Czechoslovak Mathematical Journal

Eunmi Pak; Juan de Dios Pérez; Carlos J. G. Machado; Changhwa Woo

Hopf hypersurfaces in complex two-plane Grassmannians with generalized Tanaka-Webster parallel normal Jacobi operator

Czechoslovak Mathematical Journal, Vol. 65 (2015), No. 1, 207–218

Persistent URL: http://dml.cz/dmlcz/144222

Czechoslovak Mathematical Journal, 65 (140) (2015), 207–218

HOPF HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WITH GENERALIZED TANAKA-WEBSTER PARALLEL NORMAL JACOBI OPERATOR

Eunmi Pak, Daegu, Juan de Dios Pérez,
Carlos J. G. Machado, Granada, Changhwa Woo, Daegu

(Received December 24, 2013)

Abstract. We study the classifying problem of immersed submanifolds in Hermitian symmetric spaces. Typically in this paper, we deal with real hypersurfaces in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ which has a remarkable geometric structure as a Hermitian symmetric space of rank 2. In relation to the generalized Tanaka-Webster connection, we consider a new concept of the parallel normal Jacobi operator for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ and prove non-existence of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with generalized Tanaka-Webster parallel normal Jacobi operator.

Keywords: real hypersurface; complex two-plane Grassmannian; Hopf hypersurface; generalized Tanaka-Webster connection; normal Jacobi operator; generalized Tanaka-Webster parallel normal Jacobi operator

MSC 2010: 53C40, 53C15

Introduction

In complex projective spaces or in quaternionic projective spaces, many differential geometers studied real hypersurfaces with parallel curvature tensor ([7]). From a new perspective, it is investigated to classify real hypersurfaces in complex two-plane Grassmannians with parallel normal Jacobi operator, that is, $\nabla \overline{R}_N = 0$ ([8], [10] and [6]).

This work was supported by grant Proj. No. NRF-2011-220-C00002 from National Research Foundation of Korea. The first author was supported by grant Proj. No. NRF-2012-R1A2A2A-01043023, the second by MCT-FEDER Grant MTM 2010-18099 and the fourth by Proj. No. NRF-2013-H1A8A1004325.

As a prevailing notion, in a Riemannian manifold $(\overline{M}, \overline{g})$, a vector field X along a geodesic γ of $\overline{M}$ is called a *Jacobi field* if it satisfies the second order Jacobi equation

$$\overline{\nabla}^2_{\dot{\gamma}} X + \overline{R}(X, \dot{\gamma})\dot{\gamma} = 0,$$

where $\dot{\gamma}$ is the vector tangent to γ. For any tangent vector field X at $x \in \overline{M}$, the Jacobi operator $\overline{R}_X$ is defined by

$$(\overline{R}_X Y)(x) = (\overline{R}(Y, X)X)(x),$$

for any vector field $Y \in T_x\overline{M}$.

On the other hand, let us put a unit normal vector field N to a hypersurface M into the curvature tensor $\overline{R}$ of the ambient space $\overline{M}$. In [8], for any tangent vector field X on M, the *normal Jacobi operator* $\overline{R}_N$ is defined by

$$\overline{R}_N(X) = \overline{R}(X, N)N.$$

The ambient space, a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ consists of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space is the unique compact irreducible Riemannian manifold equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J. Then, naturally, we could consider two geometric conditions for hypersurfaces M in $G_2(\mathbb{C}^{m+2})$: that both the one-dimensional distribution $[\xi] = \mathrm{Span}\{\xi\}$ and the three-dimensional distribution $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator A of M ([3]), where the *Reeb vector field* ξ is defined by $\xi = -JN$, N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$ and the *almost contact 3-structure* vector fields ξ_ν are defined by $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$, where $\{J_1, J_2, J_3\}$ denotes a local basis of $\mathfrak{J}$. The distribution $\mathfrak{D}$ denotes the orthogonal complement of $\mathfrak{D}^\perp$ in T_xM, $x \in M$ which becomes the maximal quaternionic subbundle of T_xM, $x \in M$. If X is a tangent vector on M, we may put

$$JX = \varphi X + \eta(X)N, \quad J_\nu X = \varphi_\nu X + \eta_\nu(X)N$$

where φX (resp. $\varphi_\nu X$) is the tangential part of JX (resp. $J_\nu X$) and $\eta(X) = g(X, \xi)$ (resp. $\eta_\nu(X) = g(X, \xi_\nu)$) is the coefficient of normal part of JX (resp. $J_\nu X$). In this case, we call φ the structure tensor field of M.

By using the result in Alekseevskij [1], Berndt and Suh [3] proved the following:

Theorem A. *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

By using the normal Jacobi operator, Jeong, Kim and Suh considered the notion of *parallel normal Jacobi operator*, that is, $\nabla_X \overline{R}_N = 0$ along any vector field X on M in $G_2(\mathbb{C}^{m+2})$. Then they gave a non-existence theorem as follows [8]:

Theorem B. *There exist no Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m\geqslant 3$, with parallel normal Jacobi operator.*

Recall that the Reeb vector field ξ is said to be *Hopf* if it is invariant under the shape operator A. The one dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. By the formulas in [8], Section 3, it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf.

Moreover, Jeong and Suh considered the general notion of the *$\mathfrak{F}$-parallel normal Jacobi operator* defined in such a way that $\nabla_{\mathfrak{F}} \overline{R}_N = 0$, $\mathfrak{F} = [\xi] \cup \mathfrak{D}^\perp$, which is weaker than the notion of the parallel normal Jacobi operator mentioned above. They gave a non-existence theorem as follows [10]:

Theorem C. *There exist no connected Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m\geqslant 3$, with $\mathfrak{F}$-parallel normal Jacobi operator, $\mathfrak{F} = [\xi] \cup \mathfrak{D}^\perp$.*

Related to the Levi-Civita connection ∇, the generalized Tanaka-Webster connection (from now on, GTW connection) for contact metric manifolds was introduced by Tanno ([13]) as a generalization of the connection defined by Tanaka in [12] and, independently, by Webster in [14]. The Tanaka-Webster connection is defined as a canonical affine connection on a non-degenerate, pseudo-Hermitian CR-manifold. A real hypersurface M in a Kähler manifold has an (integrable) CR-structure associated with the almost contact structure (φ, ξ, η, g) induced on M by the Kähler structure, but, in general, this CR-structure is not guaranteed to be pseudo-Hermitian. Cho defined GTW connection for a real hypersurface of a Kähler manifold (see [4], [5]) by

$$\widehat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\varphi AX, Y)\xi - \eta(Y)\varphi AX - k\eta(X)\varphi Y,$$

with a constant $k \in \mathbb{R} \setminus \{0\}$ (see [5], [9]).

Using this GTW connection $\widehat{\nabla}^{(k)}$, we consider the new notion of *generalized Tanaka-Webster parallel normal Jacobi operator* (in short, GTW parallel normal Jacobi operator), that is, $\widehat{\nabla}^{(k)}_X \overline{R}_N = 0$ for any vector field $X \in T_xM$. In Section 1 we will prove the following Main Theorem.

Main Theorem. *There exist no Hopf hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$, with GTW parallel normal Jacobi operator.*

In Section 2 we define a new notion called the *GTW Reeb-parallel* defined by $(\widehat{\nabla}^{(k)}_\xi \overline{R}_N)Y = 0$ for any tangent vector field Y on M. It is weaker than the GTW parallel normal Jacobi operator. As an interesting result, for $\xi \in \mathfrak{D}^\perp$, any Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ admits a natural GTW Reeb-parallel normal Jacobi operator.

In this paper, we refer to [1], [2], [3], [8], and [11] for Riemannian geometric structures of $G_2(\mathbb{C}^{m+2})$ and its geometric quantities.

1. Proof of Main Theorem

Let us denote by $\overline{R}(X,Y)Z$ the curvature tensor in $G_2(\mathbb{C}^{m+2})$. Then the normal Jacobi operator $\overline{R}_N$ of M in $G_2(\mathbb{C}^{m+2})$ can be defined by $\overline{R}_N X = \overline{R}(X,N)N$ for any vector field $X \in T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, where the distribution $\mathfrak{D}$ denotes the orthogonal complement of $\mathfrak{D}^\perp$ in T_xM, $x \in M$ (see [8]).

In [8] and [10], the derivative of the normal Jacobi operator is written as

$$
\begin{aligned}
(\nabla_X \overline{R}_N)Y &= 3g(\varphi AX, Y)\xi + 3\eta(Y)\varphi AX \\
&\quad + 3\sum_{\nu=1}^{3}\{g(\varphi_\nu AX, Y)\xi_\nu + \eta_\nu(Y)\varphi_\nu AX\} \\
&\quad - \sum_{\nu=1}^{3}[2\eta_\nu(\varphi AX)(\varphi_\nu\varphi Y - \eta(Y)\xi_\nu) - g(\varphi_\nu AX, \varphi Y)\varphi_\nu\xi \\
&\quad - \eta(Y)\eta_\nu(AX)\varphi_\nu\xi - \eta_\nu(\varphi Y)(\varphi_\nu\varphi AX - g(AX,\xi)\xi_\nu)]
\end{aligned}
\tag{1.1}
$$

for any tangent vector fields X and Y on M.

In [5], the author defined the GTW connection $\widehat{\nabla}^{(k)}$ for M as follows:

$$
\widehat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\varphi AX, Y)\xi - \eta(Y)\varphi AX - k\eta(X)\varphi Y \tag{1.2}
$$

for a non-zero real number k. By using (1.2), we have

$$
\begin{aligned}
(\widehat{\nabla}^{(k)}_X \overline{R}_N)Y &= \widehat{\nabla}^{(k)}_X(\overline{R}_N Y) - \overline{R}_N(\widehat{\nabla}^{(k)}_X Y) \\
&= \nabla_X(\overline{R}_N Y) + g(\varphi AX, \overline{R}_N Y)\xi - \eta(\overline{R}_N Y)\varphi AX - k\eta(X)\varphi\overline{R}_N Y \\
&\quad - \overline{R}_N(\nabla_X Y + g(\varphi AX, Y)\xi - \eta(Y)\varphi AX - k\eta(X)\varphi Y).
\end{aligned}
$$

From this, together with the fact that M is Hopf, we obtain

$$
\begin{aligned}
(1.3)\qquad (\widehat{\nabla}^{(k)}_X \overline{R}_N)Y = \sum_{\nu=1}^{3}\{&3g(\varphi_\nu AX, Y)\xi_\nu + 3\eta_\nu(Y)\varphi_\nu AX \\
&- 2\eta_\nu(\varphi AX)\varphi_\nu\varphi Y + 5\eta_\nu(\varphi AX)\eta(Y)\xi_\nu \\
&+ g(\varphi_\nu AX, \varphi Y)\varphi_\nu\xi + \eta_\nu(\varphi Y)\varphi_\nu\varphi AX \\
&- \alpha\eta(X)\eta_\nu(\varphi Y)\xi_\nu + 3\eta_\nu(\varphi AX)\eta_\nu(Y)\xi \\
&- \eta_\nu(\xi)g(\varphi AX, \varphi_\nu\varphi Y)\xi + \eta_\nu(\xi)\eta_\nu(\varphi AX)\eta(Y)\xi \\
&- \alpha\eta_\nu(\xi)\eta(X)\eta_\nu(\varphi Y)\xi + \eta_\nu(AX)\eta_\nu(\varphi Y)\xi \\
&- 4\eta_\nu(\xi)\eta_\nu(Y)\varphi AX - 4k\eta(X)\eta_\nu(Y)\varphi_\nu\xi \\
&+ k\eta_\nu(\xi)\eta(X)\varphi\varphi_\nu\varphi Y - k\eta_\nu(\xi)\eta(X)\eta(Y)\varphi_\nu\xi \\
&- k\eta_\nu(\xi)\eta(X)\eta_\nu(\varphi Y)\xi + 4k\eta(X)\eta_\nu(\varphi Y)\xi_\nu \\
&- 4\eta_\nu(\xi)g(\varphi AX, Y)\xi_\nu + \eta_\nu(\xi)\eta(Y)\varphi_\nu AX \\
&+ k\eta_\nu(\xi)\eta(X)\varphi_\nu Y\}
\end{aligned}
$$

for any tangent vector fields X and Y on M.

Let us assume that the normal Jacobi operator $\overline{R}_N$ on a Hopf hypersurface M in a complex two-plane Grassmann manifold $G_2(\mathbb{C}^{m+2})$ is *GTW parallel*, that is,

$$(*)\qquad (\widehat{\nabla}^{(k)}_X \overline{R}_N)Y = 0$$

for any tangent vector fields X and Y on M.

Here, it is the main goal to show that the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or its orthogonal complement $\mathfrak{D}^\perp$ such that $TM = \mathfrak{D} \oplus \mathfrak{D}^\perp$ in $G_2(\mathbb{C}^{m+2})$ when the normal Jacobi operator is GTW parallel.

From now on, we may write the Reeb vector field ξ as

$$(**)\qquad \xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$$

for some unit vector fields $X_0 \in \mathfrak{D}$ and $\xi_1 \in \mathfrak{D}^\perp$.

By putting $X = \xi$ in (1.3) and using the condition $(*)$, we have

$$
\begin{aligned}
(1.4)\qquad 0 = (\widehat{\nabla}^{(k)}_\xi \overline{R}_N)Y = \sum_{\nu=1}^{3}\{&-4\alpha\eta_\nu(\varphi Y)\xi_\nu + 4\alpha\eta_\nu(Y)\varphi_\nu\xi \\
&- 4k\eta_\nu(Y)\varphi_\nu\xi + k\eta_\nu(\xi)\varphi\varphi_\nu\varphi Y - k\eta_\nu(\xi)\eta(Y)\varphi_\nu\xi \\
&- k\eta_\nu(\xi)\eta_\nu(\varphi Y)\xi + 4k\eta_\nu(\varphi Y)\xi_\nu + k\eta_\nu(\xi)\varphi_\nu Y\}
\end{aligned}
$$

for any tangent vector field Y on M. Taking the inner product with ξ in (1.4), this becomes

$$4(\alpha - k)\eta(X_0)\eta(\xi_1)g(Y, \varphi_1 X_0) = 0$$

for any tangent vector field Y on M, since $\varphi\xi_1 = \eta(X_0)\varphi_1 X_0$. Replacing Y by $\varphi_1 X_0$ in the above equation, we obtain

$$(\alpha - k)\eta(X_0)\eta(\xi_1) = 0.$$

Thus there are 3 cases:

Case 1: $\eta(X_0) = 0$, which means that ξ belongs to the distribution $\mathfrak{D}^\perp$.

Case 2: $\eta(\xi_1) = 0$, which means that ξ belongs to the distribution $\mathfrak{D}$.

Finally, in the case of $\eta(X_0)\eta(\xi_1) \neq 0$, the only possible situation is the following one:

Case 3: $\alpha = k$. In this case, α becomes a non-zero constant real number. From [3], Section 4, we get

$$Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3} \eta_\nu(\xi)\eta_\nu(\varphi Y)$$

for any Y tangent to M. This gives

$$0 = \eta(\xi_1)\varphi\xi_1 = \eta(\xi_1)\varphi_1\xi = \eta(\xi_1)\eta(X_0)\varphi_1 X_0.$$

Because of the assumptions in Case 3, this yields $\varphi_1 X_0 = 0$. Therefore $-X_0 + \eta(X_0)\xi_1 = 0$. That is, $X_0 = \eta(X_0)\xi_1$, which is impossible. Thus we have just proved that the Reeb vector field ξ belongs either to the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.

First of all, we consider the case $\xi \in \mathfrak{D}^\perp$. Without loss of generality, we may put $\xi = \xi_1$.

Lemma 1.1. *Let M be a Hopf hypersurface of $G_2(\mathbb{C}^{m+2})$ with GTW parallel normal Jacobi operator. If the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$, then $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

P r o o f. Since ξ belongs to the distribution $\mathfrak{D}^\perp$, using (1.3) and the assumption $(*)$, we have

$$\begin{aligned}(1.5)\qquad 0 = \sum_{\nu=1}^{3}\{&3g(\varphi_\nu AX, Y)\xi_\nu + 3\eta_\nu(Y)\varphi_\nu AX - 2\eta_\nu(\varphi AX)\varphi_\nu\varphi Y \\ &+ 5\eta_\nu(\varphi AX)\eta(Y)\xi_\nu + g(\varphi_\nu AX, \varphi Y)\varphi_\nu\xi + \eta_\nu(\varphi Y)\varphi_\nu\varphi AX\end{aligned}$$

$$
\begin{aligned}
&- \alpha\eta(X)\eta_\nu(\varphi Y)\xi_\nu + 3\eta_\nu(\varphi AX)\eta_\nu(Y)\xi + \eta_\nu(AX)\eta_\nu(\varphi Y)\xi \\
&- 4k\eta(X)\eta_\nu(Y)\varphi_\nu\xi + 4k\eta(X)\eta_\nu(\varphi Y)\xi_\nu\} \\
&- g(\varphi AX, \varphi\varphi_1 Y)\xi - 4\eta_1(Y)\varphi AX + k\eta(X)\varphi\varphi_1\varphi Y \\
&- 4g(\varphi AX, Y)\xi_1 + \eta(Y)\varphi_1 AX + k\eta(X)\varphi_1 Y
\end{aligned}
$$

for any tangent vector fields X and Y on M.

Restricting Y to the distribution $\mathfrak{D}$, (1.5) can be read as

$$
\begin{aligned}
0 &= 3g(\varphi_1 AX, Y)\xi_1 + 3g(\varphi_2 AX, Y)\xi_2 + 3g(\varphi_3 AX, Y)\xi_3 \\
&\quad - 2\eta_2(\varphi AX)\varphi_2\varphi Y - 2\eta_3(\varphi AX)\varphi_3\varphi Y - g(\varphi_2 AX, \varphi Y)\xi_3 \\
&\quad + g(\varphi_3 AX, \varphi Y)\xi_2 - g(AX, \varphi_1 Y)\xi - 4g(\varphi AX, Y)\xi_1
\end{aligned} \tag{1.6}
$$

for any tangent vector field X on M.

Taking the inner product with ξ_2, we get

$$3g(\varphi_2 AX, Y) + g(\varphi_3 AX, \varphi Y) = 0$$

for any tangent vector fields X on M and $Y \in \mathfrak{D}$, that is,

$$-3A\varphi_2 Y - A\varphi_3\varphi Y = 0.$$

Replacing Y by $\varphi Y \in \mathfrak{D}$ in the above equation, we obtain

$$A\varphi_3 Y = 3A\varphi_2\varphi Y. \tag{1.7}$$

Taking the inner product with ξ_3 in (1.6), we get

$$3g(\varphi_3 AX, Y) - g(\varphi_2 AX, \varphi Y) = 0$$

for any tangent vector fields X on M and $Y \in \mathfrak{D}$. In other words,

$$3A\varphi_3 Y = A\varphi_2\varphi Y. \tag{1.8}$$

Combining (1.7) and (1.8), we get

$$A\varphi_3 Y = 9A\varphi_3 Y$$

for any tangent vector field $Y \in \mathfrak{D}$.

Replacing Y by $\varphi_3 Y$ in the above equation, we have

$$AY = 0.$$

Hence, $g(AY, \xi_\nu) = 0$ for $\nu = 1, 2, 3$ and any $Y \in \mathfrak{D}$, that is, $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$. □

In the case of $\xi \in \mathfrak{D}$, from [11] we know that M must be locally congruent to a real hypersurface of type (B) under our assumptions. So, we see that M is locally congruent to a model space either of type (A) or type (B) in Theorem A under the assumption of our Main Theorem.

Hence it remains to check whether the normal Jacobi operator $\overline{R}_N$ of real hypersurfaces of type (A) or type (B) satisfies the condition $(*)$ for any tangent vector field Y on M or not.

Now, consider $\xi \in \mathfrak{D}^{\perp}$. According to the following proposition from [3], a real hypersurface M of type (A) has four distinct constant principal curvatures as follows:

Proposition A. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^{\perp}$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then M has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r), \quad \beta = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m-2 = m(\mu),$$

and the corresponding eigenspaces are

$$\begin{aligned}
T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1 = \operatorname{Span}\{\xi\} = \operatorname{Span}\{\xi_1\},\\
T_\beta &= \mathbb{C}^{\perp}\xi = \mathbb{C}^{\perp}N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3 = \operatorname{Span}\{\xi_2,\ \xi_3\},\\
T_\lambda &= \{X;\ X \perp \mathbb{H}\xi,\ JX = J_1X\},\\
T_\mu &= \{X;\ X \perp \mathbb{H}\xi,\ JX = -J_1X\},
\end{aligned}$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$ and $\mathbb{H}\xi$ denote, respectively, the real, complex and quaternionic span of the structure vector field ξ and $\mathbb{C}^{\perp}\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

Using this, we consider a unit eigenvector $X \in T_\lambda$, $Y = \xi_2$ and assuming $\xi = \xi_1 \in \mathfrak{D}^{\perp}$, we obtain from (1.3)

$$3\lambda\varphi_2 X - \lambda\varphi_3\varphi X = 0.$$

Since X belongs to T_λ, φX is a tangent vector field on T_λ, that is, $\varphi X = \varphi_1 X$.

Thus we have $2\lambda\varphi_2 X = 0$. Taking the inner product with $\varphi_2 X$, we get $\lambda = 0$. This gives a contradiction. So we know that no real hypersurface of type (A) in $G_2(\mathbb{C}^{m+2})$ admits a GTW parallel normal Jacobi operator in the case of ξ belonging to the distribution $\mathfrak{D}^{\perp}$. We make the following remark.

Remark 1.2. If the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^{\perp}$, then there exists no hypersurface of type (A) in $G_2(\mathbb{C}^{m+2})$ with GTW parallel normal Jacobi operator.

Now we check the case $\xi \in \mathfrak{D}$ supposing that M has a GTW parallel normal Jacobi operator. In order to do this we introduce a proposition due to Berndt and Suh [3]:

Proposition B. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$\begin{aligned}
&T_\alpha = \mathbb{R}\xi = \operatorname{Span}\{\xi\},\\
&T_\beta = \mathfrak{J}J\xi = \operatorname{Span}\{\xi_\nu;\ \nu = 1,2,3\},\\
&T_\gamma = \mathfrak{J}\xi = \operatorname{Span}\{\varphi_\nu\xi;\ \nu = 1,2,3\},\\
&T_\lambda, \quad T_\mu,
\end{aligned}$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^{\perp}, \quad \mathfrak{J}T_\lambda = T_\lambda, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

The distribution $(\mathbb{H}\mathbb{C}\xi)^{\perp}$ is the orthogonal complement of $\mathbb{H}\mathbb{C}\xi$, where

$$\mathbb{H}\mathbb{C}\xi = \mathbb{R}\xi \oplus \mathbb{R}J\xi \oplus \mathfrak{J}\xi \oplus \mathfrak{J}J\xi.$$

If we consider a unit eigenvector $X \in T_\lambda$, $Y = \xi_2$ in (1.3), it becomes

$$\sum_{\nu=1}^{3}\{3\lambda\eta_\nu(\xi_2)\varphi_\nu X + \lambda g(\varphi_\nu X, \varphi\xi_2)\varphi_\nu\xi\} = 0.$$

So we have

$$3\lambda\varphi_2 X = 0.$$

Taking the inner product with $\varphi_2 X$, we get $\lambda = 0$. This gives a contradiction. So this case cannot occur. Also we make the following remark.

Remark 1.3. If the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$, then there exists no hypersurface of type (B) in $G_2(\mathbb{C}^{m+2})$ with GTW parallel normal Jacobi operator.

Hence summing up Lemma 1.1 and Remarks 1.2, 1.3, we complete the proof of Main Theorem. □

2. GTW Reeb-parallel normal Jacobi operator

In this section, we consider a new notion which differs from the GTW parallel normal Jacobi operator.

Let us assume that the normal Jacobi operator $\overline{R}_N$ on Hopf hypersurfaces M in complex two-plane Grassmann manifolds $G_2(\mathbb{C}^{m+2})$ is *GTW Reeb-parallel* defined by

$$(\widehat{\nabla}^{(k)}_{\xi}\overline{R}_N)Y = 0 \tag{2.1}$$

for any tangent vector field Y on M. From this notion, together with the proof of Main Theorem we see that the Reeb vector field ξ belongs either to the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^{\perp}$. For $\xi\in\mathfrak{D}^{\perp}$, we will prove that any Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ always has a GTW Reeb-parallel normal Jacobi operator.

Proposition 2.1. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$, such that $\xi \in \mathfrak{D}^{\perp}$. Then the normal Jacobi operator $\overline{R}_N$ is GTW Reeb-parallel.*

Proof. Putting $X = \xi$ and $\xi = \xi_1$ in (1.3), it becomes

$$\begin{aligned}(\widehat{\nabla}^{(k)}_{\xi}\overline{R}_N)Y = \sum_{\nu=1}^{3}\{&3g(\varphi_\nu A\xi, Y)\xi_\nu + 3\eta_\nu(Y)\varphi_\nu A\xi - 2\eta_\nu(\varphi A\xi)\varphi_\nu\varphi Y\\ &+ 5\eta_\nu(\varphi A\xi)\eta(Y)\xi_\nu + g(\varphi_\nu A\xi, \varphi Y)\varphi_\nu\xi + \eta_\nu(\varphi Y)\varphi_\nu\varphi A\xi\\ &- \alpha\eta(\xi)\eta_\nu(\varphi Y)\xi_\nu + 3\eta_\nu(\varphi A\xi)\eta_\nu(Y)\xi - \eta_\nu(\xi)g(\varphi A\xi, \varphi_\nu\varphi Y)\xi\\ &+ \eta_\nu(\xi)\eta_\nu(\varphi A\xi)\eta(Y)\xi - \alpha\eta_\nu(\xi)\eta(\xi)\eta_\nu(\varphi Y)\xi + \eta_\nu(A\xi)\eta_\nu(\varphi Y)\xi\\ &- 4\eta_\nu(\xi)\eta_\nu(Y)\varphi A\xi - 4k\eta(\xi)\eta_\nu(Y)\varphi_\nu\xi + k\eta_\nu(\xi)\eta(\xi)\varphi\varphi_\nu\varphi Y\\ &- k\eta_\nu(\xi)\eta(\xi)\eta(Y)\varphi_\nu\xi - k\eta_\nu(\xi)\eta(\xi)\eta_\nu(\varphi Y)\xi + 4k\eta(\xi)\eta_\nu(\varphi Y)\xi_\nu\\ &- 4\eta_\nu(\xi)g(\varphi A\xi, Y)\xi_\nu + \eta_\nu(\xi)\eta(Y)\varphi_\nu A\xi + k\eta_\nu(\xi)\eta(\xi)\varphi_\nu Y\}\end{aligned}$$

for any tangent vector field Y on M. Together with the fact that M is Hopf, it can be written as

$$(\widehat{\nabla}^{(k)}_{\xi}\overline{R}_N)Y = \sum_{\nu=1}^{3}\{3\alpha g(\varphi_\nu\xi, Y)\xi_\nu + 3\alpha\eta_\nu(Y)\varphi_\nu\xi - 2\alpha\eta_\nu(\varphi\xi)\varphi_\nu\varphi Y$$

$$
\begin{aligned}
&+5\alpha\eta_\nu(\varphi\xi)\eta(Y)\xi_\nu + \alpha g(\varphi_\nu\xi,\varphi Y)\varphi_\nu\xi + \alpha\eta_\nu(\varphi Y)\varphi_\nu\varphi\xi\\
&-\alpha\eta(\xi)\eta_\nu(\varphi Y)\xi_\nu + 3\alpha\eta_\nu(\varphi\xi)\eta_\nu(Y)\xi - \alpha\eta_\nu(\xi)g(\varphi\xi,\varphi_\nu\varphi Y)\xi\\
&+\alpha\eta_\nu(\xi)\eta_\nu(\varphi\xi)\eta(Y)\xi - \alpha\eta_\nu(\xi)\eta(\xi)\eta_\nu(\varphi Y)\xi + \alpha\eta_\nu(\xi)\eta_\nu(\varphi Y)\xi\\
&-4\alpha\eta_\nu(\xi)\eta_\nu(Y)\varphi\xi - 4k\eta(\xi)\eta_\nu(Y)\varphi_\nu\xi + k\eta_\nu(\xi)\eta(\xi)\varphi\varphi_\nu\varphi Y\\
&-k\eta_\nu(\xi)\eta(\xi)\eta(Y)\varphi_\nu\xi - k\eta_\nu(\xi)\eta(\xi)\eta_\nu(\varphi Y)\xi + 4k\eta(\xi)\eta_\nu(\varphi Y)\xi_\nu\\
&-4\alpha\eta_\nu(\xi)g(\varphi\xi,Y)\xi_\nu + \alpha\eta_\nu(\xi)\eta(Y)\varphi_\nu\xi + k\eta_\nu(\xi)\eta(\xi)\varphi_\nu Y\}\\
=&\sum_{\nu=1}^{3}\{3\alpha g(\varphi_\nu\xi,Y)\xi_\nu + 3\alpha\eta_\nu(Y)\varphi_\nu\xi + \alpha g(\varphi_\nu\xi,\varphi Y)\varphi_\nu\xi\\
&-\alpha\eta_\nu(\varphi Y)\xi_\nu - 4k\eta_\nu(Y)\varphi_\nu\xi + k\eta_\nu(\xi)\varphi\varphi_\nu\varphi Y\\
&-k\eta_\nu(\xi)\eta(Y)\varphi_\nu\xi - k\eta_\nu(\xi)\eta_\nu(\varphi Y)\xi + 4k\eta_\nu(\varphi Y)\xi_\nu\\
&+\alpha\eta_\nu(\xi)\eta(Y)\varphi_\nu\xi + k\eta_\nu(\xi)\varphi_\nu Y\}
\end{aligned}
$$

for any tangent vector field Y on M.

By using (2.1) and (2.8) in [11], Section 2, we have

$$
(\widehat{\nabla}^{(k)}_\xi \overline{R}_N)Y = \sum_{\nu=1}^{3}\{-4\alpha\eta_\nu(\varphi Y)\xi_\nu + 4\alpha\eta_\nu(Y)\varphi_\nu\xi - 4k\eta_\nu(Y)\varphi_\nu\xi + 4k\eta_\nu(\varphi Y)\xi_\nu\}
$$

for any tangent vector field Y on M.

Because of (2.3) in [11], Section 2, we get

$$
\begin{aligned}
(\widehat{\nabla}^{(k)}_\xi \overline{R}_N)Y = &-4(\alpha-k)\{\eta_1(\varphi Y)\xi_1 + \eta_2(\varphi Y)\xi_2 + \eta_3(\varphi Y)\xi_3\\
&+\eta_1(Y)\varphi_1\xi + \eta_2(Y)\varphi_2\xi + \eta_3(Y)\varphi_3\xi\} = 0
\end{aligned}
$$

for any tangent vector field Y on M. Thus from (2.1), the normal Jacobi operator $\overline{R}_N$ is GTW Reeb-parallel. □

Acknowledgment. The authors would like to express their sincere gratitude to the referee for his/her careful reading of our manuscript and many kinds of valuable comments.

References

[1] *D. V. Alekseevskij*: Compact quaternion spaces. Funkts. Anal. Prilozh. *2* (1968), 11–20. (In Russian.) zbl MR

[2] *J. Berndt, Y. J. Suh*: Real hypersurfaces with isometric Reeb flow in complex two-plane Grassmannians. Monatsh. Math. *137* (2002), 87–98. zbl MR

[3] *J. Berndt, Y. J. Suh*: Real hypersurfaces in complex two-plane Grassmannians. Monatsh. Math. *127* (1999), 1–14. zbl MR

[4] *J. T. Cho*: Levi-parallel hypersurfaces in a complex space form. Tsukuba J. Math. *30* (2006), 329–343. zbl MR

[5] *J. T. Cho*: CR structures on real hypersurfaces of a complex space form. Publ. Math. *54* (1999), 473–487. zbl MR

[6] *J. de Dios Pérez, I. Jeong, Y. J. Suh*: Real hypersurfaces in complex two-plane Grassmannians with commuting normal Jacobi operator. Acta Math. Hung. *117* (2007), 201–217. zbl MR

[7] *J. de Dios Pérez, Y. J. Suh*: Real hypersurfaces of quaternionic projective space satisfying $\nabla_{U_i} R = 0$. Differ. Geom. Appl. *7* (1997), 211–217. zbl MR

[8] *I. Jeong, H. J. Kim, Y. J. Suh*: Real hypersurfaces in complex two-plane Grassmannians with parallel normal Jacobi operator. Publ. Math. *76* (2010), 203–218. zbl MR

[9] *I. Jeong, M. Kimura, H. Lee, Y. J. Suh*: Real hypersurfaces in complex two-plane Grassmannians with generalized Tanaka-Webster Reeb parallel shape operator. Monatsh. Math. *171* (2013), 357–376. zbl MR

[10] *I. Jeong, Y. J. Suh*: Real hypersurfaces in complex two-plane Grassmannians with $\mathfrak{F}$-parallel normal Jacobi operator. Kyungpook Math. J. *51* (2011), 395–410. zbl MR

[11] *H. Lee, Y. J. Suh*: Real hypersurfaces of type B in complex two-plane Grassmannians related to the Reeb vector. Bull. Korean Math. Soc. *47* (2010), 551–561. zbl MR

[12] *N. Tanaka*: On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections. Jap. J. Math., new Ser. *2* (1976), 131–190. zbl MR

[13] *S. Tanno*: Variational problems on contact Riemannian manifolds. Trans. Am. Math. Soc. *314* (1989), 349–379. zbl MR

[14] *S. M. Webster*: Pseudo-Hermitian structures on a real hypersurface. J. Differ. Geom. *13* (1978), 25–41. zbl MR

Authors' addresses: Eunmi Pak, Changhwa Woo, Department of Mathematics, Kyungpook National University, 80 Daehakro, Bukgu, Daegu 702-701, Korea, e-mail: `empak@hanmail.net`, `legalgwch@knu.ac.kr`; Juan de Dios Pérez, Carlos J. G. Machado, Departmento de Geometría y Topología, Faculta de Ciencias, Universidad de Granada, 18071-Granada, Spain, e-mail: `jdperez@ugr.es`, `cgmachado@gmail.com`.