\in\partial U_2$ such that $y\in\lambda N(y)$ for some $\lambda\in[0,1)$, we conclude that N has at least one fixed point which is an integral solution of equation (1.2). □

5. Application

The objective of this section is to apply the controllability results of the previous section to the reaction-diffusion inclusion of parabolic type

$$
(5.1)\qquad \begin{cases} \dfrac{\partial}{\partial t}[y(t,x)-g(y_t(\cdot,x))]-\Delta[y(t,x)-g(y_t(\cdot,x))] \\ \qquad\qquad \in Q(t,y(t-r,x))+(Bu)(t) & \text{for } t\in[0,a],\ x\in[0,\pi], \\ y(t,0)=y(t,\pi) & \text{for } t\in[0,a], \\ y(\theta,x)=\varphi(\theta,x) & \text{for } \theta\in[-r,0],\ x\in[0,\pi], \end{cases}
$$

where Δ is the Laplacian operator on $[0,\pi]$, $\varphi\in C([-r,0];C([0,\pi];\mathbb{R}))$, $g\colon C([-r,0];C([0,\pi];\mathbb{R}))\to C([0,\pi];\mathbb{R})$ is Lipshitz continuous, that is, there exists $k_0>0$ such that

$$\|g\varphi-g\psi\|\leqslant k_0\|\varphi-\psi\| \quad \text{for } \varphi,\psi\in C([-r,0];C([0,\pi];\mathbb{R})),$$

$Q\colon [0,a]\times[0,\pi]\to\mathcal{P}(\mathbb{R})$ is a multivalued map with compact values satisfying

$$\exists k_1>0\colon\ H_d(Q(t,x_1),Q(t,x_2))\leqslant k_1\|x_1-x_2\| \quad \text{for } t\in[0,a] \text{ and } x_1,x_2\in[0,\pi],$$

and

$$d(0,Q(t,0))\leqslant k_1 \quad \text{for } t\in[0,a].$$

$B\colon\mathcal{U}\to C([0,\pi];\mathbb{R})$ is a bounded linear operator defined on a Banach space $\mathcal{U}$ and $u\in L^2([0,a];\mathcal{U})$.

It is well known from [11] that Δ possesses the following properties:

$$
\begin{cases} \overline{D(\Delta)}=\{u\in C([0,\pi];\mathbb{R})\colon\ u(0)=u(\pi)=0\}, \\ (0,\infty)\subset\varrho(\Delta), \\ \|(\lambda-\Delta)^{-1}\|\leqslant\dfrac{1}{\lambda} \quad \text{for } \lambda>0. \end{cases}
$$

Hence, assumption (H_1) is verified.

Also, if $k_0<1$, then the function g satisfies assumption (H_2).

Define on $[0,a]\times C([-r,0];C([0,\pi];\mathbb{R}))$ a multivalued operator F as

$$F(t,\varphi)(x)=Q(t,\varphi(-r)(x)).$$

Then F satisfies (H_4).

Let $(T_0(t))_{t\geqslant0}$ be the strongly continuous semigroup generated by the part of Δ in $\overline{D(\Delta)}$, define the operator $W\colon\ L^2([0,a];\mathcal{U})\to C([0,\pi];\mathbb{R})$ as

$$Wu=\lim_{\lambda\to\infty}\int_0^a T_0(a-s)A_\lambda Bu(s)\,\mathrm{d}s$$

and assume that W^{-1} exists and takes values in $L^2([0,a];\mathcal{U})\setminus\operatorname{Ker}W$.

Let $M_1, M_2 \geqslant 0$ be such that

$$\|B\| \leqslant M_1 \text{ and } \|W^{-1}\| \leqslant M_2.$$

Theorem 5.1. *Let φ be such that $\varphi(0) - g(\varphi) \in \overline{D(\Delta)}$.*

If $(1 + aM_1M_2\mathrm{e}^{\omega a})k_0 < 1$ then the partial neutral functional differential inclusion (5.1) *is controllable on* $[0, a]$.

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