Basudeb Dhara; Sukhendu Kar; Sachhidananda Mondal
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GENERALIZED DERIVATIONS ON LIE IDEALS IN PRIME RINGS

Basudeb Dhara, Belda, Sukhendu Kar,
Sachhidananda Mondal, Kolkata

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Abstract. Let R be a prime ring with its Utumi ring of quotients U and extended centroid C. Suppose that F is a generalized derivation of R and L is a noncentral Lie ideal of R such that $F(u)[F(u),u]^n = 0$ for all $u \in L$, where $n \geqslant 1$ is a fixed integer. Then one of the following holds:

(1) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;
(2) R satisfies s_4 and $F(x) = ax + xb$ for all $x \in R$, with $a, b \in U$ and $a - b \in C$;
(3) $\operatorname{char}(R) = 2$ and R satisfies s_4.

As an application we also obtain some range inclusion results of continuous generalized derivations on Banach algebras.

Keywords: prime ring; derivation; generalized derivation; extended centroid; Utumi quotient ring; Lie ideal; Banach algebra

MSC 2010: 16W25, 16W80, 16N60

1. Introduction

Let R be an associative ring with center $Z(R)$. For $x, y \in R$, the commutator of x, y is denoted by $[x, y]$ and defined by $[x, y] = xy - yx$. By d we mean a derivation of R. An additive mapping F from R to R is called a generalized derivation if there exists a derivation d from R to R such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.

Throughout this paper, R will always represent a prime ring with center $Z(R)$, extended centroid C, and U is its Utumi quotient ring. A well known result proved by Posner [25] states that if the commutators satisfy $[d(x), x] \in Z(R)$ for all $x \in R$, then either $d = 0$ or R is commutative. Later this result of Posner was generalized in many directions by a number of authors. Posner's theorem was extended to Lie ideals in prime rings by Lee [18] and then by Lanski [16]. In [4], Carini and De

Filippis studied the situation in more generalized form considering power values. They proved that if $\mathrm{char}(R) \neq 2$ and $[d(x), x]^n \in Z(R)$ for all $x \in L$, where L is a noncentral Lie ideal of R and $n \geqslant 1$ a fixed integer, then $d = 0$ or R satisfies s_4. De Filippis [6] proved a result replacing d by a generalized derivation F of R. More precisely, he proved the following result:

Let R be a prime ring of characteristic not equal 2 with right Utumi quotient ring U and extended centroid C, $F \neq 0$ a generalized derivation of R, L a noncentral Lie ideal of R and $n \geqslant 1$. If $[F(u), u]^n = 0$ for all $u \in L$, then there exists an element $a \in C$ such that $F(x) = ax$ for all $x \in R$, unless R satisfies s_4 and there exists an element $b \in U$ such that $F(x) = bx + xb$ for all $x \in R$.

Recently, De Filippis et al. [7] studied the situation $[F(u), u]F(u) = 0$ for all u in a noncentral Lie ideal of R. More precisely, they obtained the following result:

Let R be a prime ring of characteristic not equal 2, U the Utumi quotient ring of R, $C = Z(U)$ the extended centroid of R, L a noncentral Lie ideal of R and F a non-zero generalized derivation of R. Suppose that $[F(u), u]F(u) = 0$ for all $u \in L$. Then one of the following holds:

(1) there exists $\alpha \in C$ such that $F(x) = \alpha x$ for all $x \in R$;
(2) R satisfies the standard identity s_4 and there exist $a \in U$ and $\alpha \in C$, such that $F(x) = ax + xa + \alpha x$ for all $x \in R$.

In the present paper we study the situation $F(u)[F(u), u]^n = 0$ for all $u \in L$, where $n \geqslant 1$ is a fixed integer and L is a noncentral Lie ideal of R.

More precisely, we prove the following results:

Theorem 1.1. *Let R be a prime ring with its Utumi ring of quotients U and extended centroid C. Suppose that F is a generalized derivation of R and L is a noncentral Lie ideal of R such that $F(u)[F(u), u]^n = 0$ for all $u \in L$, where $n \geqslant 1$ is a fixed integer. Then one of the following holds:*

(1) *there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*
(2) *R satisfies s_4 and $F(x) = ax + xb$ for all $x \in R$, with $a, b \in U$ and $a - b \in C$;*
(3) $\mathrm{char}(R) = 2$ *and R satisfies s_4.*

Theorem 1.2. *Let R be a noncommutative prime ring with its Utumi ring of quotients U and extended centroid C. Suppose that F is a generalized derivation of R such that $F(x)[F(x), x]^n = 0$ for all $x \in R$, where $n \geqslant 1$ is a fixed integer. Then one of the following holds:*

(1) *there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*
(2) $\mathrm{char}(R) = 2$ *and R satisfies s_4.*

In the last section we apply the above results to Banach algebras. Here A will denote a complex noncommutative Banach algebra. By a Banach algebra we shall mean a complex normed algebra A whose underlying vector space is a Banach space. By $\operatorname{rad}(A)$ we denote the Jacobson radical of A, which is the intersection of all primitive ideals of A. A is said to be semisimple, if $\operatorname{rad}(A) = 0$.

In 1955, Singer and Wermer [27] gave an interesting result. They proved that every continuous derivation on a commutative Banach algebra maps the algebra into its radical. After thirty years, Thomas [28] proved the same result without considering continuity of derivation. It is clear that the same result does not hold in noncommutative Banach algebras because of inner derivations. It is still an open question whether the above result of Singer and Wermer is true or not in the noncommutative Banach algebra. Some partial solutions of this open question have been obtained by a number of authors under certain conditions for noncommutative Banach algebras.

Let A be a noncommutative Banach algebra and D a continuous derivation on A. Yood [30] proved that if $[D(x), y]$ lies in $\operatorname{rad}(A)$ for all $x, y \in A$, then D maps A into $\operatorname{rad}(A)$. Later, Brešar and Vukman [3] generalized Yood's result by stating that the same conclusion holds under the weaker condition $[D(x), x] \in \operatorname{rad}(A)$ for all $x \in A$. A similar result was also obtained by Mathieu and Murphy [23] if $[D(x), x] \in Z(A)$ for all $x \in A$. In [22], Mathieu proved the same conclusion if $[D(x), x]D(x) \in \operatorname{rad}(A)$ for all $x \in A$. Vukman [29] proved that the same conclusion holds if $[D(x), x]_3 \in \operatorname{rad}(A)$ for all $x \in A$.

Continuing along this line, in [15] Kim proved that if d is a continuous linear Jordan derivation in a Banach algebra A, such that $[d(x), x]d(x)[d(x), x] \in \operatorname{rad}(A)$ for all $x \in A$, then d maps A into $\operatorname{rad}(A)$. In [14] he obtained the same conclusion in the case $d(x)[d(x), x]d(x) \in \operatorname{rad}(A)$ for all $x \in A$. More recently in [24], Park proves that if d is a derivation of a noncommutative Banach algebra A such that $[[d(x), x], d(x)] \in \operatorname{rad}(A)$ for all $x \in A$, then again d maps A into $\operatorname{rad}(A)$. Recently, Kim [13] proved that if d is a continuous linear Jordan derivation in a Banach algebra A such that $d(x)^3[d(x), x] \in \operatorname{rad}(A)$ for all $x \in A$, then d maps A into $\operatorname{rad}(A)$. In [7], De Filippis et al. proved the following result:

Let F be a continuous generalized derivation of R such that $F(x) = ax + d(x)$ for some element $a \in A$, and d a derivation of A. If $[F(x), x]F(x) \in \operatorname{rad}(A)$ for all $x \in R$, then $d(A) \subseteq \operatorname{rad}(A)$ and $[a, A] \subseteq \operatorname{rad}(A)$.

In the last section, finally we provide a result about continuous generalized derivations on Banach algebras which is as follows:

Theorem 1.3. *Let A be a noncommutative Banach algebra, $\zeta = L_a + d$ a continuous generalized derivation of A and n a fixed positive integer. If $\zeta(x)[\zeta(x), x]^n \in \operatorname{rad}(A)$ for all $x \in A$, then $d(A) \subseteq \operatorname{rad}(A)$ and $[a, A] \subseteq \operatorname{rad}(A)$.*

2. Results on Lie ideals

First we give the following remarks:

Remark 2.1. Let R be a prime ring and L a noncentral Lie ideal of R. If $\mathrm{char}(R) \neq 2$, by [2], Lemma 1, there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. If $\mathrm{char}(R) = 2$ and $\dim_C RC > 4$, i.e., $\mathrm{char}(R) = 2$ and R does not satisfy s_4, then by [17], Theorem 13, there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. Thus if either $\mathrm{char}(R) \neq 2$ or $\mathrm{char}(R) = 2$ and R does not satisfy s_4, then we may conclude that there exists a nonzero ideal I of R such that $[I, I] \subseteq L$.

Remark 2.2. Let R be a prime ring and U the Utumi quotient ring of R, and $C = Z(U)$ the center of U (see [1] for more details). It is well known that any derivation of R can be uniquely extended to a derivation of U. In [19], Theorem 3, Lee proved that every generalized derivation g on a dense right ideal of R can be uniquely extended to a generalized derivation of U. Furthermore, the extended generalized derivation g has the form $g(x) = ax + d(x)$ for all $x \in U$, where $a \in U$ and d is a derivation of U.

Now we begin with the following lemmas.

Lemma 2.3. *Let R be a prime ring with extended centroid C and $a, b \in R$. If $(a[x_1,x_2] + [x_1,x_2]b)(a[x_1,x_2]^2 + [x_1,x_2](b-a)[x_1,x_2] - [x_1,x_2]^2 b)^n = 0$ for all $x_1, x_2 \in R$, where $n \geqslant 1$ is a fixed integer, then either R satisfies a nontrivial generalized polynomial identity* (GPI) *or $a, b \in C$.*

Proof. Assume that R does not satisfy any nontrivial GPI. Let $T = U *_C C\{x_1, x_2\}$, the free product of U and $C\{x_1, x_2\}$, the free C-algebra in noncommuting indeterminates x_1 and x_2. If R is commutative, then R satisfies trivially a nontrivial GPI, a contradiction. So, R must be noncommutative.

Then,

$$(a[x_1,x_2] + [x_1,x_2]b)(a[x_1,x_2]^2 + [x_1,x_2](b-a)[x_1,x_2] - [x_1,x_2]^2 b)^n = 0 \tag{2.1}$$

in $T = U *_C C\{x_1, x_2\}$. If $b \notin C$, then b and 1 are linearly independent over C. Thus, (2.1) implies

$$(a[x_1,x_2]+[x_1,x_2]b)(a[x_1,x_2]^2+[x_1,x_2](b-a)[x_1,x_2]-[x_1,x_2]^2b)^{n-1}(-[x_1,x_2]^2b) = 0$$

and so

$$([x_1,x_2]b)(-[x_1,x_2]^2 b)^n = 0$$

in T, implying $b=0$, a contradiction. Therefore, we conclude that $b\in C$ and hence (2.1) reduces to

$$(a+b)[x_1,x_2][a[x_1,x_2],[x_1,x_2]]^n=0 \tag{2.2}$$

in T. If $a\notin C$, then in (2.2), $(a+b)[x_1,x_2](a[x_1,x_2]^2)^n$ appears nontrivially, a contradiction. Therefore, we have $a\in C$. □

Lemma 2.4. *Let $R=M_2(F)$ be the set of all 2×2 matrices over a field F of characteristic different from 2 and $a,b\in R$. If $(a[x_1,x_2]+[x_1,x_2]b)(a[x_1,x_2]^2+[x_1,x_2](b-a)[x_1,x_2]-[x_1,x_2]^2b)^n=0$ for all $x_1,x_2\in R$, where $n\geqslant 1$ is a fixed integer, then $(a-b)\in F\cdot I_2$.*

Proof. Since $[x_1,x_2]^2\in Z(M_2(F))$ for all $x_1,x_2\in M_2(F)$, the identity reduces to

$$(a[x_1,x_2]+[x_1,x_2]b)([(a-b),[x_1,x_2]][x_1,x_2])^n=0 \tag{2.3}$$

for all $x_1,x_2\in R$. Again for all $x_1,x_2\in M_2(F)$ since $[x_1,x_2]^2\in Z(M_2(F))$, we have $0=[(a-b),[x_1,x_2]^2]=[(a-b),[x_1,x_2]][x_1,x_2]+[x_1,x_2][(a-b),[x_1,x_2]]$ and hence $[(a-b),[x_1,x_2]][x_1,x_2]=-[x_1,x_2][(a-b),[x_1,x_2]]$. Thus (2.3) reduces to

$$(-1)^{n(n+1)/2}(a[x_1,x_2]+[x_1,x_2]b)[x_1,x_2]^n[(a-b),[x_1,x_2]]^n=0, \tag{2.4}$$

that is

$$(a[x_1,x_2]+[x_1,x_2]b)[x_1,x_2]^n[(a-b),[x_1,x_2]]^n=0 \tag{2.5}$$

for all $x_1,x_2\in R$.

Choose $x_1,x_2\in R$ such that $[x_1,x_2]^2\neq 0$. Now $[(a-b),[x_1,x_2]]^2\in Z(R)$. If $[(a-b),[x_1,x_2]]^2\neq 0$, then $[(a-b),[x_1,x_2]]^2$ is invertible in R. We multiply in (2.5) from the right by $([(a-b),[x_1,x_2]]^n)^{-1}$ or $[(a-b),[x_1,x_2]]([(a-b),[x_1,x_2]]^{n+1})^{-1}$ according as n is even or odd and then get

$$(a[x_1,x_2]+[x_1,x_2]b)[x_1,x_2]^n=0. \tag{2.6}$$

Right multiplying (2.6) by $([x_1,x_2]^n)^{-1}$ or $[x_1,x_2]([x_1,x_2]^{n+1})^{-1}$ according as n is even or odd we obtain that

$$a[x_1,x_2]+[x_1,x_2]b=0. \tag{2.7}$$

Right multiplying by $[x_1, x_2]$ in (2.7) we have

$$a[x_1, x_2]^2 + [x_1, x_2]b[x_1, x_2] = 0. \tag{2.8}$$

By using the fact $[x_1, x_2]^2 \in Z(R)$, we conclude

$$[x_1, x_2]^2 a + [x_1, x_2]b[x_1, x_2] = 0. \tag{2.9}$$

Left multiplying (2.7) by $[x_1, x_2]$, we get

$$[x_1, x_2]a[x_1, x_2] + [x_1, x_2]^2 b = 0. \tag{2.10}$$

Subtracting (2.9) from (2.10) results in

$$[x_1, x_2](a - b)[x_1, x_2] - [x_1, x_2]^2(a - b) = 0, \tag{2.11}$$

that is

$$[x_1, x_2][a - b, [x_1, x_2]] = 0. \tag{2.12}$$

Left multiplying the above relation by $[x_1, x_2]$, we have $[a - b, [x_1, x_2]] = 0$ and hence $[a - b, [x_1, x_2]]^2 = 0$.

Thus up to now we have proved that if for some $x_1, x_2 \in R$, $[x_1, x_2]^2 \neq 0$, then $[p, [x_1, x_2]]^2 = 0$, where $p = a - b$. We choose $x_1 = e_{12}$, $x_2 = e_{21}$. Then we get $[x_1, x_2] = e_{11} - e_{22}$ and $[x_1, x_2]^2 = I_2 \neq 0$. Hence $0 = [p, [x_1, x_2]]^2 = [p, [e_{12}, e_{21}]]^2 = -4p_{12}p_{21}I_2$. This implies that either $p_{12} = 0$ or $p_{21} = 0$. Without loss of generality we assume that $p_{12} = 0$. Now we choose $x_1 = e_{11}$ and $x_2 = e_{12} - e_{21}$. Then we have $[x_1, x_2]^2 = I_2 \neq 0$ and hence $0 = [p, [x_1, x_2]]^2 = [p, [e_{11}, e_{12} - e_{21}]]^2 = \{p_{21}^2 - (p_{11} - p_{22})^2\}I_2$. This implies that $p_{21}^2 - (p_{11} - p_{22})^2 = 0$. Similarly, by choosing $x_1 = e_{11}$ and $x_2 = e_{12} + e_{21}$, we can show that $p_{21}^2 + (p_{11} - p_{22})^2 = 0$. Addition and substraction of $p_{21}^2 - (p_{11} - p_{22})^2 = 0$ and $p_{21}^2 + (p_{11} - p_{22})^2 = 0$ implies $p_{21} = 0$ and $p_{11} = p_{22}$. So $p = (a - b) \in F \cdot I_2$. □

Lemma 2.5. *Let R be a prime ring with extended centroid C and $a, b \in R$. If $(a[x_1, x_2] + [x_1, x_2]b)(a[x_1, x_2]^2 + [x_1, x_2](b - a)[x_1, x_2] - [x_1, x_2]^2 b)^n = 0$ for all $x_1, x_2 \in R$, where $n \geqslant 1$ is a fixed integer, then one of the following holds:*

(1) *$a, b \in C$;*
(2) *R satisfies s_4 and $a, b \in U$ with $(a - b) \in C$;*
(3) $\mathrm{char}(R) = 2$ *and R satisfies s_4.*

P r o o f. We have that R satisfies the generalized polynomial identity

$$f(x_1,x_2)=(a[x_1,x_2]+[x_1,x_2]b)(a[x_1,x_2]^2 + [x_1,x_2](b-a)[x_1,x_2]-[x_1,x_2]^2b)^n. \tag{2.13}$$

If R does not satisfy any nontrivial GPI, by Lemma 2.3 we obtain $a,b\in C$ which gives the conclusion. So, we assume that R satisfies a nontrivial GPI. Since R and U satisfy the same generalized polynomial identities (see [5]), U satisfies $f(x_1,x_2)$. In case C is infinite, we have $f(x_1,x_2)=0$ for all $x_1,x_2\in U\otimes_C\overline{C}$, where $\overline{C}$ is the algebraic closure of C. Moreover, both U and $U\otimes_C\overline{C}$ are prime and centrally closed algebras [9]. Hence, replacing R by U or $U\otimes_C\overline{C}$ according to C being finite or infinite, without loss of generality we may assume that $C=Z(R)$ and R is C-algebra centrally closed. By Martindale's theorem [21], R is then a primitive ring having nonzero socle $\operatorname{soc}(R)$ with C as the associated division ring. Hence, by Jacobson's theorem [11], page 75, R is isomorphic to a dense ring of linear transformations of a vector space V over C.

If $\dim_C V=2$, then $R\cong M_2(C)$, that is, R satisfies s_4. In this case if $\operatorname{char}(R)\neq 2$, then by Lemma 2.4 we obtain conclusion (2), otherwise conclusion (3) holds.

Next we assume that $\dim_C V\geqslant 3$.

We show that for any $v\in V$, v and bv are linearly C-dependent. Suppose that v and bv are linearly independent for some $v\in V$. Since $\dim_C V\geqslant 3$, there exists $w\in V$ such that v, bv, w are a linearly C-independent set of vectors. By density, there exist $x_1,x_2\in R$ such that

$$x_1v=v,\quad x_1bv=-bv,\quad x_1w=0,\quad x_2v=0,\quad x_2bv=w,\quad x_2w=v.$$

Then $0=(a[x_1,x_2]+[x_1,x_2]b)(a[x_1,x_2]^2+[x_1,x_2](b-a)[x_1,x_2]-[x_1,x_2]^2b)^nv=w$, a contradiction. Hence, v and bv are linearly C-dependent for all $v\in V$. Thus for each $v\in V$, $bv=\alpha_v v$ for some $\alpha_v\in C$. It is very easy to prove that α_v is independent of the choice of $v\in V$. Thus we can write $bv=\alpha v$ for all $v\in V$ and $\alpha\in C$ fixed. Now let $r\in R$, $v\in V$. Since $bv=\alpha v$,

$$[b,r]v=(br)v-(rb)v=b(rv)-r(bv)=\alpha(rv)-r(\alpha v)=0.$$

Thus $[b,r]v=0$ for all $v\in V$, i.e., $[b,r]V=0$. Since $[b,r]$ acts faithfully as a linear transformation on the vector space V, $[b,r]=0$ for all $r\in R$. Therefore, $b\in C$. Thus our identity reduces to

$$(a+b)[x_1,x_2]([a,[x_1,x_2]][x_1,x_2])^n=0$$

for all $x_1,x_2\in R$. Then either $a+b=0$ or $a\in C$ (see the proof of Theorem 2.2 for inner derivation case in [8]). Both cases lead to $a\in C$. Thus conclusion (1) is obtained. $\square$

P r o o f of Theorem 1.1. If $\mathrm{char}(R) = 2$ and R satisfies s_4, we obtain our conclusion (3). So we assume that either $\mathrm{char}(R) \neq 2$ or R does not satisfy s_4. Since L is a noncentral Lie ideal of R, by Remark 2.1 there exists a nonzero ideal I of R such that $[I, I] \subseteq L$. Hence, by our assumption we have

$$F([x_1, x_2])[F([x_1, x_2]), [x_1, x_2]]^n = 0$$

for all $x_1, x_2 \in I$. Since I, R and U satisfy the same generalized polynomial identities (see [5]) as well as the same differential identities (see [20]), they also satisfy the same generalized differential identities by Remark 2.2. Hence,

$$F([x_1, x_2])[F([x_1, x_2]), [x_1, x_2]]^n = 0$$

for all $x_1, x_2 \in U$, where $F(x) = ax + d(x)$ for some $a \in U$ and a derivation d of U. Hence, U satisfies

$$(a[x_1, x_2] + d([x_1, x_2]))[a[x_1, x_2] + d([x_1, x_2]), [x_1, x_2]]^n = 0. \tag{2.14}$$

Now we divide the proof into two cases:

Case I: Let $d(x) = [b, x]$ for all $x \in U$ and for some $b \in U$, i.e., d is an inner derivation of U. Then from (2.14), we obtain that U satisfies

$$((a + b)[x_1, x_2] - [x_1, x_2]b)[(a + b)[x_1, x_2] - [x_1, x_2]b, [x_1, x_2]]^n = 0. \tag{2.15}$$

By Lemma 2.5, one of the following holds:

(1) $a + b$, $b \in C$. In this case $F(x) = ax + [b, x] = ax$ for all $x \in U$ and so for all $x \in R$, where $a \in C$. Thus conclusion (1) is obtained.

(2) R satisfies s_4 and $a+2b \in C$. By assumption in this case $\mathrm{char}(R)$ must be $\neq 2$. Thus $F(x) = ax + [b, x] = (a + b)x - xb$ for all $x \in U$ and so for all $x \in R$. This is our conclusion (2).

Case II: Next we assume that d is not an inner derivation of U. From (2.14), we have that U satisfies

$$(a[x_1, x_2] + [d(x_1), x_2] + [x_1, d(x_2)])[a[x_1, x_2] + [d(x_1), x_2] + [x_1, d(x_2)], [x_1, x_2]]^n = 0.$$

This gives by Kharchenko's theorem [12] that U satisfies

$$(a[x_1, x_2] + [y_1, x_2] + [x_1, y_2])[a[x_1, x_2] + [y_1, x_2] + [x_1, y_2], [x_1, x_2]]^n = 0. \tag{2.16}$$

If U does not satisfy s_4, we replace y_1 by $[q, x_1]$ and y_2 by $[q, x_2]$ for some $q \notin C$ in the above relation and then applying Lemma 2.5, we get $a+q, q \in C$, a contradiction.

If U satisfies s_4, then $\operatorname{char}(R) \neq 2$. In this case, for $y_1 = 0$ and $y_2 = 0$ we have from (2.16) that U satisfies

$$a[x_1, x_2]([a, [x_1, x_2]][x_1, x_2])^n = 0. \tag{2.17}$$

By Lemma 2.5, $a \in C$. Then from (2.16) we have a contradiction for $x_1 = e_{11}$, $x_2 = e_{12}$, $y_2 = -e_{21}$, $y_1 = 0$ that $0 = (a[x_1, x_2] + [y_1, x_2] + [x_1, y_2])[a[x_1, x_2] + [y_1, x_2] + [x_1, y_2], [x_1, x_2]]^n = (ae_{12} + e_{21})[e_{21}, e_{12}]^n = (ae_{12} + e_{21})(e_{22} + (-1)^n e_{11}) = ae_{12} + (-1)^n e_{21}$.

Thus Theorem 1.1 is proved. □

P r o o f of Theorem 1.2. By Theorem 1.1, we have only to consider the case when R satisfies s_4. If $\operatorname{char}(R) = 2$, we have the conclusion (2). Let $\operatorname{char}(R) \neq 2$. Then $F(x) = ax + xb$ for all $x \in R$ with $a - b \in C$, that is $F(x) = ax + xa + \lambda x$ for all $x \in R$ and for some $\lambda \in C$. In this case $R \subseteq M_2(F)$ for some field F, and $M_2(F)$ satisfies

$$(ax + xa + \lambda x)[ax + xa, x]^n = 0,$$

that is

$$(ax + xa + \lambda x)[a, x^2]^n = 0. \tag{2.18}$$

We may assume n is even; if not so, we multiply by $[a, x^2]$ from the right to make it even. Since we know that $[x, y]^2 \in Z(M_2(F))$ for all $x, y \in M_2(F)$, we get by right and left multiplying by x in (2.18) that for all $x \in M_2(F)$,

$$(ax + xa + \lambda x)x[a, x^2]^n = 0 \tag{2.19}$$

and

$$x(ax + xa + \lambda x)[a, x^2]^n = 0. \tag{2.20}$$

Subtracting one from the other, we have $[a, x^2]^{n+1} = 0$ and so $[a, x^2]^{n+2} = 0$. Since $n + 2$ is even, it yields that $[a, x^2]^2 = 0$ for all $x \in M_2(F)$. Replacing x by e_{11}, we have $0 = e_{22}[a, e_{11}]^2 e_{22} = -a_{21}a_{12}e_{22}$, that is $a_{21}a_{12} = 0$. Let φ and χ be two inner automorphisms defined by $\varphi(x) = (1 + e_{21})x(1 - e_{21})$ and $\chi(x) = (1 - e_{21})x \times (1 + e_{21})$. Since $\varphi(a)$ and $\chi(a)$ possess the same properties as a does, we conclude that $\varphi(a)_{12}\varphi(a)_{21} = 0$ and $\chi(a)_{12}\chi(a)_{21} = 0$. Both these cases give $a_{12}(-a_{22} + a_{11} - a_{12}) = 0$ and $a_{12}(a_{22} - a_{11} - a_{12}) = 0$. Adding these two relations yields $2a_{12}^2 = 0$, implying $a_{12} = 0$. Similarly, we can prove $a_{21} = 0$. Thus a is diagonal. Therefore, $\varphi(a) = (1 + e_{21})a(1 - e_{21}) = a + (a_{11} - a_{22})e_{21}$ is diagonal, which implies that $a_{11} = a_{22}$. Hence, a is central. Thus $F(x) = (2a + \lambda)x$ for all $x \in R$, which is our conclusion (1). □

3. Results on Banach algebras

Here A will denote a complex noncommutative Banach algebra. Our final result in this paper is about continuous generalized derivations on noncommutative Banach algebras.

We need the following results to prove our theorem.

Remark 3.1 (see [26]). Any continuous derivation of a Banach algebra leaves the primitive ideals invariant.

Remark 3.2 (see [27]). Any continuous linear derivation on a commutative Banach algebra maps the algebra into its radical.

Remark 3.3 (see [10]). Any linear derivation on a semisimple Banach algebra is continuous.

Proof of Theorem 1.3. By the hypothesis, ζ is continuous. Again, since L_a, the left multiplication by some element $a \in A$, is continuous, we find that the derivation d is also continuous. By Remark 3.1, for any primitive ideal P of A we have $\zeta(P) \subseteq aP + d(P) \subseteq P$. It means that the continuous generalized derivation ζ leaves the primitive ideal invariant. Denote $\bar{A} = A/P$ for any primitive ideal P. Thus we can define the generalized derivation $\zeta_P\colon \bar{A} \to \bar{A}$ by $\zeta_P(\bar{x}) = \zeta_P(x+P) = \zeta(x) + P$ for all $\bar{x} \in \bar{A}$, where $A/P = \bar{A}$. Since P is a primitive ideal, $\bar{A}$ is primitive and so it is prime. The hypothesis $\zeta(x)[\zeta(x), x]^n \in \operatorname{rad}(A)$ yields that $\zeta_P(\bar{x})[\zeta_P(\bar{x}), \bar{x}]^n = \bar{0}$ for all $\bar{x} \in \bar{A}$. Now from Theorem 1.2, it is immediate that $d = \bar{0}$ and $\bar{a} \in Z(\bar{A})$, that is, $d(A) \subseteq P$ and $[a, A] \subseteq P$. Since the radical of A is the intersection of all primitive ideals, we arrive at the required conclusions. $\square$

From the above we have the following concluding result:

Corollary 3.4. *Let A be a noncommutative semisimple Banach algebra, $\zeta = L_a + d$ a continuous generalized derivation of A and n a fixed positive integer. If $\zeta(x)[\zeta(x), x]^n = 0$ for all $x \in A$, then $\zeta(x) = \alpha x$ for some $\alpha \in Z(A)$.*

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Authors' addresses: Basudeb Dhara, Department of Mathematics, Belda College, Belda, Paschim Medinipur, West Bengal-721 424, India, e-mail: `basu_dhara@yahoo.com`; Sukhendu Kar, Sachhidananda Mondal, Department of Mathematics, Jadavpur University, 188, Raja S. C. Mallick Road., Jadavpur, Kolkata-700 032, India, e-mail: `karsukhendu@yahoo.co.in`, `sachhidananda.mondal@gmail.com`.