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MODEL ANALYSIS OF BPX PRECONDITIONER BASED ON SMOOTHED AGGREGATION

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Abstract. We prove nearly uniform convergence bounds for the BPX preconditioner based on smoothed aggregation under the assumption that the mesh is regular. The analysis is based on the fact that under the assumption of regular geometry, the coarse-space basis functions form a system of macroelements. This property tends to be satisfied by the smoothed aggregation bases formed for unstructured meshes.

Keywords: smoothed aggregation; parallel preconditioner; BPX preconditioner

MSC 2010: 65F10, 65M55

1. Introduction

The classical multigrid is a multiplicative method of Schwarz type with inexact subspace solvers given by smoothers [1]. As a consequence, its fundamental components suffer from inner dependencies and have to be performed in a sequence, preventing truly large-scale parallelism. Unlike standard multigrid, the so-called BPX preconditioner frame of Bramble, Pasciak, and Xu [2] is fully additive, allowing for fine-grain parallelism on the level of a single coarse-space basis function. The sufficient conditions for its convergence and the mathematical requirements on its efficient implementation are, however, different from the ones for multiplicative multilevel iterative methods, despite the fact that the sufficient conditions look similar.

Our smoothed aggregation algebraic multigrid coarsening technique was proved to be very efficient in the context of solving large scale systems of linear algebraic equations arising from the discretization of elliptic problems and their singular perturbations (see [4], [7], [6], [8]). The smoothed aggregation method was, however,

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developed and analyzed in the context of traditional multiplicative multigrid. In this paper, we use smoothed aggregation in the BPX frame and analyze the convergence of the resulting iterative method applied to a model example.

In the unpublished technical report [10], we made a first attempt to analyze the smoothed aggregation method in the context of standard multigrid. The report contains merely a sketch of the theory. The most difficult part was to establish the resolution-independent equivalence of discrete and continuous L_2-norms

$$\left\| \sum_i x_i \varphi_i^l \right\|_{L_2} \approx \text{ scaling } \left(\sum_i x_i^2 \right)^{1/2} \tag{1.1}$$

for the hierarchy of coarse-spaces $\operatorname{span}\{\varphi_i^l\}_i$ (l denotes the level, φ_i^l a basis function). The equivalence was used to prove the weak approximation property needed to verify the assumptions of the regularity-free abstract multigrid convergence theory of [1].

In [7], for the standard multigrid, we found a way to avoid the need for this equivalence by fully algebraic means. The convergence proof of [7] only requires the equivalence of discrete and continuous L_2-norms to hold for disaggregated functions, which is satisfied trivially, because aggregation-based prolongators are, after scaling, orthogonal matrices.

In the context of the BPX preconditioner, however, equivalence (1.1) is unavoidable. This follows from the fact that the efficient implementation of the BPX preconditioner requires the computationally cheap implementation of the approximate l_2-projections onto coarse-spaces; such implementation must avoid the action of the inverse of the Gram matrix. Thus, for the coarse-space basis, we need a Gram matrix that has an inverse that can be approximated by the inverse of its diagonal. For this reason, we returned to the method of analysis outlined in [10] and developed it fully for the case of model geometry. Our proof of (1.1) is non-constructive, based on a compactness argument (Rellich's theorem).

For the BPX preconditioner based on smoothed aggregation we prove, assuming model geometry and H^1-equivalent form, that the condition number of the preconditioned system grows at most as $O(L^2)$, where L is the number of levels.

The presented theory requires the coarse-space bases (or their supports) to form a system of disjoint macroelements covering the entire computational domain. The macroelement function is spanned solely by the set of associated basis functions; no other basis functions are allowed to intersect the macroelement with their supports. Such macroelements are obviously formed in the case of regular geometry. In the general case, however, the smoothed aggregation coarse-space bases tend to form the macroelements as well. The equivalence of discrete and continuous L_2-norms is therefore very likely to hold for unstructured aggregation formed on unstructured meshes.

The interpolation estimates (the weak approximation property of the coarse-space bases with the l_2-norm of the left-hand side measured on the finest level) are more or less standard variations on the finite element theory of [3], used in a variety of forms in many works, for example [7]. In a BPX context, those estimates had to be carried out for smoothed aggregation coarse-space basis functions, which we, using an algebraic trick, avoided in [7]. Here, only the weak approximation property of pure aggregations had to be proved.

The proof of uniform equivalence (1.1) is, up to its sketch in an unpublished technical report [10], new.

In what follows, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the Euclidean l_2-inner product and the Euclidean norm, respectively, in the relevant vector space. Assume A is a symmetric, positive definite matrix. We use the symbols $\langle\cdot,\cdot\rangle_A$ and $\|\cdot\|_A$ for the usual A-inner product $\langle\cdot,\cdot\rangle_A = \langle A\cdot,\cdot\rangle$ and A-norm $\|\cdot\|_A = \langle\cdot,\cdot\rangle_A^{1/2}$. Let $\mathcal{I}$ be an index set. We employ the notation $\langle\cdot,\cdot\rangle_{l_2(\mathcal{I})}$ and $\|\cdot\|_{l_2(\mathcal{I})}$ for the Euclidean inner product and the norm on a discrete domain $\mathcal{I}$ defined as

$$\langle\mathbf{x},\mathbf{y}\rangle_{l_2(\mathcal{I})} = \sum_{i\in\mathcal{I}} x_i y_i, \quad \|\cdot\|_{l_2(\mathcal{I})} = \langle\cdot,\cdot\rangle_{l_2(\mathcal{I})}^{1/2},$$

respectively. Here, $\mathbf{x}$ and $\mathbf{y}$ are vectors such that their entries $x_i, y_i \in \mathbb{R}$ are defined for all $i \in \mathcal{I}$. On $\mathbb{R}^N$, $\{1,\ldots,N\} \supset \mathcal{I}$, $\langle\cdot,\cdot\rangle_{l_2(\mathcal{I})}$ is a semi-product and $\|\cdot\|_{l_2(\mathcal{I})}$ is a semi-norm. If U is a Banach space, $\|\cdot\|_U$ is understood as the norm in U. For U being a Hilbert space, $(\cdot,\cdot)_U$ stands for inner product on U. Assume $(U,\|\cdot\|_U)$ and $(V,\|\cdot\|_V)$ are Banach spaces and $L\colon U \to V$ a linear mapping. We introduce the operator norm of L by

$$\|L\|_{U\to V} = \sup_{u\in U\setminus\{0\}} \frac{\|Lu\|_V}{\|u\|_U}.$$

For a symmetric, positive definite matrix A we define a condition number

$$\operatorname{cond}(A) = \lambda_{\max}(A)/\lambda_{\min}(A).$$

Similarly, for symmetric positive definite matrices A, B, the mutual condition number is given by $\operatorname{cond}(A,B) = \lambda_{\max}(BA)/\lambda_{\min}(BA)$.

We use generic constants $C, c > 0$ in the way usual in partial differential equations theory. This means, for example, for $\|\mathbf{u}\| \leqslant C\|\mathbf{v}\|$ and $\|\mathbf{v}\| \leqslant C\|\mathbf{w}\|$ we simply write $\|\mathbf{u}\| \leqslant C\|\mathbf{w}\|$. Typically, C, c are constants independent of the finest level mesh size and, whenever relevant, also of the level and the number of levels. In the local estimates, constants are also independent of the local index in the mesh (macroelement number, basis function number). In the abstract estimate of [2] presented in Section 2, the constants are independent of the level and the number of levels.

2. BPX preconditioner in operator setting

In this section we define the BPX preconditioner and give a convergence bound. At the end of the section, we describe the implementation of the method, assuming the system of prolongators is given. Up to minor technical details that suit our purpose, this section follows [2].

Let $(U, (\cdot,\cdot)_U, \|\cdot\|_U)$ be a Hilbert space. Consider a problem

$$\text{find}\, u \in U\colon\ a(u,v) = f(v) \quad \forall\, v \in U.$$

Here, $a(\cdot,\cdot)$ is a symmetric bilinear form, coercive and continuous on U and $f(\cdot) \in U^{-1}$ (U^{-1} is the dual space). Let

$$U = U_1 \supset U_2 \supset \ldots \supset U_L$$

be a hierarchy of nested Hilbert spaces with the inner product inherited from U. For each $l = 1, \ldots, L$, define operator $\mathscr{A}_l\colon\ U_l \to U_l$ by

$$a(u_l, v_l) = (\mathscr{A}_l u_l, v_l)_U \quad \forall\, u_l, v_l \in U_l. \tag{2.1}$$

In what follows, we often use the symbol $\mathscr{A}$ for $\mathscr{A}_1$. Denote σ_l to be the largest eigenvalue of $\mathscr{A}_l$. Assume $\overline{\sigma}_l \geqslant \sigma_l$, $l = 1, \ldots, L$, is an upper bound, $\overline{\sigma}_{l+1} \leqslant \overline{\sigma}_l$. Let $\mathscr{Q}_l\colon\ U \to U_l$ be an orthogonal projection and $\widetilde{\mathscr{Q}}_l\colon\ U \to U_l$ its spectrally equivalent approximation. The BPX preconditioner is defined by

$$\mathscr{B} = \frac{1}{\overline{\sigma}_1}\widetilde{\mathscr{Q}}_1 + \sum_{l=2}^{L}\Big(\frac{1}{\overline{\sigma}_l} - \frac{1}{\overline{\sigma}_{l-1}}\Big)\widetilde{\mathscr{Q}}_l. \tag{2.2}$$

Since $\mathscr{Q}_1 = \mathscr{I}$, where $\mathscr{I}$ denotes the identity mapping, we can also set $\widetilde{\mathscr{Q}}_1 = \mathscr{Q}_1 = \mathscr{I}$. Note that by rearranging the sums and setting $\widetilde{\mathscr{Q}}_{L+1} = 0$ we get

$$\begin{aligned}\mathscr{B} &= \frac{1}{\overline{\sigma}_1}\widetilde{\mathscr{Q}}_1 + \sum_{l=2}^{L}\frac{1}{\overline{\sigma}_l}\widetilde{\mathscr{Q}}_l - \sum_{l=2}^{L}\frac{1}{\overline{\sigma}_{l-1}}\widetilde{\mathscr{Q}}_l \\ &= \sum_{l=1}^{L}\frac{1}{\overline{\sigma}_l}\widetilde{\mathscr{Q}}_l - \sum_{l=1}^{L-1}\frac{1}{\overline{\sigma}_l}\widetilde{\mathscr{Q}}_{l+1} - \frac{1}{\overline{\sigma}_L}\widetilde{\mathscr{Q}}_{L+1} \\ &= \sum_{l=1}^{L}\frac{1}{\overline{\sigma}_l}(\widetilde{\mathscr{Q}}_l - \widetilde{\mathscr{Q}}_{l+1}).\end{aligned} \tag{2.3}$$

In the following theorem, we give a convergence bound of [2]. Since the proof is relatively simple and we prove a slightly different statement than the authors of [2] (with the upper bounds $\overline{\sigma}_l$ in the place of the actual maximal eigenvalues σ_l), we provide the proof in detail for readers' convenience.

Theorem 2.1 ([2]). *Assume there is a constant C_1 independent of l and L such that for every $u \in U$ and every level $l = 1, \ldots, L$, the exact projections $\mathscr{Q}_l$, $\mathscr{Q}_{L+1} = 0$, satisfy*

$$\|(\mathscr{I} - \mathscr{Q}_{l+1})u\|_U^2 \leqslant \frac{C_1}{\overline{\sigma}_l} a(u,u). \tag{2.4}$$

In addition, we assume that the operators $\widetilde{\mathscr{Q}}_l$, $l = 1, \ldots, L$, are symmetric, spectrally equivalent to projections $\mathscr{Q}_l$ in the sense that

$$c_2(\mathscr{Q}_l u, u)_U \leqslant (\widetilde{\mathscr{Q}}_l u, u)_U \leqslant C_2(\mathscr{Q}_l u, u)_U \quad \forall\, u \in U,\ l = 1, \ldots, L, \tag{2.5}$$

with constants $C_2 \geqslant c_2 > 0$ independent of l and L, $l = 1, \ldots, L-1$. Last, we assume that $\overline{\sigma}_{l+1} \leqslant \overline{\sigma}_l$ for all levels $l = 1, \ldots, L-1$. Then $\mathscr{B}$ is symmetric, $\mathscr{B}\mathscr{A}$ is $a(\cdot,\cdot)$-symmetric, and

$$\frac{c_2}{C_1 L} a(u,u) \leqslant a(\mathscr{B}\mathscr{A} u, u) \leqslant C_2 L a(u,u) \quad \forall\, u \in U \tag{2.6}$$

with constants $c_2/(C_1 L)$ and $C_2 L$ being the lower and the upper estimates of the lower and the upper spectral bound of $\mathscr{B}\mathscr{A}$, respectively.

The following proof is a masterpiece by Bramble, Pasciak, and Xu. The upper bound is more or less straightforward. The proof of coercivity (the lower bound) is similar to the proof of Lion's lemma.

Proof. The symmetry of $\mathscr{B}$ is obvious from the symmetry of $\widetilde{\mathscr{Q}}_l$, $l = 1, \ldots, L$ and definition of $\mathscr{B}$ (2.2). The $a(\cdot,\cdot)$-symmetry of $\mathscr{B}\mathscr{A}$ follows by a standard argument

$$a(\mathscr{B}\mathscr{A} u, v) = (\mathscr{A}\mathscr{B}\mathscr{A} u, v)_U = (\mathscr{A} u, \mathscr{B}\mathscr{A} v)_U = a(u, \mathscr{B}\mathscr{A} v).$$

Let us set $\mathscr{B}_{\mathrm{ex}}$ to be the operator $\mathscr{B}$ with $\widetilde{\mathscr{Q}}_l = \mathscr{Q}_l$ for all levels l. Let $u \in U$. By (2.2) and (2.1), we have

$$\begin{aligned} a(\mathscr{B}\mathscr{A} u, u) = {} & \frac{1}{\overline{\sigma}_1}(\widetilde{\mathscr{Q}}_1 \mathscr{A} u, \mathscr{A} u)_U \\ & + \sum_{l=2}^{L} \Big(\frac{1}{\overline{\sigma}_l} - \frac{1}{\overline{\sigma}_{l-1}}\Big)(\widetilde{\mathscr{Q}}_l \mathscr{A} u, \mathscr{A} u)_U \end{aligned}$$

and

$$a(\mathscr{B}_{\text{ex}}\mathscr{A}u, u) = \frac{1}{\overline{\sigma}_1}(\mathscr{Q}_1\mathscr{A}u, \mathscr{A}u)_U + \sum_{l=2}^{L}\Big(\frac{1}{\overline{\sigma}_l} - \frac{1}{\overline{\sigma}_{l-1}}\Big)(\mathscr{Q}_l\mathscr{A}u, \mathscr{A}u)_U$$

with

$$c_2(\mathscr{Q}_l\mathscr{A}u, \mathscr{A}u)_U \leqslant (\widetilde{\mathscr{Q}}_l\mathscr{A}u, \mathscr{A}u)_U \leqslant C_2(\mathscr{Q}_l\mathscr{A}u, \mathscr{A}u)_U$$

by (2.5). Therefore,

$$c_2 a(\mathscr{B}_{\text{ex}}\mathscr{A}u, u) \leqslant a(\mathscr{B}\mathscr{A}u, u) \leqslant C_2 a(\mathscr{B}_{\text{ex}}\mathscr{A}u, u) \quad \forall\, u \in U.$$

It is therefore sufficient to prove (2.6) with $\mathscr{B}_{\text{ex}}$ in the place of $\mathscr{B}$ and $c_2 = C_2 = 1$.

Set $U_{L+1} = \emptyset$. Define W_l to be the orthogonal complement of U_{l+1} in U_l, that is,

$$W_l = \{u \in U_l\colon\ (u, w)_U = 0 \quad \forall\, w \in U_{l+1}\}, \quad l = 1, \ldots, L.$$

Clearly, spaces W_l, $l = 1, \ldots, L$, form an orthogonal decomposition of U and the operators $\mathscr{Q}_l - \mathscr{Q}_{l+1}$ are orthogonal projections onto the respective spaces W_l. As a consequence of this orthogonality, (2.1) and (2.3), using the properties of orthogonal projections

$$\mathscr{Q}_l - \mathscr{Q}_{l+1} = (\mathscr{Q}_l - \mathscr{Q}_{l+1})^2 = (\mathscr{Q}_l - \mathscr{Q}_{l+1})^*$$

(* denotes the adjoint operator) and the Pythagorean Theorem,

$$\begin{aligned}
a(\mathscr{B}_{\text{ex}}\mathscr{A}u, u) &= \sum_{l=1}^{L}\frac{1}{\overline{\sigma}_l}a((\mathscr{Q}_l - \mathscr{Q}_{l+1})\mathscr{A}u, u)\\
&= \sum_{l=1}^{L}\frac{1}{\overline{\sigma}_l}((\mathscr{Q}_l - \mathscr{Q}_{l+1})\mathscr{A}u, \mathscr{A}u)_U\\
&= \sum_{l=1}^{L}\frac{1}{\overline{\sigma}_l}((\mathscr{Q}_l - \mathscr{Q}_{l+1})^2\mathscr{A}u, \mathscr{A}u)_U\\
&= \sum_{l=1}^{L}\frac{1}{\overline{\sigma}_l}((\mathscr{Q}_l - \mathscr{Q}_{l+1})\mathscr{A}u, (\mathscr{Q}_l - \mathscr{Q}_{l+1})\mathscr{A}u)_U\\
&= \sum_{l=1}^{L}\frac{1}{\overline{\sigma}_l}\|(\mathscr{Q}_l - \mathscr{Q}_{l+1})\mathscr{A}u\|_U^2 \qquad (2.7)\\
&= \sum_{l=1}^{L}\frac{1}{\overline{\sigma}_l}(\|\mathscr{Q}_l\mathscr{A}u\|_U^2 - \|\mathscr{Q}_{l+1}\mathscr{A}u\|_U^2)\\
&\leqslant \sum_{l=1}^{L}\frac{1}{\overline{\sigma}_l}\|\mathscr{Q}_l\mathscr{A}u\|_U^2. \qquad (2.8)
\end{aligned}$$

Let $\mathscr{P}_l$ be $a(\cdot,\cdot)$-orthogonal projection onto U_l, $l = 1, \ldots, L$, and $u, v \in U$. Since $\mathscr{P}_l$ is $a(\cdot,\cdot)$-symmetric, $\mathscr{P}_l = \mathscr{I}$ on U_l, $\mathscr{Q}_l$ is symmetric and $\mathscr{I} - \mathscr{Q}_l$ is the orthogonal projection onto $U_l^\perp$ ($\perp$ denotes the orthogonal complement),

$$\begin{aligned}
(\mathscr{Q}_l \mathscr{A} u, v)_U &= (\mathscr{A} u, \mathscr{Q}_l v)_U = (\mathscr{A} u, \mathscr{P}_l \mathscr{Q}_l v)_U = (\mathscr{A} \mathscr{P}_l u, \mathscr{Q}_l v)_U \\
&= (\mathscr{A}_l \mathscr{P}_l u, \mathscr{Q}_l v)_U = (\mathscr{A}_l \mathscr{P}_l u, \mathscr{Q}_l v)_U + (\mathscr{A}_l \mathscr{P}_l u, (\mathscr{I} - \mathscr{Q}_l) v)_U \\
&= (\mathscr{A}_l \mathscr{P}_l u, v)_U,
\end{aligned}$$

hence,

$$\mathscr{Q}_l \mathscr{A} = \mathscr{A}_l \mathscr{P}_l$$

and therefore,

$$\|\mathscr{Q}_l \mathscr{A} u\|_U^2 = \|\mathscr{A}_l \mathscr{P}_l u\|_U^2 \leqslant \overline{\sigma}_l a(\mathscr{P}_l u, \mathscr{P}_l u) \leqslant \overline{\sigma}_l a(u, u).$$

This estimate together with (2.8) prove the upper bound of (2.6) with $\mathscr{B}_{\mathrm{ex}}$ in the place of $\mathscr{B}$ and $C_2 = 1$.

To establish the lower bound of (2.6), we estimate using

$$\mathscr{I} = \sum_{l=1}^{L} (\mathscr{Q}_l - \mathscr{Q}_{l+1}),$$

the fact that $\mathscr{I} - \mathscr{Q}_l$ is an orthogonal projection onto $U_l^\perp$ and the Cauchy-Schwarz inequality,

$$\begin{aligned}
a(u, u) &= \sum_{l=1}^{L} a((\mathscr{Q}_l - \mathscr{Q}_{l+1}) u, u) \\
&= \sum_{l=1}^{L} ((\mathscr{Q}_l - \mathscr{Q}_{l+1}) u, \mathscr{A} u)_U \\
&= \sum_{l=1}^{L} ((\mathscr{Q}_l - \mathscr{Q}_{l+1}) u, (\mathscr{Q}_l - \mathscr{Q}_{l+1}) \mathscr{A} u)_U \\
&= \sum_{l=1}^{L} [((\mathscr{I} - \mathscr{Q}_l) u, (\mathscr{Q}_l - \mathscr{Q}_{l+1}) \mathscr{A} u)_U \\
&\qquad + ((\mathscr{Q}_l - \mathscr{Q}_{l+1}) u, (\mathscr{Q}_l - \mathscr{Q}_{l+1}) \mathscr{A} u)_U] \\
&= \sum_{l=1}^{L} ((\mathscr{I} - \mathscr{Q}_{l+1}) u, (\mathscr{Q}_l - \mathscr{Q}_{l+1}) \mathscr{A} u)_U \\
&\leqslant \sum_{l=1}^{L} \|(\mathscr{I} - \mathscr{Q}_{l+1}) u\|_U \|(\mathscr{Q}_l - \mathscr{Q}_{l+1}) \mathscr{A} u\|_U.
\end{aligned}$$

Thus, by assumption (2.4), the Cauchy-Schwarz inequality, and (2.7) we get

$$
\begin{aligned}
a(u,u) &\leqslant \sum_{l=1}^{L}\Big(\frac{C_1}{\overline{\sigma}_l}\Big)^{1/2} a^{1/2}(u,u)\|(\mathscr{Q}_l-\mathscr{Q}_{l+1})\mathscr{A}u\|_U \\
&\leqslant C_1^{1/2}a^{1/2}(u,u)\bigg(\sum_{l=1}^{L}\frac{1}{\overline{\sigma}_l}\|(\mathscr{Q}_l-\mathscr{Q}_{l+1})\mathscr{A}u\|_U^2\bigg)^{1/2}\bigg(\sum_{l=1}^{L}1^2\bigg)^{1/2} \\
&= (C_1L)^{1/2}a^{1/2}(u,u)a(\mathscr{B}_{\mathrm{ex}}u,u)^{1/2}.
\end{aligned}
$$

Assume $u \neq 0$. Dividing the above estimate by $a^{1/2}(u,u)$ and squaring the result, we get

$$(\mathscr{B}_{\mathrm{ex}}u,u) \geqslant \frac{1}{C_1L}a(u,u),$$

proving the first inequality of (2.6) for $\mathscr{B}_{\mathrm{ex}}$ in the place of $\mathscr{B}$ and $c_2 = 1$. For $u = 0$, (2.6) holds trivially. This completes our proof. $\square$

In the rest of this section, we describe the implementation of the method assuming the system of prolongators is given.

Let

$$A\mathbf{x} = \mathbf{b}$$

be the system of linear algebraic equations with an $n \times n$ symmetric, positive definite matrix A and $\mathbf{b} \in \mathbb{R}^n$. Set $n_1 = n$. We assume the system of injective linear prolongators

$$I_{l+1}^{l}\colon\ \mathbb{R}^{n_{l+1}} \to \mathbb{R}^{n_l} n_{l+1} < n_l, \quad l = 1,\dots,L-1,$$

is given. We set $(U,(\cdot,\cdot)_U,\|\cdot\|_U)$ to be the Euclidean space $(\mathbb{R}^n,\langle\cdot,\cdot\rangle,\|\cdot\|)$ and

$$a(\cdot,\cdot) = \langle A\cdot,\cdot\rangle. \tag{2.9}$$

We introduce *composite prolongators*

$$I_l^1 = I_2^1 I_3^2 \dots I_l^{l-1}, \quad l = 1,\dots,L.$$

The coarse-spaces are defined by

$$U_l = \mathrm{Range}(I_l^1), \quad l = 1,\dots,L$$

and coarse-level matrices by

$$A_l = (I_l^1)^{\mathrm{T}} A I_l^1.$$

Note that the matrix A_l is the operator $\mathscr{A}_l$ defined by (2.9) and (2.1), represented with respect to the basis given by the columns of I_l^1. The exact projection operators in the matrix form are

$$\mathscr{Q}_l = Q_l = I_l^1((I_l^1)^{\mathrm{T}} I_l^1)^{-1}(I_l^1)^{\mathrm{T}}, \quad l = 1, \ldots, L. \tag{2.10}$$

We choose the inexact projections to be the operators Q_l with the matrix $(I_l^1)^{\mathrm{T}} I_l^1$ replaced by its diagonal, that is

$$\widetilde{\mathscr{Q}}_l = \widetilde{Q}_l = I_l^1 D_l^{-1}(I_l^1)^{\mathrm{T}}, \quad D_l = \operatorname{diag}((I_l^1)^{\mathrm{T}} I_l^1), \quad l = 1, \ldots, L. \tag{2.11}$$

The action of the BPX preconditioner (2.2) is given by the following algorithm:

Algorithm 1. Given $\mathbf{x} \in \mathbb{R}^n$, evaluate the action $\mathbf{y} = B\mathbf{x} \in \mathbb{R}^n$ of the preconditioner $B = \mathscr{B}$ by

$$\mathbf{y} = \frac{1}{\overline{\sigma}_1}\mathbf{x} + \sum_{l=2}^{L}\Big(\frac{1}{\overline{\sigma}_l} - \frac{1}{\overline{\sigma}_{l-1}}\Big) I_l^1 D_l^{-1}(I_l^1)^{\mathrm{T}}\mathbf{x}, D_l = \operatorname{diag}((I_l^1)^{\mathrm{T}} I_l^1). \tag{2.12}$$

In (2.12), $\overline{\sigma}_l$ is an upper bound of

$$\sigma_l = \lambda_{\max}(\mathscr{A}_l) = \sup_{\mathbf{x}\in \operatorname{Range}(I_l^1)\setminus\{\mathbf{0}\}} \frac{\|\mathbf{x}\|_A^2}{\|\mathbf{x}\|^2} = \sup_{\mathbf{x}\in\mathbb{R}^{n_l}\setminus\{\mathbf{0}\}} \frac{\|I_l^1\mathbf{x}\|_A^2}{\|I_l^1\mathbf{x}\|^2}, \quad l = 1, \ldots, L. \tag{2.13}$$

The choice of $\overline{\sigma}_l$ is, for our model example, addressed in Remark 4.18.

Denote $\mathbf{f}_i^l$ to be the i-th column of I_l^1. Note that vectors $\{\mathbf{f}_i^l\}_{i=1}^{n_l}$ form a natural basis of $U_l = \operatorname{Range}(I_l^1)$. It is straightforward to see that the operation $\mathbf{y} = \widetilde{Q}_l\mathbf{x}$ can be implemented using the parallel loop

$$\mathbf{y} = \mathbf{0};\ \text{for } i = 1, \ldots, n_l \text{ do in parallel } \mathbf{y} \leftarrow \mathbf{y} + \frac{\langle \mathbf{x}, \mathbf{f}_i^l\rangle}{\|\mathbf{f}_i^l\|^2}\mathbf{f}_i^l$$

with the update of $\mathbf{y}$ being a critical section. (Only one of the parallel processes is allowed to perform the critical section at any moment.) Algorithm 1 can be therefore implemented in parallel using the operation of sparse inner product $\langle\cdot,\cdot\rangle_{l_2(\mathcal{I})}$ as follows:

Algorithm 2.

▷ Setup: given composite prolongators I_l^1, $l = 2, \ldots, L$, set $\mathbf{f}_i^l$ to be the i-th column of I_l^1 and evaluate D_{ii}^l, $l = 2, \ldots, L$, $i = 1, \ldots, n_l$, as follows:

⊳⊳ for all $l = 2, \ldots, L$, $i = 1, \ldots, n_l$, do in parallel

$$\text{set } D_{ii}^l = \langle \mathbf{f}_i^l, \mathbf{f}_i^l\rangle_{l_2(\operatorname{supp}(\mathbf{f}_i^l))}.$$

▷ Action: given $\mathbf{x} \in \mathbb{R}^{n_1}$, evaluate $\mathbf{y} = B\mathbf{x}$ as follows:

⧐ set $\mathbf{y} = \overline{\sigma}_1^{-1}\mathbf{x}$,

⧐ for all $l = 2, \ldots, L$, $i = 1, \ldots, n_{l+1}$ do in parallel

$$\mathbf{y} \leftarrow \mathbf{y} + ((\overline{\sigma}_l^{-1} - \overline{\sigma}_{l-1}^{-1})/D_{ii}^l)\langle \mathbf{f}_i^l, \mathbf{x}\rangle_{l_2(\mathrm{supp}(\mathbf{f}_i^l))}\mathbf{f}_i^l,$$

with the update of $\mathbf{y}$ being a critical section.

Note that in practice, the critical section can be avoided by colouring the graph of the overlaps of the supports of $\mathbf{f}_i^l$ and by performing the update of $\mathbf{y}$ colour by colour.

3. Smoothed aggregation prolongators in model case

In the smoothed aggregation method ([4], [5], [7]) we create prolongator I_{l+1}^l (assuming prolongators $I_2^1, \ldots, I_l^{l-1}$ are already given) in the form

$$I_{l+1}^l = S_l P_{l+1}^l.$$

Here, S_l is an $n_l \times n_l$ sparse linear *prolongator smoother*, being the first degree polynomial in $A_l = (I_l^1)^{\mathrm{T}} A I_l^1$, and P_{l+1}^l is an $n_l \times n_{l+1}$ *tentative prolongator* given by unknowns aggregation. The tentative prolongator is responsible for the approximation, while the prolongator smoother enforces the smoothness of the coarse-level spaces. The simplest prolongator P_{l+1}^l will be given in this section. For the most general form of tentative prolongator applicable to non-scalar problems on unstructured meshes, see [7].

Let $\Omega = (0,1) \times (0,1)$ be a computational domain. We consider a model problem

$$\text{find } u \in H_0^1(\Omega)\colon\ a(u,v) = f(v) \quad \forall\, v \in H_0^1(\Omega). \tag{3.1}$$

Here, $a(\cdot,\cdot) = (\nabla\cdot, \nabla\cdot)_{L_2(\Omega)}$ and $f(\cdot) \in (H_0^1(\Omega))^{-1}$. The problem is discretized by $P1$ elements on a uniform triangular mesh obtained from a regular square mesh when each square is broken by connecting its left lower and right upper vertices with a straight edge. We assume the number of interior nodes in the direction of both axes x and y is 3^{L-1}.

On the finest level, we form the aggregates (index sets of vertices) $\{\mathcal{A}_i^1\}_{i=1}^{9^{L-2}}$ by grouping the mesh vertices into 3×3 regular, square groups. For each aggregate, the central vertex represents the aggregate on the second level. Thus, we have mesh vertices on level 2 and the procedure can be repeated, giving rise to the hierarchy of the aggregates $\{\mathcal{A}_i^l\}_{i=1}^{9^{L-l-1}}$, $l = 1, \ldots, L-1$, and the hierarchy of nodal points

$\{\mathbf{v}_i^l\}_{i=1}^{n_l}$, $l = 1, \dots, L$, with $\mathbf{v}_i^l$ being the central point of the aggregate $\mathcal{A}_i^{l-1}$. Then, we define the tentative prolongators

$$(P_{l+1}^l)_{ij} = \begin{cases} 1 & \text{for } i \in \mathcal{A}_j^l, \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

$i = 1, \dots, n_l$, $j = 1, \dots, n_{l+1}$, $n_l = 9^{L-l}$, $l = 1, \dots, L-1$. Thus, P_{l+1}^l is a 0/1 matrix with disjoint non-zero structure. Each column of P_{l+1}^l corresponds to the disaggregation of one $\mathbb{R}^{n_{l+1}}$ variable into nine $\mathbb{R}^{n_l}$ variables. Thus, P_{l+1}^l can be thought of as a piecewise constant coarsening in a discrete sense.

Next we specify the prolongator smoother. Let $\bar{\lambda}_1 \geqslant \lambda_{\max}(A)$ be an available upper bound. We set

$$\bar{\lambda}_l = \bar{\lambda}_1 \tag{3.3}$$

for all levels $l = 2, \dots, L$. In Lemma 4.1, we will show that $\lambda_{\max}(A_l) \leqslant \bar{\lambda}_l$. Define prolongator smoothers S_l by

$$S_l = I - \frac{4}{3}\frac{1}{\bar{\lambda}_l} A_l, \quad l = 1, \dots L-1. \tag{3.4}$$

The parameter $\frac{4}{3}$ is chosen because, in a certain sense, it minimizes the upper bound of $\lambda_{\max}(S_l^2 A_l)$. The details are obvious from (4.5) in the proof of Lemma 4.1.

The choice of the upper bounds $\overline{\sigma}_l \geqslant \sigma_l$ needed in (2.12) is addressed by Remark 4.18.

4. Verification of the assumptions of the abstract theory

Define the coarse-level basis functions $\varphi_i^l = \pi_1 I_l^1 \mathbf{e}_i^l$, $l = 1, \dots, L$, $i = 1, \dots, n_l$. Here, π_1 is the finest level finite element interpolator

$$\pi_1\colon\ \mathbf{x} \in \mathbb{R}^n \mapsto \sum_i x_i \varphi_i^1$$

with $\{\varphi_i^1\}_{i=1}^n$ being the finest level finite element basis and $\mathbf{e}_i^l$ the i-th canonical basis vector of $\mathbb{R}^{n_l}$.

Lemma 4.1. *Assume $\bar{\lambda}_1 \geqslant \lambda_{\max}(A)$ is an available upper bound satisfying $\bar{\lambda}_1 \leqslant C\lambda_{\max}(A)$. Set $\bar{\lambda}_l = \bar{\lambda}_1$ for all levels $l = 1, \dots, L$. There is a constant $C > 0$*

independent of the mesh size h, level l and basis function number i such that for all $l = 1, \dots, L$, $i = 1, \dots, n_l$,

$$\|\varphi_i^l\|_{H^1(\Omega)} \leqslant C, \tag{4.1}$$

$$\|\varphi_i^l\|_{L_2(\Omega)} \leqslant Ch_l, \quad h_l = 3^{l-1}h, \tag{4.2}$$

and

$$\lambda_{\max}(A_l) \leqslant \bar{\lambda}_l \leqslant C. \tag{4.3}$$

P r o o f. Assume the first inequality of (4.3) holds for level l, $1 \leqslant l < L$. We will show that $\bar{\lambda}_{l+1} = \bar{\lambda}_l = \bar{\lambda}_1$ satisfies the first inequality of (4.3) as well. We estimate

$$\begin{aligned} \lambda_{\max}(A_{l+1}) &= \sup_{\mathbf{x} \in \mathbb{R}^{n_{l+1}} \setminus \{\mathbf{0}\}} \frac{\langle A_{l+1}\mathbf{x}, \mathbf{x}\rangle}{\|\mathbf{x}\|^2} \\ &= \sup_{\mathbf{x} \in \mathbb{R}^{n_{l+1}} \setminus \{\mathbf{0}\}} \frac{\langle A_l S_l P_{l+1}^l \mathbf{x}, S_l P_{l+1}^l \mathbf{x}\rangle}{\|\mathbf{x}\|^2} \\ &= \sup_{\mathbf{x} \in \mathbb{R}^{n_{l+1}} \setminus \{\mathbf{0}\}} \frac{\langle S_l^2 A_l P_{l+1}^l \mathbf{x}, P_{l+1}^l \mathbf{x}\rangle}{\|\mathbf{x}\|^2} \\ &\leqslant \lambda_{\max}(S_l^2 A_l) \sup_{\mathbf{x} \in \mathbb{R}^{n_{l+1}} \setminus \{\mathbf{0}\}} \frac{\|P_{l+1}^l \mathbf{x}\|^2}{\|\mathbf{x}\|^2}. \end{aligned} \tag{4.4}$$

Next we estimate $\lambda_{\max}(S_l^2 A_l)$ in terms of $\bar{\lambda}_l$. By the spectral mapping theorem,

$$\begin{aligned} \lambda_{\max}(S_l^2 A_l) &= \lambda_{\max}\Big(\Big(I - \frac{4}{3}\frac{1}{\bar{\lambda}_l} A_l\Big)^2 A_l\Big) \\ &= \max_{\lambda \in \sigma(A_l)} \Big(1 - \frac{4}{3}\frac{1}{\bar{\lambda}_l}\lambda\Big)^2 \lambda \\ &= \bar{\lambda}_l \max_{\lambda \in \sigma(A_l)} \Big(1 - \frac{4}{3}\frac{\lambda}{\bar{\lambda}_l}\Big)^2 \frac{\lambda}{\bar{\lambda}_l} \\ &\leqslant \bar{\lambda}_l \max_{t \in [0,1]} \Big(1 - \frac{4}{3}t\Big)^2 t \\ &= \frac{1}{9}\bar{\lambda}_l. \end{aligned} \tag{4.5}$$

Since each aggregate contains exactly 9 degrees of freedom, it holds that

$$\frac{\|P_{l+1}^l \mathbf{x}\|^2}{\|\mathbf{x}\|^2} = 9 \tag{4.6}$$

for all $\mathbf{x} \in \mathbb{R}^{n_l}$. This identity, (4.4), and (4.5) give

$$\bar{\lambda}_{l+1} \equiv \bar{\lambda}_l \geqslant \lambda_{\max}(A_{l+1}).$$

The proof of the first inequality of (4.3) follows by induction with $\bar{\lambda}_1 \geqslant \lambda_{\max}(A)$. Now, the well-known bound $\lambda_{\max}(A) \leqslant C$ ([3]) with C independent of h, gives the second inequality of (4.3).

The estimate (4.1) is a consequence (4.3). Indeed,

$$\begin{aligned} |\varphi_i^l|^2_{H^1(\Omega)} &= (\nabla \pi_1 I_l^1 \mathbf{e}_i^l, \nabla \pi_1 I_l^1 \mathbf{e}_i^l)_{L_2(\Omega)} = \langle A_1 I_l^1 \mathbf{e}_i^l, I_l^1 \mathbf{e}_i^l \rangle \\ &= \langle A_l \mathbf{e}_i^l, \mathbf{e}_i^l \rangle \leqslant \lambda_{\max}(A_l) \|\mathbf{e}_i^l\|^2 \leqslant C. \end{aligned}$$

Since the $H^1(\Omega)$-norm and the $H^1(\Omega)$-seminorm are equivalent on $H_0^1(\Omega)$ by Friedrichs' inequality, we get the statement (4.1).

Let us prove (4.2). It is well-known ([3]) that

$$ch\|\mathbf{x}\| \leqslant \|\pi_1 \mathbf{x}\|_{L_2(\Omega)} \leqslant Ch\|\mathbf{x}\| \quad \forall \mathbf{x} \in \mathbb{R}^{n_1} \tag{4.7}$$

with constants $C \geqslant c > 0$ independent of h. Finally estimate using $\varrho(S_l) = \lambda_{\max}(I - \frac{4}{3}{\lambda_l}^{-1} A) \leqslant 1$, (4.6), and (4.7),

$$\begin{aligned} \|\varphi_i^l\|_{L_2(\Omega)} &= \|\pi_1 I_l^1 \mathbf{e}_i^l\|_{L_2(\Omega)} \leqslant Ch\|I_l^1 \mathbf{e}_i^l\| \\ &= Ch\|S_1 P_2^1 I_l^2 \mathbf{e}_i^l\| \leqslant Ch\varrho(S_1)\|P_2^1 I_l^2 \mathbf{e}_i^l\| \\ &\leqslant Ch\|P_2^1 I_l^2 \mathbf{e}_i^l\| = 3Ch\|I_l^2 \mathbf{e}_i^l\| \\ &= \ldots = 3^{l-1} Ch\|I_l^l \mathbf{e}_i^l\| = Ch_l. \end{aligned}$$

This constitutes the proof of (4.2). □

To make our theory comprehensible, we introduce the notion of macroelement. The macroelement has two aspects: the set of associated basis functions $\{\varphi_i\}_{i\in\tau}$ (τ is an index set) and the geometrical domain T (understood closed) that contains the intersection of their supports, that is,

$$T \supset \bigcap_{i\in\tau} \operatorname{supp} \varphi_i. \tag{4.8}$$

The essential properties of the macroelements $\langle T, \{\varphi_j\}_{j\in\tau} \rangle$ are:

1. property (4.8),
2. the closed domains T have disjoint interiors and cover the entire computational domain,
3. except for the basis functions φ_i, $i \in \tau$, associated with the macroelement, no other basis functions are allowed to intersect $\operatorname{int} T$ with their supports.

The function on the macroelement is therefore a linear combination of macroelement basis functions $\{\varphi_i\}$, $i \in \tau$, satisfying (4.8) with no other basis functions involved. The rigorous definition follows:

Definition 4.2. Consider a computational domain Ω and a system of basis functions $\{\varphi_i\}$, with (well-defined) supports contained in $\overline{\Omega}$. Let $\{T_i\}$ be a family of closed domains $T_i \subset \overline{\Omega}$ such that

a)

$$\bigcup_i T_i = \overline{\Omega} \quad \text{and} \quad \operatorname{int} T_i \cap \operatorname{int} T_j = \emptyset \quad \text{for } i \neq j, \tag{4.9}$$

b) for every T_i there is an index set τ_i such that the corresponding set of basis functions $\{\varphi_j\}_{j\in\tau_i}$ satisfies

$$\bigcap_{j\in\tau_i} \operatorname{supp} \varphi_j \subset T_i \quad \text{and} \quad \operatorname{supp} \varphi_j \cap \operatorname{int} T_i = \emptyset \quad \forall\, j \notin \tau_i. \tag{4.10}$$

Then we call the system $\{\langle T_i,\ \{\varphi_j\}_{j\in\tau_i}\rangle\}_i$ a system of macroelements on Ω.

Remark 4.3. Clearly, the finite elements as concieved in [3] are, according to Definition 4.2, also macroelements.

It is a matter of routine to show that for Poisson equation in 1D discretized using $P1$ elements on a uniform grid, the coarse-space basis functions obtained by smoothed aggregation using the aggregates consisting of three neighbouring nodes are in fact $P1$ basis functions as well, see [9]. The coarse-level resolution is $3\times$ the fine-level resolution. The macroelements are then formed by overlaps of supports of two adjacent basis functions and are identical with coarse $P1$ elements.

Before introducing our macroelements and proving their properties, we need several auxiliary statements:

Lemma 4.4. *The coarse-level spaces satisfy the following properties:*

a) *The coarse-level matrices follow the nine-point scheme; entry a^l_{ij} of $A_l = (I^1_l)^{\mathrm{T}} A I^1_l$ can be non-zero only for directly adjacent (in the 9-point scheme) aggregates $\mathcal{A}^{l-1}_i$ and $\mathcal{A}^{l-1}_j$ on the level $l-1$. On the first level, the adjacency of the aggregates is considered assuming an underlying 9-point scheme instead of a 7-point scheme.*

b) *Apart from the vertices directly adjacent to the boundary with essential boundary condition, the vector of ones $\mathbf{1}_l \in \mathbb{R}^{n_l}$ forms the kernel of A_l, i.e.,*

$$(A_l \mathbf{1}_l)_i = 0 \tag{4.11}$$

for all vertices $\mathbf{v}_i^l$ *not adjacent to the boundary with essential boundary condition. Adjacency to the boundary is considered assuming an underlying* 9*-point scheme extended to the boundary nodes, see Fig.* 4.1.

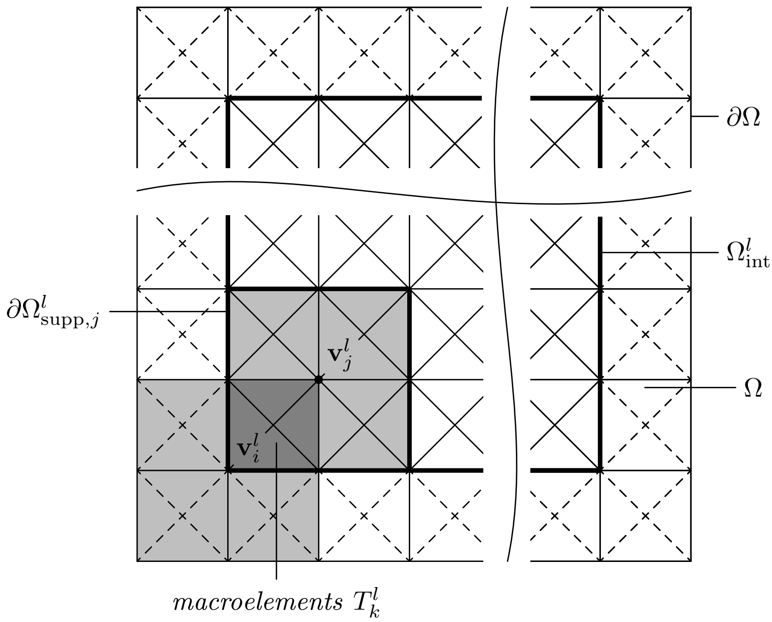

Fig. 4.1. Coarse-level geometry.

c) *Apart from the boundary with an essential boundary condition, the discrete basis functions* $I_l^1\mathbf{e}_i^l$ *form a decomposition of unity in the sense that*

$$(I_l^1\mathbf{1}_l)_i = \left(\sum_{j=1}^{n_l} I_l^1\mathbf{e}_j\right)_i = 1 \tag{4.12}$$

for all fine-level vertices $\mathbf{v}_i^1$ *that belong to the (closed) square* $\overline{\Omega}_{\text{int}}^l$ *with vertices* $\mathbf{v}_{i_{c_1}}^l, \mathbf{v}_{i_{c_2}}^l, \mathbf{v}_{i_{c_3}}^l, \mathbf{v}_{i_{c_4}}^l$ *adjacent to the corners of the unit square* Ω*. The continuous basis functions* $\varphi_i^l = \pi_1 I_l^1\mathbf{e}_i^l$ *satisfy*

$$\sum_{i=1}^{n_l} \varphi_i^l = 1 \quad \text{on } \overline{\Omega}_{\text{int}}^l, \tag{4.13}$$

see Fig. 4.1.

d) *The support of each of the basis functions* $\varphi_i^l = \pi_1 I_l^1\mathbf{e}_i^l$ *satisfies*

$$\operatorname{supp}\varphi_i^l \subset \Omega_{\text{supp},i}^l,$$

with $\Omega^l_{\mathrm{supp},i}$ being a (closed) square with side length $2h_l = 2.3^{l-1}h$ and the center of gravity located in $\mathbf{v}^l_i$. Apart from $\partial\Omega$, the vertices and edge midpoints of $\Omega^l_{\mathrm{supp},i}$ are vertices $\mathbf{v}^l_j$, $j \in \mathcal{N}^l_i \setminus \{i\}$, see Fig. 4.1. Here, $\mathcal{N}^l_i$ denotes the neighbourhood of i in the nine-point scheme.

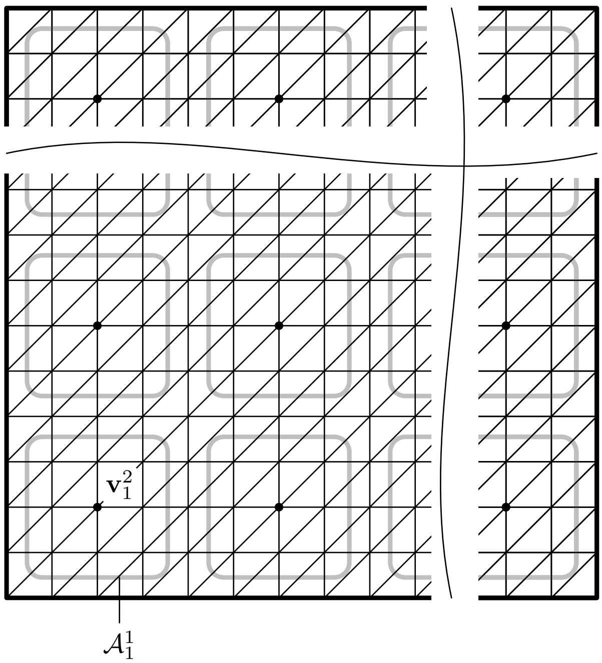

Fig. 4.2. Coarse-level geometry.

P r o o f. Let us prove statement a). Assume the stencil of A^{l-1} follows the nine-point scheme. Let $\mathcal{A}^{l-1}_i$ and $\mathcal{A}^{l-1}_j$ be two aggregates. Clearly, a^l_{ij} of $A_l = (I^{l-1}_l)^{\mathrm{T}} A_{l-1} I^{l-1}_l$ can be non-zero only if $\operatorname{supp} I^{l-1}_l \mathbf{e}^l_i$ and $\operatorname{supp} I^{l-1}_l \mathbf{e}^l_j$ are directly adjacent sets (in the 9-point scheme) of vertices on level $l-1$. Since $I^{l-1}_l = S_{l-1} P^{l-1}_l$, where S_l is a first-degree polynomial in A_{l-1} and P^{l-1}_l is given by disaggregation ($\operatorname{supp} P^{l-1}_l \mathbf{e}^l_i = \mathcal{A}^{l-1}_i$), the supports $\operatorname{supp} I^{l-1}_l \mathbf{e}^l_i$ and $\operatorname{supp} I^{l-1}_l \mathbf{e}^l_j$ are adjacent only for two directly adjacent aggregates $\mathcal{A}^{l-1}_i$ and $\mathcal{A}^{l-1}_j$. The proof of a) now follows by induction, with the fact that the matrix $A = A_1$, being a finite element stiffness matrix, follows the seven-point scheme which is a subset of the nine-point scheme.

Let us prove statement b). Assume vertex $\mathbf{v}^l_i$ is not adjacent to the boundary with essential boundary condition. Recall that matrices A_l on all levels follow the nine-point scheme. Assume statement b) holds on the level $l-1$. To prove our statement on the level l, it is sufficient to establish that

$$\sum_{j \in \mathcal{N}^l_i} a^l_{ij} = 0,$$

where $\mathcal{N}_i^l$ is the neighbourhood of i in the nine-point scheme. Clearly,

$$
\begin{aligned}
(4.14) \qquad \sum_{j\in\mathcal{N}_i^l} a_{ij}^l &= \sum_{j\in\mathcal{N}_i^l} \langle A^{l-1}\mathbf{e}_i^l, \mathbf{e}_j^l\rangle \\
&= \sum_{j\in\mathcal{N}_i^l} \langle A_{l-1}S_{l-1}^2 P_l^{l-1}\mathbf{e}_i^l, P_l^{l-1}\mathbf{e}_j^l\rangle \\
&= \Big\langle A_{l-1}S_{l-1}^2 P_l^{l-1}\mathbf{e}_i^l, \sum_{j\in\mathcal{N}_i^l} P_l^{l-1}\mathbf{e}_j^l\Big\rangle.
\end{aligned}
$$

Further, $\operatorname{supp} A_{l-1}S_{l-1}^2 P_l^{l-1}\mathbf{e}_i^l = \operatorname{supp} A_{l-1}(I-\omega A_{l-1})^2 P_l^{l-1}\mathbf{e}_i^l$ is contained in $\mathcal{A}_i^{l-1}$ with 3 layers added, which equals $\mathcal{A}_i^{l-1}$ with adjacent aggregates added. Therefore, we have

$$
(4.15) \qquad \Big(\sum_{j\in\mathcal{N}_i^l} P_l^{l-1}\mathbf{e}_j^l\Big)_k = 1 \quad \text{for} \quad k \in \mathcal{N} \equiv \bigcup_{j\in\mathcal{N}_i^l} \mathcal{A}_j^{l-1} \supset \operatorname{supp} A_{l-1}S_{l-1}^2 P_l^{l-1}\mathbf{e}_i^l.
$$

Denote $\operatorname{int}\mathcal{N}$ to be the interior of the above set $\mathcal{N} \subset \{1,\dots,n_{l-1}\}$, defined as

$$
\operatorname{int}\mathcal{N} = \{k\colon\ \mathcal{N}_k^{l-1} \subset \mathcal{N}\} \cap \{k\colon\ \mathbf{v}_k^{l-1} \text{ is not a vertex adjacent to the boundary } \partial\Omega\}.
$$

Clearly,

$$
\operatorname{supp} S_{l-1}^2 P_l^{l-1}\mathbf{e}_i^l \subset \operatorname{int}\mathcal{N}.
$$

From this, (4.14), and (4.15) it follows that

$$
\begin{aligned}
\sum_{j\in\mathcal{N}_i^l} a_{ij}^l &= \langle A_{l-1}S_{l-1}^2 P_l^{l-1}\mathbf{e}_i^l, \mathbf{1}_{l-1}\rangle = \langle S_{l-1}^2 P_l^{l-1}\mathbf{e}_i^l, A_{l-1}\mathbf{1}_l\rangle \\
&= \langle S_{l-1}^2 P_l^{l-1}\mathbf{e}_i^l, A_{l-1}\mathbf{1}_l\rangle_{l_2(\operatorname{int}\mathcal{N})} = 0
\end{aligned}
$$

as

$$
(A_{l-1}\mathbf{1}_{l-1})_k = 0 \quad \forall\, k \in \operatorname{int}\mathcal{N} \subset \operatorname{int}\{1,\dots,n_{l-1}\},
$$

by assumption, proving b) for level l. The proof of b) on all levels follows by induction, with the fact that the finite element stiffness matrix satisfies b).

Let us prove c). Consider a set $\mathcal{M} \subset \{1,\dots,n_l\}$. Let $\mathbf{x} \in \mathbb{R}^{n_l}$ be a vector such that $x_i^l = 1$ for all $i \in \mathcal{M}$. By property b), $(A_l\mathbf{x})_k = 0$ for all $k \in \operatorname{int}\mathcal{M}$ and therefore, $\mathbf{y}_l = S_l\mathbf{x} = (I-\omega A_l)\mathbf{x}$ satisfies $y_i = 1$ for all $i \in \operatorname{int}\mathcal{M}$. Assume c) holds for an intermediate I_l^k with some $k \in \{2,\dots,l\}$, that is,

$$
(I_l^k\mathbf{1}_l)_p = 1 \quad \forall\, p\colon\ \mathbf{v}_p^k \in \overline{\Omega}_{\text{int}}^l.
$$

See c). Then, $(P_k^{k-1} I_l^k \mathbf{1}_l)_p = 1$ for all vertices $\mathbf{v}_p^{k-1} \in \overline{\Omega}_{\text{int}}^l$, with one layer of vertices added. The vector

$$\mathbf{y} = I_l^{k-1} \mathbf{1}_l = S_{k-1} P_k^{k-1} I_l^k \mathbf{1}_l$$

therefore satisfies $y_p = 1$ for all p such that $\mathbf{v}_p^l \in \overline{\Omega}_{\text{int}}^l$. The proof now follows by induction, with the fact that $I_l^l \mathbf{1}_l = \mathbf{1}_l$ satisfies c). Property (4.13) is a direct consequence.

To prove d), it is sufficient to show that

$$\{\mathbf{v}_j^1,\ j \in \operatorname{supp} I_l^1 \mathbf{e}_i^l\} \subset \operatorname{int} \Omega_{\text{supp},i}^l. \tag{4.16}$$

Assume

$$\{\mathbf{v}_j^k,\ j \in \operatorname{supp} I_l^k \mathbf{e}_i^l\} \subset \operatorname{int} \Omega_{\text{supp},i}^l \tag{4.17}$$

for some $k \in \{2, \ldots, l\}$. Consider the set

$$\omega_i^{l,k-1} = \operatorname{supp} P_k^{k-1} I_{l-1}^k \mathbf{e}_i^l = \bigcup_{j \in \operatorname{supp} I_{l-1}^k \mathbf{e}_i^l} \mathcal{A}_j^{k-1}.$$

Obviously, for the set $\widetilde{\omega}_i^{l,k-1}$ consisting of $\omega_i^{l,k-1}$ with one layer of surrounding vertices added, that is,

$$\widetilde{\omega}_i^{l,k-1} = \omega_i^{l,k-1} \cup \{j \in \mathcal{N}_p^{k-1},\ p \in \omega_i^{l,k-1}\},$$

the corresponding set of vertices is contained in $\operatorname{int} \Omega_{\text{supp},i}^l$. The proof of (4.17) for $k-1$ in place of k is completed by the fact that

$$\operatorname{supp} I_l^{k-1} \mathbf{e}_i^l = \operatorname{supp}(I - \omega A_{k-1}) P_k^{k-1} I_l^k \mathbf{e}_i^l \subset \widetilde{\omega}_i^{l,k-1}.$$

The proof of (4.17) for all $k \in 1, \ldots, l$ follows by induction, with the fact that (4.17) obviously holds for $k = l$. $\square$

For $l > 1$, let us connect vertices $\mathbf{v}_j^l$, $j = 1, \ldots, n_l$, by the regular square mesh extended to the boundary $\partial\Omega$, see Fig. 4.1. This mesh consists of squares; let us choose a numbering of those squares (including those adjacent to the boundary) and denote them $\{T_i^l\}$. For each square T_i^l define an index set τ_i^l of numbers of vertices $\mathbf{v}_i^l$ that are its corner vertices. (Note that there are no vertices $\mathbf{v}_i^l$ located at $\partial\Omega$.)

Lemma 4.5. *For $l > 1$, the system $\{\langle T_i^l, \{\varphi_j^l\}_{j\in\tau_i^l}\rangle\}_i$, $\varphi_j^l = \pi_1 I_l^1 \mathbf{e}_j^l$, is a system of macroelements on Ω.*

P r o o f. We verify the conditions of Definition 4.2.

Obviously, squares T_j^l have disjoint interiors and cover the entire computational domain Ω. Thus, (4.9) holds.

By Lemma 4.4 d), vertices $\mathbf{v}_i^l$ are located at the centers of gravity of squares $\Omega_{\mathrm{supp},i}^l \supset \operatorname{supp}\varphi_i^l$. Clearly,

$$T_i^l = \bigcap_{j\in\tau_i^l} \Omega_{\mathrm{supp},j}^l \supset \bigcap_{j\in\tau_i^l} \operatorname{supp}\varphi_j^l$$

and for $j \notin \tau_i^l$

$$\operatorname{int} T_i^l \cap \operatorname{supp}\varphi_j^l \subset \operatorname{int} T_i^l \cap \Omega_{\mathrm{supp},j}^l = \emptyset.$$

This proves (4.10). □

Lemma 4.6. *For $l > 1$, basis functions φ_i^l, $i = 1,\dots,n_l$, satisfy the following properties:*

1. *The $H^1(\Omega)$-seminorm and $L_2(\Omega)$-norm of each φ_i^l, $l = 1,\dots,L$, $i = 1,\dots,n_l$, are bounded by*

$$|\varphi_i^l|_{H^1(\Omega)} \leqslant C \tag{4.18}$$

   and

$$\|\varphi_i^l\|_{L_2(\Omega)} \leqslant Ch_l. \tag{4.19}$$

   Here, $C > 0$ is a constant independent of the mesh size h, level l, and basis function number i.

2. *For T_i^l that is not adjacent to a boundary with an essential boundary condition, the quadruple of associated basis functions ($\tau_i^l = \{i_1, i_2, i_3, i_4\}$) satisfies*

$$\sum_{j=1}^{4} \varphi_{i_j}^l = 1 \quad \text{on } T_i^l.$$

3. *On the edges of an interior macroelement T_i^l, the traces of basis functions satisfy*

$$\begin{aligned} \operatorname{tr}\varphi_{i_1}^l &= 0 \quad \text{on } e_2\cup e_3, & \operatorname{tr}\varphi_{i_2}^l &= 0 \quad \text{on } e_3\cup e_4,\\ \operatorname{tr}\varphi_{i_3}^l &= 0 \quad \text{on } e_4\cup e_1, & \operatorname{tr}\varphi_{i_4}^l &= 0 \quad \text{on } e_1\cup e_2. \end{aligned}$$

   In the above identity, the edges e_j of T_i^l (in the local numbering) and vertices $\mathbf{v}_{i_j}^l$ are numbered in the same way as on the reference element as depicted in Fig. 4.3.

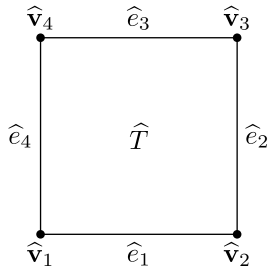

Fig. 4.3. Reference square.

P r o o f. Statement No. 1 follows directly from Lemma 4.1.

Statement No. 3 of our lemma follows by Lemma 4.4 d). The decomposition of unity (Statement No. 2) follows by Lemma 4.4 c) and d). This completes our proof. $\square$

Lemma 4.7. *Let $\widehat{T}$ be a unit square with edges and vertices as depicted in Fig. 4.3. Define the set $\mathcal{G} = \{(\widehat{\varphi}_1, \widehat{\varphi}_2, \widehat{\varphi}_3, \widehat{\varphi}_4)^{\mathrm{T}}\} \subset [H^1(\widehat{T})]^4$, where each function $\widehat{\varphi}_i$ is associated with vertex $\widehat{\mathbf{v}}_i$, by the following properties:*

1. *There is a positive finite constant C such that*

$$\|\widehat{\varphi}_i\|_{H^1(\widehat{T})} \leqslant C, \quad i = 1, \ldots, 4. \tag{4.20}$$

2. *The functions $\widehat{\varphi}_i$, $i = 1, \ldots, 4$, satisfy the decomposition of unity*

$$\sum_{i=1}^{4} \widehat{\varphi}_i = 1 \quad \text{on } \widehat{T}. \tag{4.21}$$

3. *The traces $\operatorname{tr} \widehat{\varphi}_i \in H^{1/2}(\partial \widehat{T})$ of functions $\widehat{\varphi}_i$, $i = 1, \ldots, 4$, on the boundary $\partial \widehat{T}$, denoted later simply as $\widehat{\varphi}_i$, satisfy*

$$\begin{aligned}
\widehat{\varphi}_1 &= 0 \quad \text{on } \widehat{e}_2 \cup \widehat{e}_3, \\
\widehat{\varphi}_2 &= 0 \quad \text{on } \widehat{e}_3 \cup \widehat{e}_4, \\
\widehat{\varphi}_3 &= 0 \quad \text{on } \widehat{e}_1 \cup \widehat{e}_4, \\
\widehat{\varphi}_4 &= 0 \quad \text{on } \widehat{e}_1 \cup \widehat{e}_2.
\end{aligned}$$

Define the Gram matrix

$$G = G(\widehat{\varphi}_1, \widehat{\varphi}_2, \widehat{\varphi}_3, \widehat{\varphi}_4) \equiv \{g_{ij} = (\widehat{\varphi}_i, \widehat{\varphi}_j)_{L_2(\widehat{T})}\}_{ij=1}^{4}.$$

There is a positive constant c dependent exclusively on C such that

$$\lambda_{\min}(G) \geqslant c > 0$$

for all quadruples $(\widehat{\varphi}_1, \widehat{\varphi}_5, \widehat{\varphi}_3, \widehat{\varphi}_4)^{\mathrm{T}} \in \mathcal{G}$.

Proof. Let us prove that the basis functions $\widehat{\varphi}_i$, $i = 1, \ldots, 4$, are linearly independent. Assume for now the contrary, i.e.,

$$\sum_{i=1}^{4} c_i \widehat{\varphi}_i = 0 \quad \text{with some } c_i \neq 0.$$

By property No. 3, $\widehat{\varphi}_3 = \widehat{\varphi}_4 = 0$ on $\widehat{e}_1$. Hence, by the assumption of the decomposition of unity (4.21),

$$\widehat{\varphi}_1 + \widehat{\varphi}_2 = 1 \quad \text{on } \widehat{e}_1 \tag{4.22}$$

and, by the assumption of the linear dependence,

$$c_1\widehat{\varphi}_1 + c_2\widehat{\varphi}_2 = 0 \quad \text{on } \widehat{e}_1, \tag{4.23}$$

with $c_1 \neq 0$ or $c_2 \neq 0$. Let us say that $c_1 \neq 0$. We will show that properties (4.22), (4.23), and $c_1 \neq 0$ exclude each other.

By (4.23) and (4.22) it follows that

$$(c_1 - c_2)\widehat{\varphi}_1 = -c_2 \quad \text{on } \widehat{e}_1.$$

Let $c_2 \neq 0$. Then $\widehat{\varphi}_1 = \text{const} \neq 0$ on $\widehat{e}_1$. Since $\widehat{\varphi}_1 = 0$ on $\widehat{e}_2$ by property No. 3, it follows that there is a jump in the trace of $\widehat{\varphi}_1$ at the point $\widehat{\mathbf{v}}_2$ and thus, $\widehat{\varphi}_1 \notin H^{1/2}(\partial\widehat{T})$, which contradicts the trace theorem, as $\|\widehat{\varphi}_1\|_{H^1(\widehat{T})} \leqslant C < \infty$.

Consider now the case of $c_2 = 0$. Then, by (4.23) $c_1\widehat{\varphi}_1 = 0$, hence by (4.22) it follows that $c_1(1 - \widehat{\varphi}_2) = 0$ on $\widehat{e}_1$. Since $c_1 \neq 0$ by the assumption, it follows that $\widehat{\varphi}_2 = 1$ on $\widehat{e}_1$. Since $\widehat{\varphi}_2 = 0$ on $\widehat{e}_4$, there is a jump in the trace of $\widehat{\varphi}_2$ at the point $\widehat{\mathbf{v}}_1$, hence $\widehat{\varphi}_2 \notin H^{1/2}(\partial\widehat{T})$, which contradicts the trace theorem. Thus, $c_1 \neq 0$ leads to contradiction for any c_2, hence $c_1 = 0$.

Due to the double axial symmetry of $\widehat{T}$ (with respect to both x and y), it follows that $\sum\limits_{i=1}^{4} c_i\widehat{\varphi}_i = 0$ implies $c_1 = c_2 = c_3 = c_4 = 0$ and therefore the basis functions $\widehat{\varphi}_i$, $i = 1, \ldots, 4$, are linearly independent.

Since G is a Gram matrix corresponding to the linearly independent basis, the functional

$$\Phi(\widehat{\varphi}_1, \widehat{\varphi}_2, \widehat{\varphi}_3, \widehat{\varphi}_4) = \lambda_{\min}(G(\widehat{\varphi}_1, \widehat{\varphi}_2, \widehat{\varphi}_3, \widehat{\varphi}_4))$$

is a positive functional on $\mathcal{G}$. By the Cauchy-Schwarz inequality, entries $g_{ij} = (\widehat{\varphi}_i, \widehat{\varphi}_j)_{L_2(\widehat{T})}$ are continuous bilinear forms on $L_2(\widehat{T})$, hence continuous functionals on $\mathcal{G}$. At the same time, eigenvalues of G depend continuously on the entries g_{ij}. Thus, Φ is a continuous, positive functional on $[L_2(\widehat{T})]^4$. In the rest of the proof we will show that the set $\mathcal{G}$ is compact in $[L_2(\widehat{T})]^4$. Clearly, the set $\mathcal{G}$ is bounded in $[H^1(\widehat{T})]^4$, hence weakly precompact. Further, the set $\mathcal{G}$ is convex. Indeed, for two functions $\widehat{\varphi}_i$, $\widehat{\varphi}_i'$ such that

$$\|\widehat{\varphi}_i\|_{H^1(\widehat{T})} \leqslant C, \quad \|\widehat{\varphi}_i'\|_{H^1(\widehat{T})} \leqslant C$$

and α, β non-negative numbers such that $\alpha + \beta = 1$, it holds that

$$\|\alpha\widehat{\varphi}_i + \beta\widehat{\varphi}_i'\|_{H^1(\widehat{T})} \leqslant \alpha\|\widehat{\varphi}_i\|_{H^1(\widehat{T})} + \beta\|\widehat{\varphi}_i\|_{H^1(\widehat{T})} \leqslant (\alpha+\beta)C = C.$$

For two quadruples of functions $\{(\widehat{\varphi}_1, \widehat{\varphi}_2, \widehat{\varphi}_3, \widehat{\varphi}_4)^{\mathrm{T}}\}$ and $\{(\widehat{\varphi}_1', \widehat{\varphi}_2', \widehat{\varphi}_3', \widehat{\varphi}_4')^{\mathrm{T}}\}$ satisfying equality constraints No. 2 and No. 3, their convex combination $\{\alpha\widehat{\varphi}_i + \beta\widehat{\varphi}_i'\}_{i=1}^4$, $\alpha + \beta = 1$, $\alpha, \beta \in \mathbb{R}_0^+$, satisfies conditions No. 2 and No. 3, too. Thus, $\mathcal{G}$ is convex and weakly precompact. Since $\mathcal{G}$ is closed in $[H^1(\widehat{T})]^4$ and convex, it is weakly closed, hence weakly compact in $[H^1(\widehat{T})]^4$. By Rellich's theorem, $\mathcal{G}$ is compact in $[L_2(\widehat{T})]^4$.

Summing up, Φ is a continuous positive functional on $\mathcal{G} \subset [L_2(\widehat{T})]^4$, with $\mathcal{G}$ being a compact set. Thus, Φ attains its positive minimum on $\mathcal{G}$, proving our statement.

$\square$

Remark 4.8. Let $\widehat{\varphi}_i$, $i = 1, \ldots, 4$, be basis functions satisfying assumptions of Lemma 4.7, G the corresponding $L_2(\widehat{T})$-Gram matrix, and $\widehat{u} = \sum_{i=1}^4 u_i\widehat{\varphi}_i$, $\mathbf{u} = (u_1, u_2, u_3, u_4)^{\mathrm{T}} \in \mathbb{R}^4$. Then,

$$\|\widehat{u}\|_{L_2(\widehat{T})}^2 = \left(\sum_{i=1}^4 u_i\widehat{\varphi}_i, \sum_{j=1}^4 u_j\widehat{\varphi}_j\right)_{L_2(\widehat{T})} = \sum_{i=1}^4\sum_{j=1}^4 (\widehat{\varphi}_i, \widehat{\varphi}_j)_{L_2(\widehat{T})} u_i u_j = \langle G\mathbf{u}, \mathbf{u}\rangle. \tag{4.24}$$

Hence, by Lemma 4.7 it follows that

$$\|\widehat{u}\|_{L_2(\widehat{T})}^2 = \|\mathbf{u}\|_G^2 \geqslant \lambda_{\min}(G)\|\mathbf{u}\|^2 \geqslant c\|\mathbf{u}\|^2. \tag{4.25}$$

At the same time, (4.24) gives

$$\|\widehat{u}\|_{L_2(\widehat{T})}^2 \leqslant \lambda_{\max}(G)\|\mathbf{u}\|^2, \tag{4.26}$$

where

$$|g_{ij}| = (\widehat{\varphi}_i, \widehat{\varphi}_j)_{L_2(\widehat{T})} \leqslant \|\widehat{\varphi}_i\|_{L_2(\widehat{T})}\|\widehat{\varphi}_j\|_{L_2(\widehat{T})} \leqslant \|\widehat{\varphi}_i\|_{H^1(\widehat{T})}\|\widehat{\varphi}_j\|_{H^1(\widehat{T})} \leqslant C.$$

Thus, by Gershgorin's theorem,

$$\lambda_{\max}(G) \leqslant C.$$

This bound, (4.26), and the coercivity estimate (4.25) yield uniform norm equivalence

$$c\|\mathbf{u}\|^2 \leqslant \|\widehat{u}\|^2_{L_2(\widehat{T})} \leqslant C\|\mathbf{u}\|^2$$

with constants $C \geqslant c > 0$ dependent exclusively on the constant C in (4.20).

Lemma 4.9. *Consider the affine mapping $\varphi(\cdot)\colon\ \mathbb{R}^2 \to \mathbb{R}^2$ that maps unit square $\widehat{T}$ onto square T with the side length H and the left lower vertex located at $\mathbf{b} \in \mathbb{R}^2$. The mapping is given by*

$$\varphi(\widehat{\mathbf{x}}) = \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \widehat{\mathbf{x}} + \mathbf{b}. \tag{4.27}$$

See Fig. 4.4. Let $u \in H^1(T)$. Define the transformed function $\widehat{u} \in H^1(\widehat{T})$ by

$$\widehat{u}(\widehat{\mathbf{x}}) = u(\varphi(\widehat{\mathbf{x}})). \tag{4.28}$$

Then it holds that

$$\|\widehat{u}\|_{L_2(\widehat{T})} = H^{-1}\|u\|_{L_2(T)}, \tag{4.29}$$

$$|\widehat{u}|_{H^1(\widehat{T})} = |u|_{H^1(T)}. \tag{4.30}$$

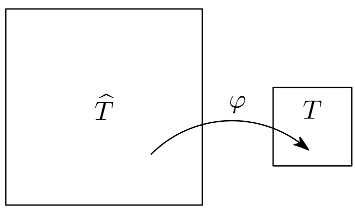

Fig. 4.4. The macroelement transformation.

P r o o f. The proof follows by the elementary transformation of the integrals. □

R e m a r k 4.10. Let T_i^l be an interior macroelement. From Lemma 4.6 and Lemma 4.9 it follows that the associated basis functions $\varphi^l_{i_j}$, $j = 1, \dots, 4$, transformed by the mapping (4.27) via (4.28) (that is, resulting functions $\widehat{\varphi}_j$, $j = 1, \dots, 4$) satisfy the assumptions of Lemma 4.7. Indeed, by Lemma 4.6 we get $\|\varphi^l_{i_j}\|_{H^1(T_i^l)} \leqslant C$. Hence, by Lemma 4.9 it follows that $\|\widehat{\varphi}_j\|_{H^1(\widehat{T})} \leqslant C$. Assumptions No. 2 and No. 3 of Lemma 4.7 follow from properties No. 2 and No. 3 proved in Lemma 4.6.

Lemma 4.11. *Define the level l interpolation operator*

$$\pi_l\colon\ \mathbf{x} \in \mathbb{R}^{n_l} \mapsto \sum_{i=1}^{n_l} x_i \varphi_i^l, \quad l = 1, \ldots, L. \tag{4.31}$$

There are positive constants $C \geqslant c$ independent of the mesh size h and level l such that for every level $l = 1, \ldots, L$ and every $\mathbf{u} \in \mathbb{R}^{n_l}$, the following norm equivalence holds:

$$c h_l \|\mathbf{u}\| \leqslant \|\pi_l \mathbf{u}\|_{L_2(\Omega)} \leqslant C h_l \|\mathbf{u}\|. \tag{4.32}$$

Proof. Let us prove first the left inequality of (4.32). Define Ω^l_{int} to be the union of all macroelements T_i^l that are not adjacent to the boundary with the essential boundary condition, and $\mathcal{T}_i^l$ to be the set of indices of basis functions associated with macroelement T_i^l. Assume T_i^l is an interior macroelement. The entries of the set $\mathcal{T}_i^l = \{j_1, j_2, j_3, j_4\}$ are ordered in the same way as the vertices in Fig. 4.3.

Let us consider the affine mapping φ_i^l that maps the unit square $\widehat{T}$ onto T_i^l as in Lemma 4.9. Consider a function $u = \pi_l \mathbf{u}$, $\mathbf{u} \in \mathbb{R}^{n_l}$. Clearly,

$$u = \sum_{j \in \mathcal{T}_i^l} u_j \varphi_j^l \quad \text{on } T_i^l.$$

Define the transformed function

$$\widehat{u}(\widehat{\mathbf{x}}) = u(\varphi_i^l(\widehat{\mathbf{x}})), \quad \widehat{\mathbf{x}} \in \widehat{T}.$$

Further, define the transformed basis

$$\widehat{\varphi}_k(\widehat{\mathbf{x}}) = \varphi^l_{j_k}(\varphi_i^l(\widehat{\mathbf{x}})), \quad k = 1, \ldots, 4.$$

Then,

$$\widehat{u}(\widehat{\mathbf{x}}) = \sum_{k=1}^{4} u_{j_k} \widehat{\varphi}_k(\widehat{\mathbf{x}}). \tag{4.33}$$

By Remark 4.10, the transformed basis functions $\{\widehat{\varphi}_k\}_{k=1}^4$ satisfy the assumptions of Lemma 4.7. Hence, denoting $G = \{(\widehat{\varphi}_i, \widehat{\varphi}_j)_{L_2(\widehat{T})}\}_{i,j=1}^4$, the Gram matrix corresponding to the transformed basis, we have the estimate

$$\Big\| \sum_{i=1}^4 w_i \widehat{\varphi}_i \Big\|^2_{L_2(\widehat{T})} = \langle G\mathbf{w}, \mathbf{w}\rangle \geqslant \lambda_{\min}(G) \|\mathbf{w}\|^2 \geqslant c \sum_{i=1}^4 w_i^2 \quad \forall\, \mathbf{w} \in \mathbb{R}^4.$$

See Remark 4.8. By this inequality, (4.33), and Lemma 4.9, it follows that

$$\|u\|^2_{L_2(T_i^l)} = h_l^2\|\widehat{u}\|^2_{L_2(\widehat{T})} = h_l^2\Big\|\sum_{k=1}^4 u_{j_k}\widehat{\varphi}_k\Big\|^2_{L_2(\widehat{T})} \geqslant ch_l^2\sum_{k=1}^4 u_{j_k}^2 = ch_l^2\sum_{j\in\mathcal{T}_i^l} u_j^2.$$

Since each degree of freedom i belongs to at least one set $\mathcal{T}_j^l$, the previous inequality gives

$$\|\mathbf{u}\|^2 \leqslant \sum_{T_i^l\subset\Omega^l_{\mathrm{int}}}\sum_{j\in\mathcal{T}_i^l} u_j^2 \leqslant C^{-1}h_l^{-2}\sum_{T_i^l\subset\Omega^l_{\mathrm{int}}}\|u\|^2_{L_2(T_i^l)} \leqslant C^{-1}h_l^{-2}\|u\|^2_{L_2(\Omega)},$$

completing the proof.

The second inequality of (4.32) is more or less trivial. Define the global Gram matrix

$$G^l = \{(\varphi_i^l,\varphi_j^l)_{L_2(\Omega)}\}_{i,j=1}^{n_l}.$$

The (minimal) constant C in the second inequality of (4.32) is then $\sqrt{\lambda_{\max}(G^l)}$. (See Remark 4.8.) The matrix G^l contains at most 9 non-zeroes per row and the non-zeroes can be estimated by the Cauchy-Schwarz inequality and Lemma 4.6 by

$$|g_{ij}^l| \leqslant \|\varphi_i^l\|_{L_2(\Omega)}\|\varphi_j^l\|_{L_2(\Omega)} \leqslant Ch_l^2.$$

By Gershgorin's theorem, $\lambda_{\max}(G^l) \leqslant Ch_l^2$ and the proof follows. □

Corollary 4.12. *For the Gram matrix $M_l = (I_l^1)^{\mathrm{T}}I_l^1$, $l=1,\dots,L$, corresponding to the discrete basis $\{I_l^1\mathbf{e}_i^l\}_{i=1}^{n_l}$, the diagonal matrix $D_l = \operatorname{diag}(M_l)$ is uniformly spectrally equivalent in the sense that the equivalence*

$$c\|\mathbf{x}\|_{M_l} \leqslant \|\mathbf{x}\|_{D_l} \leqslant C\|\mathbf{x}\|_{M_l} \quad \forall\,\mathbf{x}\in\mathbb{R}^{n_l} \tag{4.34}$$

holds with constants $C \geqslant c > 0$ independent of h and l. As a consequence, assumption (2.5) holds for $\widetilde{\mathscr{Q}}_l = \widetilde{Q}^l$ and $\mathscr{Q}_l = Q_l$ given by (2.10) and (2.11).

P r o o f. From (4.32) follows the uniform norm equivalence

$$c3^{l-1}\|\mathbf{x}\| \leqslant \|I_l^1\mathbf{x}\| = \|\mathbf{x}\|_{M_l} \leqslant C3^{l-1}\|\mathbf{x}\| \quad \forall\,\mathbf{x}\in\mathbb{R}^{n_l}.$$

Hence, M_l is well-conditioned. The eigenvalues of D_l satisfy

$$\lambda_i(D_l) \equiv (D_l)_{ii} = \|\mathbf{e}_i^l\|^2_{M_l} = \frac{\|\mathbf{e}_i^l\|^2_{M_l}}{\|\mathbf{e}_i^l\|^2} \in [\lambda_{\min}(M_l),\lambda_{\max}(M_l)] \subset [c3^{l-1}, C3^{l-1}],$$

proving (4.34).

From (4.34) follows $C^{-1}\|\cdot\|_{M_l^{-1}} \leqslant \|\cdot\|_{D_l^{-1}} \leqslant c^{-1}\|\cdot\|_{M_l^{-1}}$ and therefore,

$$\begin{aligned} C^{-1}\langle Q_l\mathbf{u},\mathbf{u}\rangle &= C^{-1}\langle M_l^{-1}(I_l^1)^{\mathrm{T}}\mathbf{u},(I_l^1)^{\mathrm{T}}\mathbf{u}\rangle \leqslant \langle D_l^{-1}(I_l^1)^{\mathrm{T}}\mathbf{u},(I_l^1)^{\mathrm{T}}\mathbf{u}\rangle \\ &= \langle \widetilde{Q}_l\mathbf{u},\mathbf{u}\rangle \leqslant c^{-1}\langle M_l^{-1}(I_l^1)^{\mathrm{T}}\mathbf{u},(I_l^1)^{\mathrm{T}}\mathbf{u}\rangle = c^{-1}\langle Q_l\mathbf{u},\mathbf{u}\rangle \quad \forall\,\mathbf{u}\in U \end{aligned}$$

with constants $C \geqslant c > 0$ from (4.34), proving assumption (2.5) of Theorem 2.1. □

Lemma 4.13 (Scaled Poincaré and Friedrichs' inequality)**.** *Let T be a square domain with side of length H. Then there is a constant $C > 0$ independent of H (and, characteristic for a square) such that (Poincaré inequality)*

$$\inf_{c\in\mathbb{R}} \|u-c\|_{L_2(T)} \leqslant CH|u|_{H^1(T)} \quad \forall\,u\in H^1(T), \tag{4.35}$$

and (Friedrichs' inequality)

$$\|u\|_{L_2(T)} \leqslant CH|u|_{H^1(T)} \quad \forall\,u\in H^1_{0,\Gamma}(T) \equiv \{u\in H^1(T)\colon\ \operatorname{tr} u = 0 \text{ on } \Gamma\}, \tag{4.36}$$

if $\Gamma\subset\partial T$ contains at least one edge of T.

P r o o f. The proof follows from Poincaré and Friedrichs' inequalities on a unit square by scaling, using Lemma 4.9. □

For each vertex $\mathbf{v}_i^l$ we introduce a ball $B_i^l\subset\Omega$ with center in $\mathbf{v}_i^l$ that has measure about $\mu(T_j^l)$, in the sense that there are constants $C \geqslant c > 0$ independent of the mesh size h, level l, and basis function number i such that

$$ch_l^2 \leqslant \mu(B_i^l) \leqslant Ch_l^2, \tag{4.37}$$

see Fig. 4.5. For convenience, we assume that the balls B_i^l do not intersect each other. We encapsulate each domain

$$T_j^l \cup \bigcup_{i\in\mathcal{T}_j^l} B_i^l$$

into (the nearly smallest possible) square $\widetilde{T}_j^l$. Clearly, the intersections of the extended macroelements $\widetilde{T}_j^l$ are bounded, that is, there is a finite integer N independent of level such that each $\mathbf{x}\in\Omega$ belongs to at most N extended macroelements $\widetilde{T}_i^l$.

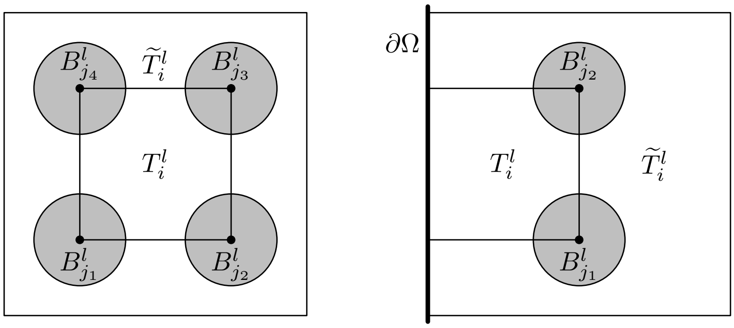

Fig. 4.5. Extended macroelement.

Both for the interior and for the boundary macroelement, define the local interpolation operator $\Pi^l_i\colon\ L_2(\widetilde{T}^l_i) \to L_2(T^l_i)$ by

$$\Pi^l_i u = \sum_{j\in\mathcal{T}^l_i} \Big(\frac{1}{\mu(B^l_j)} \int_{B^l_j} u \,\mathrm{d}\mathbf{x}\Big) \varphi^l_j. \tag{4.38}$$

Here, $\mathcal{T}^l_i$ is an index set of the numbers of basis functions associated with T^l_i.

Next we prove $L_2(\Omega)$-stability of the local interpolation operator.

Lemma 4.14. *Both for the interior and for the boundary macroelement T^l_i, the mapping Π^l_i is stable in the L_2-norm, in the sense that*

$$\|\Pi^l_i u\|_{L_2(T^l_i)} \leqslant C\|u\|_{L_2(\widetilde{T}^l_i)} \quad \forall\, u \in L_2(\widetilde{T}^l_i) \tag{4.39}$$

with a positive constant C independent of the level l, mesh size h, and macroelement number i.

P r o o f. We estimate using the definition of Π^l_i in (4.38), the triangle inequality, the Cauchy-Schwarz inequality, L_2-bound (4.2), and (4.37). We have

$$\begin{aligned}
\|\Pi^l_i u\|_{L_2(T^l_i)} &= \Big\| \sum_{j\in\mathcal{T}^l_i} \Big(\frac{1}{\mu(B^l_j)} \int_{B^l_j} u\,\mathrm{d}\mathbf{x}\Big)\varphi^l_j \Big\|_{L_2(T^l_i)} \\
&\leqslant \sum_{j\in\mathcal{T}^l_i} \Big(\frac{1}{\mu(B^l_j)} \int_{B^l_j} u\,\mathrm{d}\mathbf{x}\Big) \|\varphi^l_j\|_{L_2(T^l_i)} \\
&\leqslant Ch_l \sum_{j\in\mathcal{T}^l_i} \frac{1}{\mu(B^l_j)} (u,1)_{L_2(B^l_j)} \\
&\leqslant Ch_l \sum_{j\in\mathcal{T}^l_i} \frac{1}{\mu(B^l_j)} \|u\|_{L_2(B^l_j)} \|1\|_{L_2(B^l_j)} \\
&\leqslant C \sum_{j\in\mathcal{T}^l_i} \|u\|_{L_2(B^l_i)}.
\end{aligned}$$

Further, by the Cauchy-Schwarz inequality,

$$\sum_{j\in\mathcal{T}_i^l}\|u\|_{L_2(B_j^l)}\leqslant\Big(\sum_{j\in\mathcal{T}_i^l}\|u\|^2_{L_2(B_j^l)}\Big)^{1/2}\Big(\sum_{j\in\mathcal{T}_i^l}1^2\Big)^{1/2}\leqslant 2\|u\|_{L_2(\widetilde{T}_i^l)}.$$

The proof follows from the last two estimates. □

The proof of the following lemma uses the key argument of finite element approximation theory.

Lemma 4.15. *For both the interior and the boundary macroelement T_i^l, the interpolation operator Π_i^l defined in* (4.38) *satisfies the estimate*

$$\|u-\Pi_i^l u\|_{L_2(T_i^l)}\leqslant Ch_l|u|_{H^1(\widetilde{T}_i^l)}\quad\forall\, u\in H^1_{0,\partial\widetilde{T}_i^l\cap\partial\Omega}(\widetilde{T}_i^l)\tag{4.40}$$

with constant $C>0$ independent of h, l, and i. In addition, the interpolation operator

$$\Pi^l\colon\ u\in H^1(\Omega)\mapsto\sum_{i=1}^{n_l}\Big(\frac{1}{\mu(B_1^l)}\int_{B_i^l}u\,\mathrm{d}\mathbf{x}\Big)\varphi_i^l\tag{4.41}$$

satisfies

$$\|u-\Pi^l u\|_{L^2(\Omega)}\leqslant Ch_l|u|_{H^1(\Omega)}\quad\forall\, u\in H^1_0(\Omega)\tag{4.42}$$

with constant $C>0$ independent of h and l.

Proof. By Lemma 4.14 it follows that

$$\|I-\Pi_i^l\|_{L^2(\widetilde{T}_i^l)\to L_2(T_i^l)}\leqslant\|I\|_{L^2(\widetilde{T}_i^l)\to L_2(T_i^l)}+\|\Pi_i^l\|_{L^2(\widetilde{T}_i^l)\to L_2(T_i^l)}\leqslant C.\tag{4.43}$$

Let T_i^l be an interior macroelement and $\mathcal{T}_i^l$ the set of basis functions associated with T_i^l. By Lemma 4.6 we obtain

$$\sum_{j\in\mathcal{T}_i^l}\varphi_i^l=1\quad\text{on } T_i^l.$$

Hence, for any constant function c defined on $\widetilde{T}_i^l$ it holds that

$$\Pi_i^l c=\sum_{j\in\mathcal{T}_i^l}\Big(\frac{1}{\mu(B_i^l)}\int_{B_i^l}c\,\mathrm{d}\mathbf{x}\Big)\varphi_j^l=c\sum_{j\in\mathcal{T}_i^l}\varphi_j^l=c\quad\text{on } T_i^l.\tag{4.44}$$

Let $u \in H^1(\widetilde{T}_i^l)$. By (4.44) and (4.43), for any constant c we get

$$\begin{aligned}
\|u - \Pi_i^l u\|_{L_2(T_i^l)} &= \|u - c - (\Pi_i^l u - c)\|_{L_2(T_i^l)} \\
&= \|u - c - (\Pi_i^l u - \Pi_i^l c)\|_{L_2(T_i^l)} \\
&= \|(I - \Pi_i^l)(u - c)\|_{L_2(T_i^l)} \\
&\leqslant \|I - \Pi_i^l\|_{L^2(\widetilde{T}_i^l) \to L_2(T_i^l)} \|u - c\|_{L_2(\widetilde{T}_i^l)} \\
&\leqslant C\|I - \Pi_i^l\|_{L^2(\widetilde{T}_i^l) \to L_2(T_i^l)} \|u - c\|_{L_2(\widetilde{T}_i^l)}.
\end{aligned}$$

In the above estimate we choose

$$c = \arg\min_{q \in \mathbb{R}} \|u - q\|_{L_2(\widetilde{T}_i^l)}.$$

Hence, by the previous inequality and the scaled Poincaré inequality (4.35), (4.40) follows.

To prove (4.40) for a boundary macroelement is even simpler.

Let $u \in H^1_{0,\partial\Omega \cap \partial\widetilde{T}_i^l}(\Omega)$. We use (4.43) and the scaled Friedrichs' inequality (4.36) to estimate

$$\|u - \Pi_i^l u\|_{L_2(T_i^l)} \leqslant \|I - \Pi_i^l\|_{L^2(\widetilde{T}_i^l) \to L_2(T_i^l)} \|u\|_{L_2(\widetilde{T}_i^l)} \leqslant Ch_l |u|^2_{H^1(\widetilde{T}_i^l)}.$$

This completes the proof of (4.40).

To prove (4.42) we use the obvious identity

$$\|u - \Pi^l u\|_{L_2(T_i^l)} = \|u - \Pi_i^l u\|_{L_2(T_i^l)},$$

the local estimate (4.40), and the fact that every point $\mathbf{x} \in \Omega$ belongs to at most $N < \infty$ extended macroelements $\widetilde{T}_i^l$. Thus,

$$\begin{aligned}
\|u - \Pi^l u\|^2_{L^2(\Omega)} &= \sum_{T_i^l \subset \Omega} \|u - \Pi^l u\|^2_{L_2(T_i^l)} = \sum_{T_i^l \subset \Omega} \|u - \Pi_i^l u\|^2_{L_2(T_i^l)} \\
&\leqslant C \sum_{T_i^l \subset \Omega} h_l^2 |u|^2_{H^1(\widetilde{T}_i^l)} \leqslant Ch_l^2 |u|^2_{H^1(\Omega)}.
\end{aligned}$$

□

Lemma 4.16. *There is a constant $c_\sigma > 0$ independent of h, l and L such that*

$$\overline{\sigma}_l \equiv \frac{c_\sigma}{9^{l-1}} \geqslant \sigma_l \tag{4.45}$$

for all $l = 1, \dots, L$. In addition, there is a constant $C > 0$ independent of h, l, and L such that for every $\mathbf{u} \in U = \mathbb{R}^{n_1}$, the exact orthogonal projections $\mathscr{Q}_l = Q_l \colon U \to U_l$, $U_{L+1} = \emptyset$, $Q_{L+1} = 0$, satisfy

$$\|\mathbf{u} - Q_{l+1}\mathbf{u}\| \leqslant \frac{C}{\sqrt{\overline{\sigma}_l}} \|\mathbf{u}\|_A, \quad l = 1, \dots, L. \tag{4.46}$$

P r o o f. To estimate the spectral bound (4.45) we use the norm equivalence proved in Lemma 4.11. By (2.13),

$$\sigma_l = \sup_{\mathbf{x} \in \mathbb{R}^{n_l} \setminus \{\mathbf{0}\}} \frac{\langle A I_l^1 \mathbf{x}, I_l^1 \mathbf{x} \rangle}{\|I_l^1 \mathbf{x}\|^2} = \sup_{\mathbf{x} \in \mathbb{R}^{n_l} \setminus \{\mathbf{0}\}} \frac{\langle A_l \mathbf{x}, \mathbf{x} \rangle}{\|I_l^1 \mathbf{x}\|^2},$$

where by (4.32) and $\pi_l = \pi_1 I_l^1$,

$$\|I_l^1 \mathbf{x}\|^2 \geqslant c h_1^{-2} \|\pi_1 I_l^1 \mathbf{x}\|^2 = c h_1^{-2} \|\pi_l \mathbf{x}\|^2_{L_2(\Omega)} \geqslant c \Big(\frac{h_l}{h_1}\Big)^2 \|\mathbf{x}\|^2 \quad \forall \mathbf{x} \in \mathbb{R}^{n_l}. \tag{4.47}$$

The previous two inequalities, together with (4.3), give

$$\sigma_l \leqslant C \Big(\frac{h_1}{h_l}\Big)^2 \sup_{\mathbf{x} \in \mathbb{R}^{n_l} \setminus \{\mathbf{0}\}} \frac{\langle A_l \mathbf{x}, \mathbf{x} \rangle}{\|\mathbf{x}\|^2} \leqslant C \Big(\frac{h_1}{h_l}\Big)^2 \lambda_{\max}(A_l) \leqslant C \Big(\frac{h_1}{h_l}\Big)^2,$$

proving (4.45).

We will prove (4.46) for approximate projections $\overline{Q}_l \colon U \to U_l$ defined by $\overline{Q}_l = \pi_1^{-1} \Pi^l \pi_1$, $l = 2, \dots, L$, and $\overline{Q}_{L+1} = 0$. The result for the exact projection then follows by the minimizing property of the orthogonal projection.

Let $l < L$ and $\mathbf{u} \in U$. We estimate using Lemma 4.15, norm equivalence (4.7), and $h_{l+1} = 3h_l$,

$$\begin{aligned}
\|\mathbf{u} - \overline{Q}_{l+1}\mathbf{u}\| &= \|(I - \pi_1^{-1}\Pi^{l+1}\pi_1)\mathbf{u}\| \\
&\leqslant C h_1^{-1} \|\pi_1 (I - \pi_1^{-1}\Pi^{l+1}\pi_1)\mathbf{u}\|_{L_2(\Omega)} \\
&= C h_1^{-1} \|(I - \Pi^{l+1})\pi_1 \mathbf{u}\|_{L_2(\Omega)} \\
&\leqslant C \frac{h_{l+1}}{h_1} |\pi_1 \mathbf{u}|_{H^1(\Omega)} \leqslant C \frac{h_l}{h_1} |\pi_1 \mathbf{u}|_{H^1(\Omega)} \\
&= C \frac{h_l}{h_1} \|\mathbf{u}\|_A.
\end{aligned} \tag{4.48}$$

For $l = L$, we have by $h_L = 1/2$ (since there is only one degree of freedom located in the center of Ω), and by Friedrichs' inequality (4.36) for $T = \Omega$,

$$\|\mathbf{u} - \overline{Q}_{L+1}\mathbf{u}\| = \|\mathbf{u}\| \leqslant Ch_1^{-1}\|\pi_1\mathbf{u}\|_{L_2(\Omega)} \leqslant h_1^{-1}C|\pi_1\mathbf{u}|_{H^1(\Omega)} \leqslant C\frac{h_L}{h_1}\|\mathbf{u}\|_A,$$

proving (4.48) for $l = L$.

Statement (4.46) follows from (4.45) and estimate (4.48). □

Now we are ready to formulate the final convergence theorem.

Theorem 4.17. *For model problem* (3.1) *and smoothed aggregation based coarse-spaces* $U_l = \mathrm{Range}(I_l^1)$ *with prolongators* I_{l+1}^l *as defined in Section* 3, *the BPX preconditioner* $\mathscr{B} = B$ *given by Algorithm* 1 (*and Algorithm* 2) *satisfies the estimate*

$$\frac{c}{L}\|\mathbf{u}\|_A^2 \leqslant \langle BA\mathbf{u}, \mathbf{u}\rangle_A \leqslant CL\|\mathbf{u}\|_A^2 \quad \forall\, \mathbf{u} \in U \tag{4.49}$$

with constants $C \geqslant c > 0$ *independent of* h *and* L.

Proof. The proof consists in the verification of the assumptions of Theorem 2.1. Assumption (2.4) follows from Lemma 4.16. Assumption (2.5) holds by virtue of Corollary 4.12. □

Remark 4.18 (Choice of $\overline{\sigma}_l$). In practice, it is relatively difficult to determine upper bounds $\overline{\sigma}_l \geqslant \sigma_l$ in (2.13) computationally. From Lemma 4.16, we know that there is a constant $c_\sigma > 0$ independent of h, l, and L such that

$$\overline{\sigma}_l \equiv \frac{c_\sigma}{9^{l-1}} \geqslant \sigma_l, \quad l = 1, \dots, L.$$

To get an efficient preconditioner, it is not neccessary to determine the constant c_σ. In (2.12), it is sufficient to use

$$\tilde{\sigma}_l = \frac{1}{9^{l-1}}, \quad l = 1, \dots, L,$$

in the place of $\overline{\sigma}_l = c_\sigma/9^{l-1}$. Obviously, this leads to the scalar multiple $\widetilde{B} = c_\sigma^{-1}B$. This simplification does not alter the convergence estimate since

$$\mathrm{cond}(A, \widetilde{B}) = \mathrm{cond}(A, c_\sigma^{-1}B) = \mathrm{cond}(A, B) \leqslant \frac{C}{c}L^2.$$

Here, C, c are the constants from (4.49).

R e m a r k 4.19. By means very similar to those used in Section 4, for operators $\widetilde{Q}_l$, it is possible to verify assumptions of the abstract convergence result of [1] with uniform constants. Then, for a standard multiplicative multigrid, we get the estimate of the convergence rate in the energy norm in the form $1 - C/L$. Compared to the former result of [7], where the convergence rate deteriorates with the power of 3 of L, this is an improvement.

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