Behrooz Khosravi; Behnam Khosravi; Bahman Khosravi; Zahra Momen
A new characterization for the simple group $\mathrm{PSL}(2, p^2)$ by order and some character degrees

Czechoslovak Mathematical Journal, Vol. 65 (2015), No. 1, 271–280

Persistent URL: `http://dml.cz/dmlcz/144226`

Czechoslovak Mathematical Journal, 65 (140) (2015), 271–280

A NEW CHARACTERIZATION FOR THE SIMPLE GROUP $\mathrm{PSL}(2,p^2)$ BY ORDER AND SOME CHARACTER DEGREES

Behrooz Khosravi, Tehran, Behnam Khosravi, Zanjan,
Bahman Khosravi, Qom, Zahra Momen, Tehran

(Received February 12, 2014)

This paper is dedicated to our parents,
Professor Amir Khosravi and Soraya Khosravi

Abstract. Let G be a finite group and p a prime number. We prove that if G is a finite group of order $|\mathrm{PSL}(2,p^2)|$ such that G has an irreducible character of degree p^2 and we know that G has no irreducible character θ such that $2p \mid \theta(1)$, then G is isomorphic to $\mathrm{PSL}(2,p^2)$.

As a consequence of our result we prove that $\mathrm{PSL}(2,p^2)$ is uniquely determined by the structure of its complex group algebra.

Keywords: character degree; order; projective special linear group

MSC 2010: 20C15, 20D05, 20D60

1. Introduction and preliminary results

Let G be a finite group, $\mathrm{Irr}(G)$ the set of irreducible characters of G, and denote by $\mathrm{cd}(G)$, the set of irreducible character degrees of G. The degree pattern of G, which is denoted by $X_1(G)$, is the set of all irreducible complex character degrees of G counting multiplicities. We note that $X_1(G)$ is the first column of the ordinary character table of G. If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$.

The first author would like to thank the Institute for Research in Fundamental Sciences (IPM) for the financial support. This research was in part supported by a grant from IPM (No. 92050120).

Many authors have been recently concerned with the following question: What can be said about the structure of a finite group G, if some information is known about the arithmetical structure of the degrees of the irreducible characters of G? (See [5], [10].)

We know that there are 2328 groups of order 2^7 and these groups have only 30 different degree patterns, and there are 538 of them with the same character degrees (see [3]).

A finite group G is called a K_3-group if $|G|$ has exactly three distinct prime divisors. It is proved that there exist eight simple K_3-groups. Recently in [18] it is proved that all simple K_3-groups are uniquely determined by their orders and one or both of their largest and second largest irreducible character degrees. In [8] the authors proved that the simple group $\mathrm{PSL}(2,p)$, where p is an odd prime number, is uniquely determined by its order and its largest and second largest irreducible character degrees. Also, in [7] finite groups with the same order and the same largest and second largest irreducible character degrees as $\mathrm{PGL}(2,9)$ are determined.

The goal of this paper is to introduce a new characterization for the simple group $\mathrm{PSL}(2,p^2)$, where p is an odd prime number. In fact we prove that if p is an odd prime number and G is a finite group such that $|G| = |\mathrm{PSL}(2,p^2)|$, $p^2 \in \mathrm{cd}(G)$ and there exists no $\theta \in \mathrm{Irr}(G)$ such that $2p \mid \theta(1)$, then $G \cong \mathrm{PSL}(2,p^2)$.

We note that if $p = 3$, then $A_6 \cong \mathrm{PSL}(2,9)$ and the result follows by [18]. So we consider $p > 3$.

By Molien's theorem, knowing the structure of the complex group algebra is equivalent to knowing the first column of the ordinary character table. In [14] with the only assumption that their complex group algebras are isomorphic, Tong-Viet proved that each classical simple group is uniquely determined by its complex group algebra.

It was shown in [16] that the symmetric groups are uniquely determined by the structure of their complex group algebras. Independently, this result was also proved by Nagl in [11]. It was conjectured that all nonabelian simple groups are uniquely determined by the structure of their complex group algebras. This conjecture was verified in [9], [12], [13], [15] for the alternating groups, the sporadic simple groups, the Tits group and the simple exceptional groups of Lie type. We note that abelian groups are not determined by the structure of their complex group algebras. In fact, the complex group algebras of any two abelian groups of the same orders are isomorphic. There are also examples of nonabelian p-groups with isomorphic complex group algebras, for example the dihedral group of order 8 and the quaternion group of order 8.

As a consequence of our results we give a new proof for the fact that $\mathrm{PSL}(2,p^2)$ is uniquely determined by the structure of its complex group algebra.

If $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$, then the inertia group of θ in G is $I_G(\theta) = \{g \in G;\ \theta^g = \theta\}$. If the character $\chi = \sum\limits_{i=1}^{k} e_i\chi_i$, where for each $1 \leqslant i \leqslant k$, $\chi_i \in \mathrm{Irr}(G)$ and e_i is a natural number, then each χ_i is called an irreducible constituent of χ.

Lemma 1.1 (Gallagher's Theorem [6], Corollary 6.17). *Let $N \trianglelefteq G$ and let $\chi \in \mathrm{Irr}(G)$ be such that $\chi_N = \theta \in \mathrm{Irr}(N)$. Then the characters $\beta\chi$ for $\beta \in \mathrm{Irr}(G/N)$ are irreducible distinct for distinct β and all of the irreducible constituents of θ^G.*

Lemma 1.2 (Ito's Theorem [6], Theorem 6.15). *Let $A \trianglelefteq G$ be abelian. Then $\chi(1)$ divides $|G : A|$ for all $\chi \in \mathrm{Irr}(G)$.*

Lemma 1.3 ([6], Theorems 6.2, 6.8, 11.29). *Let $N \trianglelefteq G$ and let $\chi \in \mathrm{Irr}(G)$. Let θ be an irreducible constituent of χ_N and suppose $\theta_1 = \theta, \ldots, \theta_t$ are the distinct conjugates of θ in G. Then $\chi_N = e\sum\limits_{i=1}^{t}\theta_i$, where $e = [\chi_N, \theta]$ and $t = |G : I_G(\theta)|$. Also $\theta(1) \mid \chi(1)$ and $\chi(1)/\theta(1) \mid |G : N|$.*

Lemma 1.4 ([18]). *Let G be a nonsolvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is the direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.*

Lemma 1.5 ([2]). *With the exceptions of the relations $239^2 - 2(13)^4 = -1$ and $3^5 - 2(11)^2 = 1$ every solution of the equation*

$$p^m - 2q^n = \pm 1; \quad p, q \text{ prime}; \ m, n > 1$$

has exponents $m = n = 2$; i.e., it comes from a unit $p - q \cdot 2^{1/2}$ of the quadratic field $Q(2^{1/2})$ for which the coefficients p and q are primes.

If n is an integer and r is a prime number, then we write $r^\alpha \nmid n$ when $r^\alpha \mid n$ but $r^{\alpha+1} \nmid n$. We denote by n_r the r-part of n, i.e., $n_r = r^\alpha$ so that $r^\alpha \nmid n$. If r is a prime number we denote by $\mathrm{Syl}_r(G)$ the set of Sylow r-subgroups of G and we denote by $n_r(G)$ the number of elements of $\mathrm{Syl}_r(G)$. All groups considered are finite and all characters are complex characters. We write H ch G if H is a characteristic subgroup of G. All the other notation is standard and we refer to [1].

Remark 2.1. We note that if p is an odd prime, then $|\mathrm{PSL}(2,p^2)| = p^2(p^2+1) \times (p^2-1)/2$ and if $p \geqslant 5$, then $\mathrm{cd}(\mathrm{PSL}(2,p^2)) = \{1, p^2-1, p^2, p^2+1, (p^2+1)/2\}$.

Lemma 2.1. *Let $p > 3$ be an odd prime and $m = p^2(p^2+1)(p+1)$. Then if m has a divisor of the form $m(k) = kp+1$, then $m(k) = 1$, $p+1$, p^2+1 or p^3+p^2+p+1. Also, if $h = p^2(p^2+1)(p-1)$ has a divisor $h(k) = 1+kp$, then $h(k) = 1$ or p^2+1, if $p \neq k^2+k+1$.*

Proof. By assumption $m(k) = (1+kp) \mid (p^2+1)(p+1)$ and so $(p^2+1)(p+1) = m(k)t$ for some $t > 0$. Therefore $p^3+p^2+p+1 = kpt+t$, and so $p \mid t-1$. Hence $t = 1$ or $p+1 \leqslant t$. If $t = 1$, then $k = p^2+p+1$, which implies that $m(k) = p^3+p^2+p+1$. Otherwise, $p+1 \leqslant t$ and so $p^3+p^2+p+1 \geqslant (kp+1)(p+1) = kp^2+kp+p+1$, which implies that $p \geqslant k$. If $k = 0$, 1, p, then $m(k) = 1$, $p+1$ and p^2+1, respectively. So let $1 < k < p$. Now we determine the greatest common divisor of $m(k)$ and m. If $a = (kp+1, p+1)$, then $a \mid k-1$. Also if $b = (kp+1, p^2+1)$, then $b \mid p-k$. Therefore $(1+kp, m) \leqslant ab \leqslant (k-1)(p-k) < kp$, which implies that $1+kp \nmid m$.

Similarly to the above we get that there exists $t > 0$ such that $p^3-p^2+p-1 = kpt+t$ and so $p-1 \leqslant t$, which implies that $p \geqslant k$. If $k = 0$, p, then $h(k) = 1$ and p^2+1, respectively. Therefore let $1 \leqslant k < p$. Similarly to the above we get that $kp+1 \mid (k+1)(p-k)$. Since $p \neq k^2+k+1$, we get that $kp+1 \neq (k+1)(p-k)$. Therefore $kp+1 \leqslant (k+1)(p-k)/2$, which is impossible. $\square$

Using Ito's theorem we can easily get the following result:

Lemma 2.2. *Let M be a finite group such that $p^j \nmid |M|$, where $1 \leqslant j \leqslant 2$. If M has an irreducible character of degree p^j, then $O_p(M) = 1$.*

Theorem 2.1. *Let G be a finite group such that $|G| = |\mathrm{PSL}(2,25)|$ and G has an irreducible character of degree 25. Then $G \cong \mathrm{PSL}(2,25)$.*

Proof. We know that $|G| = 5^2 2^3 \times 13 \times 3$. If G is a solvable group, let H be a Hall subgroup of G such that $|G:H| = 13$. Then $G/H_G \hookrightarrow S_{13}$ and since the order of each solvable subgroup of S_{13} whose order is divisible by 13 divides 13×12, we conclude that $|H_G| = 5^2 2^i 3^j$, where $1 \leqslant i \leqslant 3$ and $0 \leqslant j \leqslant 1$. On the other hand, H_G has an irreducible character of degree 25 by Lemma 1.3, which implies that $|H_G| \geqslant 1+5^4$ and this is a contradiction. Therefore G is not a solvable group and by Lemma 1.4 we get that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is the direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. Now using the classification of finite simple groups and [1] we get that K/H is isomorphic to A_5 or $\mathrm{PSL}(2,25)$. If $K/H \cong A_5$, then $|H| = 65$ or 130. If $|H| = 65$, then H has an irreducible character of degree 5 by Lemma 1.3, which is a contradiction since $O_5(H) \neq 1$. If $|H| = 130$, then H has a normal subgroup of order 65 and we get a contradiction similarly. Therefore $K/H \cong \mathrm{PSL}(2,25)$ and so $G/K = 1$ and $H = 1$. Hence $G \cong \mathrm{PSL}(2,25)$ and the result follows. $\square$

Theorem 2.2. *Let $p > 5$ be an odd prime number. If G is a finite group such that*

(i) $|G| = |\mathrm{PSL}(2,p^2)|$,

(ii) $p^2 \in \mathrm{cd}(G)$,

(iii) *there exists no $\theta \in \mathrm{Irr}(G)$ such that $2p \mid \theta(1)$,*

then $G \cong \mathrm{PSL}(2,p^2)$.

Proof. Let $\chi \in \mathrm{Irr}(G)$ be such that $\chi(1) = p^2$. By Lemma 2.2, we know that $O_p(G) = 1$. Also it follows that if $N \lhd G$, then $O_p(N) = 1$. In particular, we get that if $p^2 \mid |N|$ and $N \lhd G$, then N has an irreducible character of degree p^2. Now we prove the main result in several steps:

Step 1. No finite group G satisfying (i)–(iii) is a solvable group.

On the contrary let G be a solvable group. If $4 \mid p-1$, then let H be a Hall subgroup of G such that $|G:H| = (p+1)/2$. Then $G/H_G \hookrightarrow S_{(p+1)/2}$ and since $(p+1)/2 < p$, it follows that $p^2 \mid |H_G|$. Let $N = H_G$ and note that $|N| \mid p^2(p^2+1)(p-1)$. Consider a Hall subgroup of N such that $|N:L| = (|N|, 2(p-1))$. Then $N/L_N \hookrightarrow S_{2p-2}$ and $2(p-1) < 2p$ implies that $p \mid |L_N|$, since $p \nmid |S_{2p-2}|$. Using Lemma 1.3, we get that $|L_N|_p \in \mathrm{cd}(L_N)$. On the other hand, $|L_N| \mid p^2(p^2+1)/2$ implies that $n_p(L_N) = 1$ by Lemma 2.1 and so $O_p(L_N) \neq 1$, which is a contradiction by Lemma 2.2.

So in the sequel let $4 \mid p+1$.

Let H be a Hall subgroup of G such that $|G:H| = (p-1)/2$. Then $G/H_G \hookrightarrow S_{(p-1)/2}$. Let $N = H_G$ and note that $|N| \mid p^2(p^2+1)(p+1)$ and $p^2 \mid |N|$. Consider $\eta \in \mathrm{Irr}(N)$ such that $[\chi_N, \eta] \neq 0$. Then $\chi(1)/\eta(1) \mid |G:N|$, by Lemma 1.3. Therefore N has an irreducible character of degree p^2.

Case I. Let $p+1 \neq 2^\beta$ for each $\beta > 0$.

Let T be a Hall subgroup of N such that $|N:T| = |N|_2 \leqslant 2(p+1)_2 \leqslant 2(p+1)/3 \leqslant p-1$. Then $N/T_N \hookrightarrow S_{p-1}$ and so $p^2 \mid |T_N|$. Now since $|T|$ is odd we get that $n_p(T_N) = 1$, by Lemma 2.1. This implies that $O_p(T_N) \neq 1$, which is a contradiction.

Case II. Let $p+1 = 2^\alpha$ for some $\alpha > 0$.

As mentioned above N is a normal subgroup of G such that $|N| \mid p^2(p^2+1)(p+1)$ and we know that $O_p(N) = 1$. If $O_2(N) \neq 1$, then $|N/O_2(N)| \mid p^2(p+1) \times (p^2+1)/2$. Let $L/O_2(N)$ be a Hall subgroup of $N/O_2(N)$ such that $|N/O_2(N) : L/O_2(N)| = |N/O_2(N)|_2 \leqslant p+1$. Then $N/R \hookrightarrow S_{p+1}$, where $R/O_2(N) =$

$\mathrm{Core}_{N/O_2(N)}(L/O_2(N))$. Therefore $p \mid |R|$. Also $|R/O_2(N)| \mid p^2(p^2+1)/2$ and by Lemma 2.1 we get that $n_p(R/O_2(N)) = 1$ and so if $Q/O_2(N) \in \mathrm{Syl}_p(R/O_2(N))$, then $Q \lhd G$ and $|Q| = |O_2(N)|p^j$, where $1 \leqslant j \leqslant 2$. Now if $|O_2(N)| < p+1$, then $O_p(Q) \neq 1$, which is a contradiction. Otherwise if $p+1 \leqslant |O_2(N)| \leqslant 2(p+1)$, then $p^2 \mid |R|$ and so $p^2 \mid |Q|$, which implies that Q has an irreducible character of degree p^2. Hence $1 + p^4 \leqslant |Q| \leqslant 2p^2(p+1)$, which is a contradiction. Therefore $O_2(N) = 1$.

If $(p^2+1)/2 = p_0^{\beta}$ for some $\beta > 0$, then by Lemma 1.5 we get that $\beta = 1$ or $\beta = 2$. If $\beta = 1$, then $p_0 = (p^2+1)/2$ and so $p \nmid p_0(p_0-1)$. Let H be a Hall subgroup of N such that $|N : H| = p_0$. Then $N/H_N \hookrightarrow S_{p_0}$ and since the order of each solvable subgroup of S_{p_0} which is divisible by p_0 divides $p_0(p_0-1)$, we get that $p^2 \mid |H_N|$. Also $|H_N| \mid 2p^2(p+1)$. Now let $\theta \in \mathrm{Irr}(H_N)$ be such that $[\chi_{H_N}, \theta] \neq 0$. Then $\theta(1) = p^2$. Therefore $2p^2(p+1) \geqslant |H_N| \geqslant 1 + \theta(1)^2 = 1 + p^4$, which is a contradiction.

If $\beta = 2$, then $p_0^2 = (p^2+1)/2$ and $p = 2^{\alpha} - 1$. Therefore $p_0^2 = 2^{2\alpha-1} - 2^{\alpha} + 1$ and so $2^{\alpha}(2^{\alpha-1} - 1) = (p_0-1)(p_0+1)$. Hence one of the following occurs: $2^{\alpha-1} \mid p_0 - 1$ and $p_0 + 1 \mid 2(2^{\alpha-1}-1)$ or $2^{\alpha-1} \mid p_0 + 1$ and $p_0 - 1 \mid 2(2^{\alpha-1}-1)$.

In the former case $p_0 - 1 = 2^{\alpha-1}r$ and $p_0 + 1 = 2(2^{\alpha-1}-1)/r$ for some $r > 0$. If $r \geqslant 3$, then we easily get a contradiction and if $r = 1$, then $p = 7$. Similarly in the latter case for $p > 7$ we get a contradiction. If $p = 7$, then $|N| \mid 5^2 7^2 2^4$. As mentioned above $O_7(N) = 1$ and $O_2(N) = 1$. Therefore $O_5(N) \neq 1$. If $|O_5(N)| = 25$, then consider $\eta \in \mathrm{Irr}(O_5(N))$ such that $[\chi_{O_5(N)}, \eta] \neq 0$. Then $\chi(1) = et\eta(1)$ by Lemma 1.3, which implies that $et = 7^2$, where $t = |N : I_N(\eta)|$. Since $t \mid |\mathrm{Aut}(O_5(N))|$, we get that $e = 49$ and $t = 1$, which is a contradiction since $7^4 = e^2 t \leqslant |N : O_5(N)| \leqslant 49 \times 16$. Therefore $|O_5(N)| = 5$. Let $M/O_5(N)$ be a Hall subgroup of $N/O_5(N)$ such that $|N/O_5(N) : M/O_5(N)| = |N/O_5(N)|_5$. Then there exists a normal subgroup R of N such that $|N : R|$ is a divisor of 20. Therefore $|R| \mid 7^2 2^4 5$ and $7^2 5 \mid |R|$. Finally, by considering a Hall subgroup of R of index 5, we get a contradiction similarly to the above discussion.

If $(p^2+1)/2 = ab$, where $1 < a < b$ are natural numbers and $(a,b) = 1$, then $a < p$. Now similarly to the previous cases we can consider a Hall subgroup of N such that $|N : M| = (a, |N|)$ and get a normal subgroup R of N such that $p^2 \mid |R|$ and $|R| \mid 2p^2(p+1)(p^2+1)/2a$. So without loss of generality, we assume that $|N| \mid 2(p+1)p^2 p_0^{\beta} = 2^{\alpha+1} p^2 p_0^{\beta}$ and $p^2 \mid |N|$, where $p_0^{\beta} \nmid (p^2+1)/2a$ for a prime number p_0. Since N is a solvable group, by the above discussion it follows that $O_{p_0}(N) \neq 1$. Now $O_{p_0}(N/O_{p_0}(N)) = 1$, and so $O_2(N/O_{p_0}(N)) \neq 1$ or $O_p(N/O_{p_0}(N)) \neq 1$. If $Q/O_{p_0}(N) = O_p(N/O_{p_0}(N)) \neq 1$, then $|Q| = |O_{p_0}(N)|p^{\gamma}$ for some $\gamma > 0$ and $|Q| \mid p^2(p^2+1)/2$, which implies that $n_p(Q) = 1$ and so $O_p(Q) \neq 1$, which is a contradiction. Therefore $R/O_{p_0}(N) = O_2(N/O_{p_0}(N)) \neq 1$. Let $|R| = 2^i |O_{p_0}(N)|$ for some $i > 0$ and consider the finite group N/R. Let L/R be a Hall subgroup of N/R such that $|N/R : L/R| = |N/R|_2 \mid 2(p+1)/2^i \leqslant p+1$. Then if $T/R = \mathrm{Core}_{N/R}(L/R)$, then $p \mid |T/R|$ and by Lemma 2.1 we get that Q/R, the Sylow p-subgroup of T/R, is a normal subgroup of T/R. Therefore $|Q| = p^j 2^i |O_{p_0}(N)|$, where $1 \leqslant j \leqslant 2$. If $2^i < p+1$, then $O_p(N) \neq 1$, which is a contradiction. Therefore $2^i \geqslant p+1$, and $|N/R : L/R| \leqslant 2$ shows that $p^2 \mid |T/R|$. Now we consider the following two cases.

(i) Let $2^i = p+1$.

Let M be a Hall subgroup of Q such that $|Q : M| = p+1$. As mentioned above $p \mid |M_Q|$ and $O_p(Q) \neq 1$, which is a contradiction.

(ii) Let $2^i = 2(p+1)$.

In this case $n_p(Q) = 1$ or $p+1$ by Lemma 2.1. Since $O_p(Q) = 1$, we get that $n_p(Q) = p+1$. Therefore $|Q : N_Q(P)| = p+1$, where $P \in \mathrm{Syl}_p(Q)$. Let $K = N_Q(P)$. Then $Q/K_Q \hookrightarrow S_{p+1}$ and similarly to the previous cases we get that $O_p(K_Q) \neq 1$, which is a contradiction.

Therefore G is not a solvable group.

Step 2. Now we prove that G is isomorphic to $\mathrm{PSL}(2, p^2)$.

By the above discussion and using Lemma 1.4 we get that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is the direct product of m copies of a nonabelian simple group S and $|G/K| \mid |\mathrm{Out}(K/H)|$.

First we claim that $p \nmid |G/K|$. Otherwise $p \mid |G/K|$ and since $\mathrm{Out}(K/H) \cong \mathrm{Out}(S) \wr S_m$, it follows that $p \mid |S_m|$ or $p \mid |\mathrm{Out}(S)|$. If $p \mid |S_m|$, then $m \geqslant p$. Now since the smallest order of a nonabelian simple group is 60, it follows that $p(p^4-1) \geqslant |K/H| \geqslant 60^p$, which is impossible. Hence $p \mid |\mathrm{Out}(S)|$, where $p \geqslant 7$. Then by [1] we get that S is not isomorphic to a sporadic simple group or an alternating group. Therefore S is a simple group of Lie type over $\mathrm{GF}(q)$, where $q = p_0^f$. By assumption, $p \mid |\mathrm{Out}(S)| = dfg$, where d, f and g are the orders of diagonal, field and graph automorphisms of S (see [1]). If $p \mid f$, then $2^p(2^{2p}-1) \leqslant q(q^2-1) \leqslant |S| \leqslant p(p^4-1)$ for each nonabelian simple group S, which is a contradiction. Also $g \leqslant 3$, and so $p \mid d$, where $S = A_n(q)$ and $d = (n+1, q-1)$ or $S = {}^2A_n(q)$ and $d = (n+1, q+1)$. In each case we get that $p \mid q+\varepsilon$, where $\varepsilon = \pm 1$ and $n \geqslant 6$. Then $p^3 \mid |S|$, which is a contradiction. Therefore $p \nmid |G/K|$.

Now we claim that $p^2 \nmid |H|$. If $p^2 \mid |H|$, choose $\eta \in \mathrm{Irr}(H)$ such that $[\chi_H, \eta] \neq 0$. Then $\chi(1)/\eta(1) \mid |G : H|$, which implies that $\eta(1) = p^2$. Therefore $\chi_H = \eta \in \mathrm{Irr}(H)$. Since K/H is the direct product of m copies of a nonabelian simple group S and by Ito-Michler theorem (see Theorem 19.10 and Remark 19.11 of [4]) S has an irreducible character of even degree we get a contradiction by Gallagher's theorem. Therefore $p \mid |K/H|$. As mentioned above K/H is a direct product of m copies of a nonabelian simple group S. Since $2p$ does not divide the degree of any irreducible character of G, we get that $m = 1$ and so K/H is a nonabelian simple group. Now we know finite simple groups whose degree graphs are not complete (see [17]). By considering them we have:

$$K/H \in \{J_1, M_{11}, M_{23}, A_8, {}^2B_2(q), \mathrm{PSL}(3,q), \mathrm{PSU}(3,q), \mathrm{PSL}(2,q)\}.$$

Since we do not have any character whose degree is divisible by $2p$ and also we know that $|K/H| \mid p^2(p^4-1)/2$, we can easily get that $K/H \notin \{J_1, M_{11}, M_{23}, A_8\}$.

Case 1. Let $K/H \cong {}^2B_2(q)$, where $q = 2^{2m+1}$. By the character degrees of ${}^2B_2(q)$, we have that $p \mid q^2+1$, since $2p$ does not divide the degree of any irreducible character of G. Also we have that $q^2 \mid (p^4-1)/2$, which implies that $q^2 \mid 2(p \pm 1)$. So $p-1 \leqslant q^2 \leqslant 2(p+1)$. If $q^2 < 2(p \pm 1)$, then $q^2 \leqslant p \pm 1$ and so $p-1 \leqslant q^2 \leqslant p+1$, which implies that $q^2 = p-1$. But this is a contradiction since $5 \mid q^2+1$ and also $5 < p$. Hence $q^2 = 2(p \pm 1)$. Since $p \mid q^2+1$, we get a contradiction easily.

Case 2. Let $K/H \cong \mathrm{PSL}(3,q)$. Then if $q = 4$, by the fact that we do not have any character whose degree is divisible by $2p$, we have $p \in \{3,7\}$, which leads to a contradiction, by the fact that $|K/H| \mid p^2(p^4-1)/2$. So $q \neq 4$. In this case, by considering the character degrees of K/H, we can see that q should be even and $p \mid q-1$. So $q^3 \mid 2(p \pm 1)$, and thus $p^3 \leqslant (q-1)^3 < q^3 \leqslant 2(p \pm 1)$, a contradiction. By the same discussion we can see that $K/H \ncong \mathrm{PSU}(3,q)$.

Case 3. Let $K/H \cong \mathrm{PSL}(2,q)$, where q is even. Then $p \mid q \pm 1$ and $q \mid 2(p \pm 1)$. If $q < 2(p \pm 1)$, then we have $p-1 \leqslant q \leqslant p+1$, which implies that $q = p \pm 1$. Let $q = p-1$, so $p = 2^m+1$ is a Fermat prime. Then $q-1 = p-2 \mid (p^4-1)/2$. So $\pi(p-2) \subseteq \{3,5\}$, and thus $m = 4$ and $p = 17$. So $|K/H| = 2^4 \cdot 17 \cdot 3 \cdot 5$ and it follows that either $|H| = 6 \cdot 29 \cdot 17$ or $|H| = 3 \cdot 17 \cdot 29$, which means that H has the normal Sylow 17-subgroup, a contradiction. Similarly if $q = p+1$, we get a contradiction. Therefore $q = 2(p \pm 1)$ and by the fact that $p \mid q \pm 1$ we get that either $p = 3$ and $q = 4$, or $p = 3$ and $q = 8$. The latter case is not possible, since $7 \nmid 3^4 - 1$. So $K/H \cong \mathrm{PSL}(2,4)$ and so either $|H| = 3$ or $|H| = 6$, and in both cases we get a contradiction easily.

Case 4. Let $K/H \cong \mathrm{PSL}(2,q)$, where q is odd. By the fact that $2p$ does not divide the degree of any irreducible character, we conclude that $q = p^b$. Since $|K/H|_p \mid p^2$, we have $b \in \{1,2\}$. If $b = 1$, then either $|H| = p(p^2+1)/2$ or $|H| = p(p^2+1)$, which leads to a contradiction, as we discussed it several times in the paper.

Hence $K/H \cong \mathrm{PSL}(2,p^2)$ and so $|H| = 1$ and $|G/K| = 1$. Therefore $G \cong \mathrm{PSL}(2,p^2)$ and the main theorem is proved. $\square$

As a consequence we get the following result:

Corollary 2.1. *Let p be a prime number and G a finite group such that $X_1(G) = X_1(\mathrm{PSL}(2,p^2))$. Then $G \cong \mathrm{PSL}(2,p^2)$.*

The following result is a new proof for the answer to the question proposed in [14].

Corollary 2.2. *Let p be a prime number and $H = \mathrm{PSL}(2,p^2)$. If G is a group such that $\mathbb{C}G \cong \mathbb{C}H$, then $G \cong \mathrm{PSL}(2,p^2)$. Thus $\mathrm{PSL}(2,p^2)$ is uniquely determined by the structure of its complex group algebra.*

Acknowledgement. The authors would like to express their sincere appreciation to the referee for very useful suggestions which improved the results seriously.

References

[1] *J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson*: Atlas of Finite Groups. Maximal Subgroups and Ordinary Characters for Simple Groups. Clarendon Press, Oxford, 1985. zbl MR
[2] *P. Crescenzo*: A diophantine equation which arises in the theory of finite groups. Adv. Math. *17* (1975), 25–29. zbl MR
[3] *B. Huppert*: Some simple groups which are determined by the set of their character degrees. I. Ill. J. Math. *44* (2000), 828–842. zbl MR
[4] *B. Huppert*: Character Theory of Finite Groups. De Gruyter Expositions in Mathematics 25, Walter de Gruyter, Berlin, 1998. zbl MR
[5] *I. M. Isaacs*: Character degree graphs and normal subgroups. Trans. Am. Math. Soc. *356* (2004), 1155–1183. zbl MR
[6] *I. M. Isaacs*: Character Theory of Finite Groups. Pure and Applied Mathematics 69, Academic Press, New York, 1976. zbl MR
[7] *B. Khosravi*: Groups with the same orders and large character degrees as $\mathrm{PGL}(2,9)$. Quasigroups Relat. Syst. *21* (2013), 239–243. zbl MR
[8] *B. Khosravi, B. Khosravi, B. Khosravi*: Recognition of $\mathrm{PSL}(2,p)$ by order and some information on its character degrees where p is a prime. Monatsh. Math. *175* (2014), 277–282. zbl MR
[9] *W. Kimmerle*: Group rings of finite simple groups. Resen. Inst. Mat. Estat. Univ. São Paulo *5* (2002), 261–278. zbl MR
[10] *M. L. Lewis, D. L. White*: Nonsolvable groups with no prime dividing three character degrees. J. Algebra *336* (2011), 158–183. zbl MR
[11] *M. Nagl*: Charakterisierung der symmetrischen Gruppen durch ihre komplexe Gruppenalgebra. Stuttgarter Mathematische Berichte, http://www.mathematik.uni-stuttgart.de/preprints/downloads/2011/2011-007.pdf (2011). (In German.)
[12] *M. Nagl*: Über das Isomorphieproblem von Gruppenalgebren endlicher einfacher Gruppen. Diplomarbeit, Universität Stuttgart, 2008. (In German.)
[13] *H. P. Tong-Viet*: Alternating and sporadic simple groups are determined by their character degrees. Algebr. Represent. Theory *15* (2012), 379–389. zbl MR
[14] *H. P. Tong-Viet*: Simple classical groups of Lie type are determined by their character degrees. J. Algebra *357* (2012), 61–68. zbl MR
[15] *H. P. Tong-Viet*: Simple exceptional groups of Lie type are determined by their character degrees. Monatsh. Math. *166* (2012), 559–577. zbl MR

[16] *H. P. Tong-Viet*: Symmetric groups are determined by their character degrees. J. Algebra *334* (2011), 275–284. zbl MR
[17] *D. L. White*: Degree graphs of simple groups. Rocky Mt. J. Math. *39* (2009), 1713–1739. zbl MR
[18] *H. Xu, G. Chen, Y. Yan*: A new characterization of simple K_3-groups by their orders and large degrees of their irreducible characters. Commun. Algebra *42* (2014), 5374–5380. zbl MR

Authors' addresses: Behrooz Khosravi, School of Mathematics, Institute for Research in Fundamental sciences (IPM), P. O. Box 19395–5746, Tehran, Iran, and Dept. of Pure Math., Faculty of Math. and Computer Sci., Amirkabir University of Technology (Tehran Polytechnic), 424, Hafez Ave., Tehran 15914, Iran, e-mail: `khosravibbb@yahoo.com`; Behnam Khosravi, Department of Mathematics, Institute for Advanced Studies in Basic Sciences, Zanjan 45137–66731, Iran, e-mail: `behnam_kho@yahoo.com`; Bahman Khosravi, Department of Basic Sciences, Qom University of Technology, Qom, Iran, e-mail: `bahman_kho@yahoo.com`; Zahra Momen, Dept. of Pure Math., Faculty of Math. and Computer Sci., Amirkabir University of Technology (Tehran Polytechnic), 424, Hafez Ave., Tehran 15914, Iran, e-mail: `zahramomen@yahoo.com`.