Czechoslovak Mathematical Journal

Shriram K. Nimbhorkar; Rupal C. Shroff

Ojective ideals in modular lattices

Czechoslovak Mathematical Journal, Vol. 65 (2015), No. 1, 161–178

Persistent URL: `http://dml.cz/dmlcz/144219`

Czechoslovak Mathematical Journal, 65 (140) (2015), 161–178

OJECTIVE IDEALS IN MODULAR LATTICES

Shriram K. Nimbhorkar, Rupal C. Shroff, Aurangabad

(Received November 12, 2013)

Dedicated to Professor N. K. Thakare and Professor T. T. Raghunathan on the occasion of their 76th birthday

Abstract. The concept of an extending ideal in a modular lattice is introduced. A translation of module-theoretical concept of ojectivity (i.e. generalized relative injectivity) in the context of the lattice of ideals of a modular lattice is introduced. In a modular lattice satisfying a certain condition, a characterization is given for direct summands of an extending ideal to be mutually ojective. We define exchangeable decomposition and internal exchange property of an ideal in a modular lattice. It is shown that a finite decomposition of an extending ideal is exchangeable if and only if its summands are mutually ojective.

Keywords: modular lattice; essential ideal; max-semicomplement; extending ideal; direct summand; exchangeable decomposition; ojective ideal

MSC 2010: 06B10, 06C05

1. Introduction and preliminaries

A module is called extending (or CS-module, i.e., complements are summands) if every submodule of it is essential in a direct summand. Akalan, Birkenmeier and Tercan [1] and Lam [11] studied extending modules and the Goldie dimension of a module. Birkenmeier, Müller and Rizvi [2] used this concept to develop the theory of fully invariant extending modules. In 1988, Kamal and Müller, see [8], [9], [10] studied the concept of extending modules over Noetherian rings and commutative domains. Hanada, Kuratomi and Oshiro [6] studied the concept of extending modules. The following open problem was posed by Harmanci and Smith [7].

Open Problem: What can be a necessary and sufficient condition for the direct sum of extending modules to be extending?

In the honor of Oshiro, in [12], Mohamed and Müller call an M-injective module an *ojective module*. They proved that mutual ojectivity is a necessary and sufficient condition for the direct sum of extending modules to be extending.

Grzeszczuk and Puczyłowski, [5], [4] developed the concept of Goldie dimension from module theory to modular lattices. In this context they defined the concept of an essential element in a lattice with the least element 0, see [5].

In this paper we formulate and answer an analogue of the above open problem for the lattice of ideals of a modular lattice L by introducing the concept of ojectivity for the ideals of L.

In the second section we introduce the concepts of an essential element, a max-semicomplement of an element and a closed element in a lattice and obtain some of their properties. The third section deals with direct summands and extending ideals. We show that in a modular lattice satisfying a certain condition, direct summands of an extending ideal are extending. Further, we define the exchangeable decomposition and the internal exchange property of an ideal in a modular lattice and show that the 2-internal exchange property passes to direct summands. In the last section, we define an ojective ideal in a modular lattice using a translation of the module-theoretical concept of ojectivity (i.e. generalized relative injectivity) in the context of the lattice of ideals in a modular lattice with a certain condition. The key point for this translation is the following result from Mohamed and Müller [12], Theorem 7 which characterizes ojective modules in terms of lattices of submodules.

Theorem 1.1. *Let M be an R-module and A, B, R-submodules of M. If $M = A \oplus B$, then B is A-ojective if and only if for any complement C of B, M decomposes as $M = C \oplus A' \oplus B'$, with $A' \leqslant A$ and $B' \leqslant B$.*

(The word complement used in this theorem has the following meaning: C is a complement of B if C is maximal with the property $C \cap B = \{0\}$.)

This characterization serves in this paper as the definition for ojectivity of ideals in a modular lattice. We give a characterization for direct summands of an ideal to be mutually ojective. We show that the direct sum of extending ideals is extending if the direct sum is exchangeable and the summands are mutually ojective.

We recall some concepts from the lattice theory, see Grätzer [3].

Definition 1.1. A lattice L is said to be modular if for $a, b, c \in L$ with $a \leqslant c$, $a \vee (b \wedge c) = (a \vee b) \wedge c$.

Definition 1.2. A nonempty subset I of a lattice L is said to be an ideal if the following two conditions hold:

(1) $a, b \in I$ implies $a \vee b \in I$.

(2) If $x \leqslant a$ for $a \in I$, $x \in L$ then $x \in I$.

We denote the set $\{x \in L\colon\ x \leqslant a\}$ by $(a]$ and call it the principal ideal generated by a. The set $\mathrm{Id}(L)$ of all ideals of a lattice L forms a lattice under set inclusion as the partial order. In fact, if L is a lattice with the least element 0 then $\mathrm{Id}(L)$ is a complete lattice.

Lemma 1.1 ([3], page 31)**.** *A lattice L is modular if and only if* $\mathrm{Id}(L)$ *is modular.*

The undefined terms are from Grätzer [3].

2. Essential extensions and closed extensions

The concepts of an essential module and a closed module are known in the theory of modules. We extend them in the context of a lattice. Some of these concepts can be found in Grzeszczuk and Puczyłowski [5], [4].

Throughout this paper L denotes a lattice with the least element 0.

In this section we discuss properties of essential extensions and closed extensions in L.

Definition 2.1. Let $a \in L$. We say that a is essential in L, if there is no nonzero $x \in L$ such that $a \wedge x = 0$. Let $a, b \in L$, $0 \neq a \leqslant b$. We say that a is essential in b (or b is an essential extension of a), if there is no nonzero $c \leqslant b$ with $a \wedge c = 0$. We then write $a \leqslant_e b$.

An ideal I of L is said to be essential in L, if it is an essential element in $\mathrm{Id}(L)$.

Definition 2.2. If $a \leqslant_e b$ and for any $c > b$, a is not essential in c, then b is called a maximal essential extension of a.

Definition 2.3. An element a is called closed in L, if a has no proper essential extension in L. Let $a, b \in L, a \leqslant b$. We say that a is closed in b, if a has no proper essential extension in b.

An ideal I of L is said to be closed in L, if it is a closed element in $\mathrm{Id}(L)$.

Definition 2.4. If $a, b \in L$ and b is a maximal element in the set $\{x\colon\ x \in L,\ a \wedge x = 0\}$, then we say that b is a max-semicomplement of a in L. An element $x \in L$ is called a max-semicomplement in L, if there exists a $y \in L$ such that x is a max-semicomplement of y in L.

Example 2.1. In the lattice L shown in Figure 1, b is essential in d, but is not essential in c and in the lattice L. Also, d is a maximal essential extension of b. The element c is essential in L and a, b are closed in c, b is not closed in d.

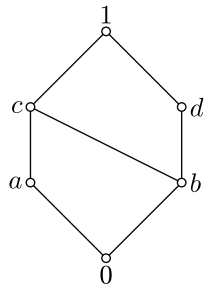

Figure 1.

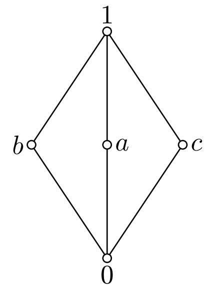

Figure 2.

The concept of a max-semicomplement of an element is different from that of a pseudocomplement of an element in a lattice. For example, in the lattice L shown in Figure 2, b, c are max-semicomplements of a but a does not have a pseudocomplement in L.

Definition 2.5. A nonzero element $x \in L$ is said to be uniform if any nonzero $y \leqslant x$ is essential in x.

A nonzero ideal I of L is called uniform, if I is a uniform element in $\mathrm{Id}(L)$.

Example 2.2. In the lattice shown in Figure 1, the element d is uniform but the element c is not uniform.

The proofs of the following results are similar to those in the module case.

Lemma 2.1. *In a lattice L the following statements hold:*

(1) *Let $a, b \in L$. Then $a \leqslant_e b$, if and only if for any $c \in L$, $a \wedge c = 0$ implies that $b \wedge c = 0$.*
(2) *Suppose that $a_i, b \in L$, $a_i \leqslant_e b$, $1 \leqslant i \leqslant n$. Then $\bigwedge_{i=1}^{n} a_i \leqslant_e b$.*
(3) *If $a, b, c \in L$ then $a \leqslant_e b$ implies $a \wedge c \leqslant_e b \wedge c$.*
(4) *If $a, b, c \in L$ then $a \leqslant_e b$, $b \leqslant_e c$ if and only if $a \leqslant_e c$.*

Definition 2.6. Let a, b, c be nonzero elements in a lattice L. The elements a, b are called direct summands of c, if $a \wedge b = 0$ and $a \vee b = c$. We then write $c = a \oplus b$ and say that c is the direct sum of a and b.

We note that for any finite number of nonzero elements $a_1, a_2, \ldots, a_n \in L$, $a_1 \vee \ldots \vee a_n$ is the direct sum if a_i's are join independent, i.e., $a_j \wedge \Big(\bigvee_{i=1, i \neq j}^{n} a_i\Big) = 0$ for each j.

The following remark follows by using modularity of a lattice.

Remark 2.1. Let L be a modular lattice and let $a, b, c \in L$ be such that $a \wedge b = 0$ and $(a \vee b) \wedge c = 0$. Then $a \wedge (b \vee c) = 0$.

Lemma 2.2. *Let L be a modular lattice. Let $0 \neq x \leqslant y \vee z$. If $x \wedge y = 0$, then $z \wedge (x \vee y) \neq 0$.*

P r o o f. Suppose $z \wedge (x \vee y) = 0$. We have

$$y = y \vee 0 = y \vee [z \wedge (x \vee y)] = (y \vee z) \wedge (x \vee y) = x \vee y \Longrightarrow x = 0.$$

□

The following lemma is from Grzeszczuk and Puczyłowski, [5], Lemma 3.

Lemma 2.3. *Let L be a modular lattice. Suppose $a \leqslant b$ and $c \leqslant d$ and $b \wedge d = 0$. Then $a \leqslant_e b$ and $c \leqslant_e d$ if and only if $a \vee c \leqslant_e b \vee d$.*

However, this result need not hold in the case of a nonmodular lattice.

Example 2.3. In the lattice shown in Figure 3, we have $a \leqslant_e c$, $b \leqslant_e d$. But $a \vee b$ is not essential in $c \vee d = 1$.

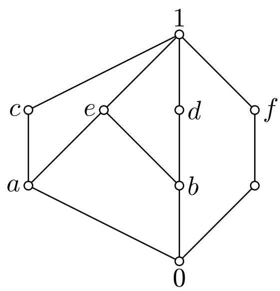

Figure 3.

The following lemma follows by using Lemma 2.3 and induction.

Lemma 2.4. *Let L be a modular lattice. Suppose $a_1, a_2, \ldots, a_t, b_1, b_2, \ldots, b_t \in L$ are such that $a_i \leqslant b_i$ for $1 \leqslant i \leqslant t$ and $b_1 \oplus b_2 \oplus \ldots \oplus b_t$ is the direct sum of $b_1, \ldots, b_t$. Then $a_i \leqslant_e b_i$ for $1 \leqslant i \leqslant t$ if and only if $a_1 \oplus a_2 \oplus \ldots \oplus a_t \leqslant_e b_1 \oplus b_2 \oplus \ldots \oplus b_t$.*

The proof of the next lemma follows by using Zorn's lemma.

Lemma 2.5. *Every ideal of a lattice L has a max-semicomplement in L.*

Remark 2.2. If a is essential in L then 0 is the only max-semicomplement of a in L.

Lemma 2.6. *Let L be a modular lattice and $a, b \in L$, $a \leqslant b$. If c is a max-semicomplement of a in L then $b \wedge c$ is a max-semicomplement of a in b.*

Proof. Since c is a max-semicomplement of a in L, $a \wedge (b \wedge c) = 0$. Let $b \wedge c \leqslant d \leqslant b$ be such that $a \wedge d = 0$. By modularity of L, we have

$$(d \vee c) \wedge a \leqslant (d \vee c) \wedge b = d \vee (c \wedge b) = d.$$

Now $a \wedge d = 0$ implies $(d \vee c) \wedge a = 0$. By the maximality of c, we get $d \vee c = c$. This implies $d \leqslant b \wedge c$. Thus, $b \wedge c$ is a max-semicomplement of a in b. □

Remark 2.3. Let $a, b \in L$ and $a \wedge b = 0$. Then b is a max-semicomplement of a in L if and only if for any $c \in L$ such that $b \leqslant c$, $a \wedge c \neq 0$.

Lemma 2.7. *Let L be a modular lattice and $a, b \in L$. If b is a max-semicomplement of a in L, then $a \vee b$ is essential in L.*

Proof. Let $c \in L$ be such that $c \wedge (a \vee b) = 0$. By Remark 2.1, $(b \vee c) \wedge a = 0$. Hence we have $b \vee c = b$ and so $c \leqslant b$. This implies $c = 0$. □

The proof of the next lemma follows from Lemma 2.1 and Zorn's lemma.

Lemma 2.8. *Every ideal of a lattice L has a maximal essential extension.*

It is clear that a closed ideal is a maximal essential extension of itself. Hence we may conclude from Lemma 2.8 that "Every ideal of a lattice L is contained in a closed ideal."

Lemma 2.9. *Every max-semicomplement in L is closed in L.*

Proof. Let a be a max-semicomplement in L. Then there exists a $b \in L$ such that $a \wedge b = 0$ and a is maximal with this property. If $a \leqslant_e c$, then by Lemma 2.1 (1), $b \wedge c = 0$. Hence by the maximality of a, we get $a = c$. Thus a is closed in L. □

We have the following characterization of closedness of an element.

Lemma 2.10. *Let L be a modular lattice and let $a, b \in L$ be such that b is a max-semicomplement of a in L. Then a is a max-semicomplement of b in L if and only if a is closed in L.*

Proof. Since a is a max-semicomplement of b in L by Lemma 2.9, a is closed in L.

Conversely suppose that a is closed in L, let $c \in L$ be such that $a \leqslant c$ and $b \wedge c = 0$. We claim that $a \leqslant_e c$. For if $d \in L$ is such that $d \leqslant c$ and $a \wedge d = 0$, then $(a \vee d) \wedge b = 0$ implies by Remark 2.1 that $a \wedge (b \vee d) = 0$. As b is a max-semicomplement of a in L, we conclude $b \vee d = b$. This implies $d \leqslant b$ and so $d = 0$. Thus, $a \leqslant_e c$, but a is closed in L so we must have $a = c$. Hence a is a max-semicomplement of b in L. □

The proof of the next lemma follows from Lemma 2.10 and Lemma 2.5.

Lemma 2.11. *Let L be a modular lattice and $A \in \operatorname{Id}(L)$. Then A is closed in L if and only if A is a max-semicomplement in L.*

Now we give a characterization for an element to be a max-semicomplement of another element.

Theorem 2.1. *Let L be a modular lattice and let $a, b \in L$ be such that $a \wedge b = 0$. Then a is a max-semicomplement of b in L if and only if a is closed in L and $a \vee b$ is essential in L.*

Proof. Let $a, b \in L$ be such that $a \wedge b = 0$. If a is a max-semicomplement of b in L, then by Lemma 2.9, a is closed in L and by Lemma 2.7, $a \vee b$ is essential in L.

Conversely, let a be closed in L and let $a \vee b$ be essential in L. Let $c \in L$ be such that $a \leqslant c$ and $b \wedge c = 0$. We claim that $a \leqslant_e c$. Let $d \in L$ be such that $d \leqslant c$ and $a \wedge d = 0$. Clearly $(a \vee d) \wedge b = 0$. Then by Remark 2.1, $d \wedge (a \vee b) = 0$. This implies $d = 0$. Thus $a \leqslant_e c$; and as a is closed in L we have $a = c$. Thus, a is a max-semicomplement of b in L. □

3. Extending ideals and direct summands

In this section we define an extending ideal in a lattice L. We show that direct summands of an extending ideal are extending.

Lemma 3.1. *Let L be a modular lattice and let $I, J, K \in \operatorname{Id}(L)$ be such that $K = I \oplus J$. Then I is a max-semicomplement of J in K and hence is closed in K.*

Proof. Let $I, J, K \in \operatorname{Id}(L)$ be such that $K = I \oplus J$. Let $P \in \operatorname{Id}(L)$ be such that $I \subseteq P \subseteq K$ and $P \cap J = (0]$. Now, by modularity of $\operatorname{Id}(L)$, we get

$$P = K \cap P = (I \vee J) \cap P = I \vee (J \cap P) = I.$$

Hence I is a max-semicomplement of J in K. □

Remark 3.1. We note that if $I, J, K \in \operatorname{Id}(L)$ are such that $I \subseteq J \subseteq K$ and I is a direct summand of K then I is also a direct summand of J.

The following proposition is a lattice theoretic analogue of a result from Lam [11], Lemma 6.41, page 222.

Proposition 3.1. *The following statements are equivalent for any $I \in \mathrm{Id}(L)$.*

(1) *Every closed ideal contained in I is a direct summand of I.*

(2) *For every ideal $K \subseteq I$, there exists a direct summand J of I such that $K \leqslant_e J$.*

Proof. (1) $\Rightarrow$ (2): Let $K \in \mathrm{Id}(L)$ be such that $K \subseteq I$. By Lemma 2.8, K has a maximal essential extension J such that $K \leqslant_e J$. But being a maximal essential extension of K, J is closed in I. Hence by (1), J is a direct summand of I. Thus (2) holds.

(2) $\Rightarrow$ (1): Let J be a closed ideal in I. Then the only essential extension of J is J itself. By (2), J is a direct summand. □

Using the equivalent conditions in Proposition 3.1, we define an extending ideal in a lattice as follows:

Definition 3.1. An ideal I of a lattice L is called extending if every ideal J contained in I is essential in a direct summand of I.

In Section 2, it is already proved that in a lattice L, every maximal essential extension is closed. Also, we have shown that if L is modular, every closed ideal is a max-semicomplement in L. Hence we have the following remark.

Remark 3.2. An ideal I of a modular lattice L is extending if every max-semi complement (or closed) ideal in I is a direct summand of I.

Example 3.1. In the lattice shown in Figure 4, consider the ideal $I = \{0, a, b, c, d, e, f, g\}$. The ideals contained in I are $I_1 = \{0, a\}$, $I_2 = \{0, b\}$, $I_3 = \{0, c\}$, $I_4 = \{0, a, b, d\}$, $I_5 = \{0, a, c, e\}$ and $I_6 = \{0, b, c, f\}$, each of which is a direct summand of I. Hence I is an extending ideal.

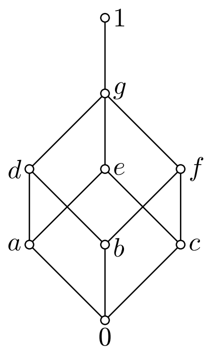

Figure 4.

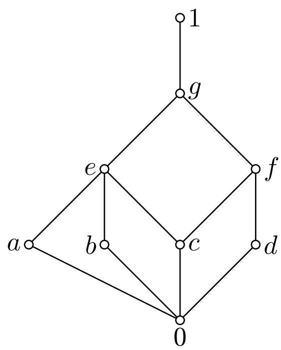

Figure 5.

Consider the ideal $I = \{0, a, b, c, d, e, f, g\}$ in the lattice shown in Figure 5. We note that the ideal $J = \{0, c\}$ is contained in I, but J is neither a direct summand of I nor is it essential in a direct summand of I. Hence the ideal I is not extending.

In the theory of modules, (e.g. Lam [11], Proposition 6.24, page 215), it is known that if A, B, C are modules of a ring R with $A \subseteq B \subseteq C$ and if A is closed in B, B is closed in C, then A is closed in C. However, in the case of a lattice, this relationship need not always hold. Hence we introduce the following concept.

Definition 3.2. We say that a lattice L satisfies the condition (A), if the following condition is satisfied in $\mathrm{Id}(L)$:

(A) Let $I, J, K \in \mathrm{Id}(L)$ be such that $I \subseteq J \subseteq K$. If I is closed in J and J is closed in K then I is closed in K.

Remark 3.3. We note that if L is a distributive lattice, then the condition (A) holds in L. Suppose that $I, J, K \in \mathrm{Id}(L)$ are such that $I \subseteq J \subseteq K$. If I is closed in J and J is closed in K then there exist nonzero ideals $I_1 \leqslant J$ and $J_1 \leqslant K$ such that $I \cap I_1 = (0]$ and $J \cap J_1 = (0]$. By the distributivity of $\mathrm{Id}(L)$, $I \cap (I_1 \vee J_1) = (0]$. Thus I is closed in K.

We give an example of a nonmodular lattice satisfying the condition (A) (Figure 6) and of a nonmodular lattice not satisfying the condition (A) (Figure 7).

However, we are unable to show that the condition (A) holds in a modular lattice.

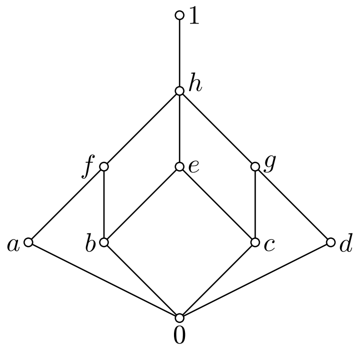

Figure 6.

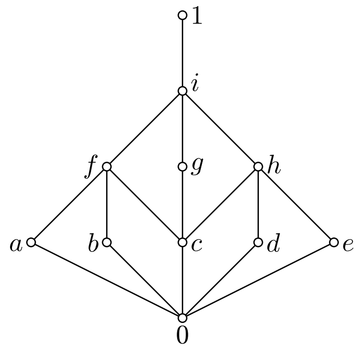

Figure 7.

Example 3.2. In the lattice L shown in Figure 6, consider the ideals $I = \{0, a\}$, $J = \{0, a, b, f\}$ and $K = \{0, a, b, c, d, e, f, g, h\}$. It is clear that I is closed in J and J is closed in K. Also I is closed in K. Similarly, we can check for all other closed ideals. Hence L satisfies the condition (A).

In the lattice L shown in Figure 7, consider the ideals $I = \{0, c\}$, $J = \{0, a, b, c, f\}$ and $K = \{0, a, b, c, d, e, f, g, h, i\}$. It is clear that I is closed in J and J is closed in K. But I is not closed in K, as there exists an ideal $Q = \{0, c, g\}$ of L such that $I \leqslant_e Q$. Hence L does not satisfy the condition (A).

Lemma 3.2. *Let L be a modular lattice satisfying the condition* (A). *Let $I \in \mathrm{Id}(L)$ be an extending ideal of L. Then every direct summand of I is extending.*

P r o o f. Let J be a direct summand of an extending ideal I of L. Let J_1 be a closed ideal of J. To show that J_1 is a direct summand of J, we note that J_1 is closed in J and J is closed in I. By condition (A), J_1 is closed in I and as I is extending, J_1 is a direct summand of I. Therefore, $I = J_1 \oplus J_2$ for some $J_2 \in \mathrm{Id}(L)$. By modularity of $\mathrm{Id}(L)$, as $J_1 \subseteq J$, we have

$$(J \cap J_2) \vee J_1 = J \cap (J_2 \vee J_1) = J \cap I = J.$$

Also, $(J \cap J_2) \cap J_1 = J \cap (J_2 \cap J_1) = J \cap (0] = (0]$. Hence J_1 is a direct summand of J. Consequently, J is extending. $\square$

Definition 3.3. Let $I, I_i \in \mathrm{Id}(L)$ for every $i \in \Lambda$. The decomposition $I = \bigoplus_{i\in\Lambda} I_i$ of I is called exchangeable if for every direct summand J of I we have $I = \Big(\bigoplus_{i\in\Lambda} I_i'\Big) \oplus J$ with $I_i' \subseteq I_i$ for all $i \in \Lambda$.

Example 3.3. In the lattice shown in Figure 8, consider the ideals $I = \{0, a, \ldots, j, k\}$, $I_1 = \{0, a, b, f\}$ and $I_2 = \{0, d, e, j\}$. Then $I = I_1 \oplus I_2$. For the direct summand $J = \{0, c\}$, there exist ideals $I_1' = \{0, a\} \subseteq I_1$ and $I_2' = \{0, d\} \subseteq I_2$ such that $I_1 \oplus I_2 = I_1' \oplus I_2' \oplus J$. Similarly, we can check for other direct summands. Hence the decomposition of the ideal I is exchangeable.

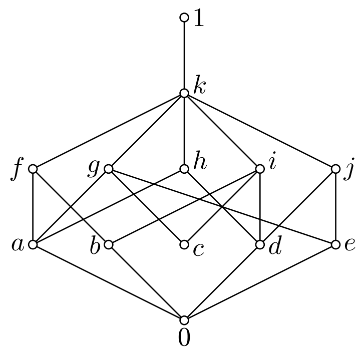

Figure 8.

Theorem 3.1. *Let L be a modular lattice and let $I, J, K \in \mathrm{Id}(L)$ be such that $K = I \oplus J'$ where $J' \subseteq J \subseteq K$. If J has an exchangeable decomposition $J = \bigoplus_{i\in\Lambda} J_i$ then $K = I \oplus \Big(\bigoplus_{i\in\Lambda} J_i'\Big)$ with $J_i' \subseteq J_i$ for all $i \in \Lambda$.*

P r o o f. By modularity of $\mathrm{Id}(L)$, we get

$$J = K \cap J = (I \vee J') \cap J = J' \vee (I \cap J).$$

We note that $(I \cap J) \cap J' = I \cap J' = (0]$. Hence $J = (I \cap J) \oplus J'$ is a direct sum.

As J has an exchangeable decomposition, for the direct summand $I \cap J$ we have $J = (I \cap J) \oplus \Big(\bigoplus\limits_{i\in\Lambda} J_i'\Big)$ with $J_i' \subseteq J_i$ for all $i \in \Lambda$. Putting $\bigoplus\limits_{i\in\Lambda} J_i' = P$, we have $J = (I \cap J) \oplus P$. Since $P \subseteq J$, this implies $I \cap P = (0]$. Also, $J' \subseteq J \subseteq K$ implies that $K = I \oplus J' = I \vee J$. Therefore,

$$K = I \vee [(I \cap J) \vee P] = I \vee P.$$

Hence $K = I \oplus P = I \oplus \Big(\bigoplus\limits_{i\in\Lambda} J_i'\Big)$ with $J_i' \subseteq J_i$ for all $i \in \Lambda$. □

Definition 3.4. An ideal I of a lattice L is said to have the n-internal exchange property if any decomposition $I = \bigoplus\limits_{i=1}^{n} I_i$ of I is exchangeable. If n is finite then I is said to have the finite internal exchange property.

Proposition 3.2. *In a modular lattice L, the 2-internal exchange property is inherited by summands.*

P r o o f. Let $I, J, K \in \mathrm{Id}(L)$ be such that $K = I \oplus J$ and let K have the 2-internal exchange property. Let $I = I_1 \oplus I_2$ and let X be a summand of I. We write $K = I_1 \oplus (I_2 \oplus J)$. It is clear that $X \oplus J$ is a summand of K. Then by the 2-internal exchange property we have

$$K = (X \oplus J) \oplus I_1' \oplus I_3 = X \oplus I_1' \oplus [I_3 \oplus J]$$

with $I_1' \subseteq I_1$, $I_3 \subseteq I_2 \oplus J$.

By using modularity of $\mathrm{Id}(L)$ and $X \vee I_1' \subseteq I$, we have

$$\begin{aligned} I = K \cap I &= [X \vee I_1' \vee [I_3 \vee J]] \cap I \\ &= (X \vee I_1') \vee ([I_3 \vee J] \cap I) \\ &= X \oplus I_1' \oplus [[I_3 \oplus J] \cap I] \end{aligned}$$

and using $I_1' \subseteq I_1$ we have

$$[I_3 \vee J] \cap I \subseteq (I_2 \vee J) \cap I = I_2 \vee (J \cap I) = I_2.$$

Hence the direct summand I of K has the 2-internal exchange property. □

4. Ojective ideals in modular lattices

In this section we obtain a lattice theoretic analogue of some results from Mohamed and Müller, [12]. They studied the concept of ojectivity for direct summands of modules. We formulate and answer the problem "What is a necessary and sufficient condition for the direct sum of extending modules to be extending?" in the context of ideals of a modular lattice.

Problem: What is a necessary and sufficient condition for the direct sum of extending ideals in a modular lattice to be extending?

We also give a characterization for direct summands of an ideal I of a modular lattice L to be extending and mutually ojective. As stated in the introduction, we use the characterization in Theorem 1.1 as the definition for ojectivity of ideals in a modular lattice.

Definition 4.1. Let $I, J, K \in \mathrm{Id}(L)$ be such that $K = I \oplus J$. The ideal J is said to be I-ojective if for any max-semicomplement C of J in K, K can be decomposed as $K = I' \oplus J' \oplus C$ with $I' \subseteq I$ and $J' \subseteq J$.

Example 4.1. In the lattice shown in Figure 8, consider the ideals $K = \{0, a, \ldots, j, k\}$, $I = \{0, a, b, f\}$ and $J = \{0, d, e, j\}$. Then $K = I \oplus J$. Also, J has a max-semicomplement $C = \{0, c\}$ and there exist ideals $I_1 = \{0, a\} \subseteq I$ and $J_1 = \{0, d\} \subseteq J$ such that $K = I \oplus J = I_1 \oplus J_1 \oplus C$. Hence J is I-ojective.

Also, if I is J-ojective and J is I-ojective for some $K = I \oplus J$ then I and J are called *mutually ojective*.

Proposition 4.1. *Let L be a modular lattice satisfying the condition* (A)*. Let $I, J, K \in \mathrm{Id}(L)$ be such that $K = I \oplus J$. Let I_1 and J_1 be direct summands of I and J, respectively. If J is I-ojective then*
(1) *J_1 is I-ojective;*
(2) *J is I_1-ojective;*
(3) *J_1 is I_1-ojective.*

P r o o f. Let L be a modular lattice satisfying the condition (A). Let $I, J, K \in \mathrm{Id}(L)$ be such that $K = I \oplus J$. Let I_1 and J_1 be direct summands of I and J, respectively. Put $I = I_1 \oplus I_2$ and $J = J_1 \oplus J_2$ for some $I_2, J_2 \in \mathrm{Id}(L)$.

(1) Put $N = I \oplus J_1$. Let X be a max-semicomplement of J_1 in N. By Theorem 2.1, X is closed in N and $X \oplus J_1 \leqslant_e N$. Then $X \oplus J_1 \oplus J_2 \leqslant_e N \oplus J_2$, that is $X \oplus J \leqslant_e K$.

Since X is closed in N and N being a direct summand of K is closed in K, we have by the condition (A) that X is closed in K. Again by Theorem 2.1, $X \oplus J \leqslant_e K$, X is closed in K implies that X is a max-semicomplement of J in K. Then using that J is I-ojective, we have the decomposition $K = X \oplus I' \oplus J'$ with $I' \subseteq I$ and $J' \subseteq J$.

Now, by $N \subseteq K$ and using the modularity of Id(L) for $X \vee I' \subseteq N$, we get

$$N = K \cap N = (X \vee I' \vee J') \cap N = X \vee I' \vee (N \cap J')$$

and

$$N \cap J' = (I \vee J_1) \cap J' \subseteq (I \vee J_1) \cap J = J_1 \vee (I \cap J) = J_1.$$

Thus we obtain a decomposition of N, as $N = X \oplus I' \oplus (N \cap J')$ with $I' \subseteq I$ and $N \cap J' \subseteq J_1$ for a max-semicomplement X of J_1 in N. Hence J_1 is I-ojective.

(2) Put $M = I_1 \oplus J$. Let Y be a max-semicomplement of J in M. It is clear that $Y \cap I_2 = (0]$. We claim that $Y \oplus I_2$ is a max-semicomplement of J in K.

For if there exists an ideal $P \subseteq K$ such that $Y \oplus I_2 \subseteq P$ and $P \cap J = (0]$, then

$$P = K \cap P = (I_2 \vee M) \cap P = I_2 \vee (M \cap P).$$

Also $I_2 \cap (P \cap M) = P \cap (I_2 \cap M) = (0]$. Hence $P = I_2 \oplus (P \cap M)$ is a direct sum. Now $Y \subseteq P \cap M$ and $(P \cap M) \cap J = M \cap (P \cap J) = (0]$ and Y is a max-semicomplement of J in M imply $Y = P \cap M$. Thus $P = I_2 \oplus Y$. Hence $I_2 \oplus Y$ is a max-semicomplement of J in K. Since J is I-ojective, we have $K = (I_2 \oplus Y) \oplus I'' \oplus J''$ with $I'' \subseteq I$ and $J'' \subseteq J$. Therefore by using the modularity of Id(L) for $Y \subseteq M$ and $J'' \subseteq M$, we have

$$\begin{aligned} M = K \cap M &= [(I_2 \vee Y) \vee I'' \vee J''] \cap M \\ &= [Y \vee I''' \vee J''] \cap M \quad (\text{where } I''' = I_2 \vee I'' = I_2 \oplus I'') \\ &= Y \vee [(I''' \vee J'') \cap M] = Y \vee [(J'' \vee I''') \cap M] \\ &= Y \vee J'' \vee (I''' \cap M) \end{aligned}$$

and

$$I''' \cap M = (I_2 \vee I'') \cap M \subseteq I \cap M = I \cap (J \vee I_1) = (I \cap J) \vee I_1 = I_1.$$

Hence $M = Y \oplus (I''' \cap M) \oplus J''$ with $I''' \cap M \subseteq I_1$ and $J'' \subseteq J$, for a max-semicomplement Y of J in M. Thus, J is I_1-ojective.

(3) Put $W = I_1 \oplus J_1$. Let T be a max-semicomplement of J_1 in W. Then Theorem 2.1 implies that T is closed in W and $T \oplus J_1 \leqslant_e W$. Also, $T \cap I_2 = (0]$. We claim that $T \oplus I_2$ is a max-semicomplement of J_1 in $W \oplus I_2 = N$.

For if there exists an ideal $Q \subseteq N$ such that $I_2 \oplus T \subseteq Q$ and $Q \cap J_1 = (0]$, then by the modularity of Id(L), as $I_2 \subseteq Q$, we get

$$\begin{aligned} Q = N \cap Q = (I \vee J_1) \cap Q &= (I_1 \vee I_2 \vee J_1) \cap Q \\ &= (I_2 \vee W) \cap Q = I_2 \vee (W \cap Q) \end{aligned}$$

and $I_2 \cap (Q \cap W) = (0]$. Hence $Q = I_2 \oplus (Q \cap W)$ is a direct sum.

Now, $T \subseteq W, T \subseteq Q$ implies that $T \subseteq Q \cap W \subseteq W$ with $(Q \cap W) \cap J_1 = (0]$. As T is a max-semicomplement of J_1 in W, we must have $T = Q \cap W$ and so $Q = I_2 \oplus T$ is a max-semicomplement of J_1 in N.

Since J_1 is I-ojective we have $N = I \oplus J_1 = (I_2 \oplus T) \oplus I' \oplus J_1'$ with $I' \subseteq I$ and $J_1' \subseteq J_1$.

Now by using the modularity of $\mathrm{Id}(L)$, as $T \subseteq W$ and $J_1' \subseteq W$ we get,

$$\begin{aligned} W = N \cap W &= [(I_2 \vee T) \vee I' \vee J_1'] \cap W \\ &= T \vee [(I_2 \vee I' \vee J_1') \cap W] \\ &= T \vee J_1' \vee [(I_2 \vee I') \cap W] \end{aligned}$$

with $J_1' \subseteq J_1$.

Put $I''' = (I_2 \oplus I') \cap W$. Then

$$I''' = (I_2 \vee I') \cap W = I'' \cap W = (I_1 \vee J_1) \cap I'' \subseteq (I_1 \vee J_1) \cap I = I_1 \vee (J_1 \cap I) = I_1.$$

Hence we obtain a decomposition $W = T \oplus I''' \oplus J_1'$ of W for a max-semicomplement T of J_1 in W. It is clear that $I''' \subseteq I_1$ and $J_1' \subseteq J_1$. Hence J_1 is I_1-ojective. $\square$

Lemma 4.1. *Let L be a modular lattice satisfying the condition* (A). *Let $I, J, K \in \mathrm{Id}(L)$ be such that $K = I \oplus J$. Let I be extending and J I-ojective. If X is a closed ideal in K with $X \cap J = (0]$, then K decomposes as $K = X \oplus I' \oplus J'$ with $I' \subseteq I$ and $J' \subseteq J$.*

Proof. Let $T = (X \oplus J) \cap I$.

By using the modularity of $\mathrm{Id}(L)$ for $J \subseteq X \vee J$, we get

$$\begin{aligned} T \vee J &= [(X \vee J) \cap I] \vee J = (X \vee J) \cap (I \vee J) \\ &= (X \vee J) \cap K = X \vee J. \end{aligned}$$

Also, $T \cap J = [(X \vee J) \cap I] \cap J = (0]$. Thus $T \oplus J = X \oplus J$ is a direct sum.

Let I_1 be a maximal essential extension of T in I. It is clear that I_1 is closed in I and $T \leqslant_e I_1$. Since I is extending I_1 is a direct summand of I. Put $I = I_1 \oplus I_2$ and define $M = I_1 \oplus J$. Then $T \leqslant_e I_1$ implies that $T \oplus J \leqslant_e I_1 \oplus J = M$, that is $X \oplus J \leqslant_e M$. Since X is closed in K, X is also closed in M. Then Theorem 2.1 implies that X is a max-semicomplement of J in M. By (2) of Proposition 4.1, J is I_1-ojective, so $M = X \oplus I_1' \oplus J'$ with $I_1' \subseteq I_1$ and $J' \subseteq J$. Then $K = M \oplus I_2 = X \oplus I_1' \oplus J' \oplus I_2$. Hence we get $K = X \oplus I' \oplus J'$ with $I_1' \oplus I_2 \subseteq I$ and $J' \subseteq J$. $\square$

In the next theorem, we give a characterization for direct summands of an ideal I of a modular lattice L to be extending and mutually ojective.

Theorem 4.1. *Let L be a modular lattice satisfying the condition* (A). *Let $I = I_1 \oplus I_2$. Then I_j is extending and is I_i-ojective for $i \neq j$ if and only if for any closed ideal X of I, I decomposes as $I = X \oplus I_1' \oplus I_2'$ with $I_i' \subseteq I_i$, $i = 1, 2$.*

Proof. Suppose that for any closed ideal X of I, I decomposes as $I = X \oplus I_1' \oplus I_2'$ with $I_1' \subseteq I_1, I_2' \subseteq I_2$. We write $I_1' \oplus I_2' = I'$. Then $I = X \oplus I'$, i.e., every closed ideal of I is a direct summand of I. Hence I is extending and by Lemma 3.2, I_1 and I_2 are extending.

Let C be a max-semicomplement of I_1 in I. By our assumption $I = C \oplus I_1' \oplus I_2'$ with $I_i' \subseteq I_i$, $i = 1, 2$. Hence I_1 is I_2-ojective.

Let D be a max-semicomplement of I_2 in I. By our assumption $I = D \oplus I_1'' \oplus I_2''$ with $I_i'' \subseteq I_i$, $i = 1, 2$. Hence I_2 is I_1-ojective.

Conversely, suppose that I_j is extending and is I_i-ojective for $i \neq j$. Let X be a closed ideal of I and let X_1 be a maximal essential extension of $X \cap I_1$ in X. Then X_1 is closed in X and as X is closed in I, by condition (A), X_1 is closed in I. Also $X \cap I_1 \leqslant_e X_1, (X \cap I_1) \cap I_2 = (0]$ implies that $X_1 \cap I_2 = (0]$. Then Lemma 4.1 implies that $I = X_1 \oplus J_1 \oplus J_2$ with $J_i \subseteq I_i, i = 1, 2$. Here we note that J_i are direct summands of I_i and hence are also extending. Put $J = J_1 \oplus J_2$, then $I = X_1 \oplus J$.

Now by using the modularity of $\mathrm{Id}(L)$ for $X_1 \subseteq X$, we get

$$X = I \cap X = (X_1 \vee J) \cap X = X_1 \vee (J \cap X)$$

where $X_1 \cap (J \cap X) = (0]$. Hence $X = X_1 \oplus (J \cap X)$ is a direct sum. Also, being a direct summand, $J \cap X$ is closed in X and as X is closed in I, by condition (A), $J \cap X$ is closed in I. Hence it follows that $J \cap X$ is closed in $J \subseteq I$. By Proposition 4.1, J_1 is J_2-ojective, then Lemma 4.1 implies that $J = (J \cap X) \oplus I_1' \oplus I_2'$ with $I_i' \subseteq J_i \subseteq I_i$. Hence

$$I = X_1 \oplus J = X_1 \oplus (J \cap X) \oplus I_1' \oplus I_2' = X \oplus I_1' \oplus I_2'$$

with $I_i' \subseteq I_i$. □

Now by using Theorem 4.1, we show that the mutual ojectivity is a sufficient condition for a direct sum of extending ideals to be extending.

Theorem 4.2. *Let L be a modular lattice satisfying the condition* (A). *Let $I = I_1 \oplus I_2$. Then I is extending and the decomposition is exchangeable if and only if I_j is extending and is I_i-ojective for $i \neq j$ and $i = 1, 2$.*

Proof. Let I be extending and let the decomposition $I = I_1 \oplus I_2$ be exchangeable. By Lemma 3.2, I_j is extending for $j = 1, 2$. Let X be a max-semicomplement of I_2 in I. Then X is closed in I. Since I is extending, X is a direct summand of I. By the definition of exchangeable decomposition we have $I = X \oplus I_1' \oplus I_2'$ with $I_i' \subseteq I_i$, $i = 1, 2$. Hence I_1 is I_2-ojective. Similarly, I_2 is I_1-ojective.

Conversely, suppose that I_j is extending and is I_i-ojective for $i \neq j$. By Theorem 4.1 above, for any max-semicomplement (or closed ideal) in I we have $I = X \oplus I_1' \oplus I_2'$ with $I_i' \subseteq I_i$, $i = 1, 2$. Thus the decomposition is exchangeable. Moreover, every closed ideal is a direct summand of I, hence I is extending. □

Proposition 4.2. *Let L be a modular lattice satisfying the condition* (A). *An extending ideal I has the 2-internal exchange property if and only if for every decomposition $I = I_1 \oplus I_2$, I_1, I_2 are mutually ojective.*

P r o o f. Since I is extending, the direct summands I_1 and I_2 are also extending. Then result follows by Theorem 4.2. □

The following result is an extension of Theorem 4.2, to a finite direct sum.

Theorem 4.3. *Let L be a modular lattice satisfying the condition* (A). *Let $I = I_1 \oplus I_2 \oplus \ldots \oplus I_n, n \geqslant 2$. Then the following statements are equivalent:*

(1) *I is extending and the decomposition is exchangeable.*

(2) *I_i are extending and $\bigoplus\limits_{j \in P} I_j$ is $\bigoplus\limits_{k \in Q} I_k$-ojective for any disjoint nonempty subsets P and Q of $\{1, 2, \ldots, n\}$.*

(3) *I_i are extending, and $I_1 \oplus I_2 \oplus \ldots \oplus I_{i-1}$ and I_i are mutually ojective for $2 \leqslant i \leqslant n$.*

P r o o f. (1) $\Rightarrow$ (2): Let I be extending and let the decomposition $I = I_1 \oplus I_2 \oplus \ldots \oplus I_n$ be exchangeable. By Lemma 3.2, I_i, $1 \leqslant i \leqslant n$ is extending. Let $N = \{1, 2, \ldots, n\}$ and let P be a nonempty subset of N. Then $N - P$ is a complement subset of P in N such that $I = \Big[\bigoplus\limits_{j \in P} I_j\Big] \oplus \Big[\bigoplus\limits_{l \in (N-P)} I_l\Big]$. As I is extending, by Theorem 4.2, $\bigoplus\limits_{j \in P} I_j$ is $\bigoplus\limits_{l \in (N-P)} I_l$-ojective. Let $Q \subseteq N - P$. Then $P \cap Q = \emptyset$ and $\bigoplus\limits_{k \in Q} I_k \subseteq \bigoplus\limits_{l \in (N-P)} I_l$. Moreover, $\bigoplus\limits_{k \in Q} I_k$ is a direct summand of $\bigoplus\limits_{l \in (N-P)} I_l$, so by Proposition 4.1, $\bigoplus\limits_{j \in P} I_j$ is $\bigoplus\limits_{k \in Q} I_k$-ojective. Hence (2) holds.

(2) $\Rightarrow$ (3): Take $P = \{1, 2, \ldots, i-1\}$ and $Q = \{i\}$. The result follows from (2).

(3) $\Rightarrow$ (1): Let each I_i be extending, and let $I_1 \oplus I_2 \oplus \ldots \oplus I_{i-1}$ and I_i be mutually ojective for $2 \leqslant i \leqslant n$. To show that I is extending and the decomposition is exchangeable, we use mathematical induction on i. Suppose that $I_1 \oplus I_2 \oplus \ldots \oplus I_{i-1}$ is extending and is an exchangeable decomposition. Let X be a closed ideal of $I_1 \oplus I_2 \oplus \ldots \oplus I_i$. By Theorem 4.1, we have $I_1 \oplus I_2 \oplus \ldots \oplus I_i = X \oplus J \oplus I_i'$ with $J \subseteq I_1 \oplus I_2 \oplus \ldots \oplus I_{i-1}, I_i' \subseteq I_i$.

Since the decomposition $I_1 \oplus I_2 \oplus \ldots \oplus I_{i-1}$ is exchangeable, by Lemma 3.1,

$$I_1 \oplus I_2 \oplus \ldots \oplus I_i = X \oplus I'_1 \oplus I'_2 \oplus \ldots \oplus I'_{i-1} \oplus I'_i$$

with $I'_j \subseteq I_j$, $1 \leqslant j \leqslant i$. Hence by induction, (1) holds for $2 \leqslant i \leqslant n$. □

In the next result we show that under some conditions the direct sum of two mutually ojective direct summands is ojective.

Theorem 4.4. *Let L be a modular lattice satisfying the condition* (A). *Let $I = I_1 \oplus I_2 \oplus I_3$ be such that I_i is extending and is I_j-ojective for $i \neq j$. If I_3 is uniform then I_3 is $(I_1 \oplus I_2)$-ojective.*

P r o o f. Let $I = I_1 \oplus I_2 \oplus I_3$ be such that I_i extending and I_j-ojective for $i \neq j$. Let I_3 be uniform and let C be a max-semicomplement of I_3 in I.

Since I_3 is closed in I, it is a max-semicomplement of C in I. By Lemma 2.6, $C' = C \cap (I_2 \oplus I_3)$ is a max-semicomplement of I_3 in $I_2 \oplus I_3$. As I_3 is I_2-ojective we have $I_2 \oplus I_3 = C' \oplus I'_2 \oplus I'_3$ with $I'_2 \subseteq I_2, I'_3 \subseteq I_3$. Then

$$I = I_1 \vee I_2 \vee I_3 = C' \vee I_1 \vee I'_2 \vee I'_3$$

and hence by using modularity of $\mathrm{Id}(L)$ for $C' \subseteq C$,

$$\begin{aligned} C = I \cap C &= (C' \vee I_1 \vee I'_2 \vee I'_3) \cap C \\ &= C' \vee [(I_1 \vee I'_2 \vee I'_3) \cap C \\ &= C' \vee C'' = C' \oplus C'' \end{aligned}$$

where $C'' = (I_1 \vee I'_2 \vee I'_3) \cap C = (I_1 \oplus I'_2 \oplus I'_3) \cap C$. Being a direct summand C'' is closed in C; as C is closed in I, so by condition (A), C'' is closed in I. As I_3 is uniform and I_3 is closed in I, $I'_3 \subseteq I_3$ implies $I'_3 = (0]$ or $I'_3 = I_3$.

If $I'_3 = (0]$ then $I = C' \oplus I_1 \oplus I'_2$ and $C'' \subseteq I_1 \oplus I'_2$.

If $I'_3 = I_3$ then

$$\begin{aligned} C \cap (I'_2 \vee I_3) &= C \cap (I'_2 \vee I'_3) \\ &= C \cap (I_2 \vee I_3) \cap (I'_2 \vee I'_3) \\ &= C' \cap (I'_2 \vee I'_3) = (0] \end{aligned}$$

but I_3 is a max-semicomplement of C so we must have $I'_2 \oplus I_3 = I_3$, that is $I'_2 = (0]$. Hence $I = C' \oplus I_1 \oplus I'_3$ and $C'' \subseteq I_1 \oplus I'_3$. Thus both the cases can be concluded as $I = C' \oplus I_1 \oplus I'_k$ and $C'' \subseteq I_1 \oplus I'_k, k = 2, 3$.

Since I_1 and I_k are mutually ojective, by Proposition 4.1 I_1 and I'_k are mutually ojective. By Theorem 4.1, $I_1 \oplus I'_k = C'' \oplus I''_1 \oplus I''_k$ with $I''_1 \subseteq I_1$, $I''_k \subseteq I'_k \subseteq I_k$. Hence

$$I = C' \oplus C'' \oplus I''_1 \oplus I''_k = C \oplus I''_1 \oplus I''_k$$

with $I''_1 \subseteq I_1$, $I''_k \subseteq I_k$. □

Acknowledgement. The authors are thankful to the referee for his/her fruitful suggestions for the improvement of the paper.

Note added in proofs: In June 2014, Professor F. Wehrung has informed the authors that the condition (A) holds in any modular lattice. Thus the results in this paper are valid for ideals in a modular lattice.

References

[1] *E. Akalan, G. F. Birkenmeier, A. Tercan*: Goldie extending modules. Commun. Algebra *37* (2009), 663–683; Corrigendum *38* (2010), 4747–4748; Corrigendum *41* (2013), 2005. zbl MR

[2] *G. F. Birkenmeier, B. J. Müller, S. T. Rizvi*: Modules in which every fully invariant submodule is essential in a direct summand. Commun. Algebra *30* (2002), 1395–1415. zbl MR

[3] *G. Grätzer*: General Lattice Theory. Birkhäuser, Basel, 1998. zbl MR

[4] *P. Grzeszczuk, E. R. Puczyłowski*: On finiteness conditions of modular lattices. Commun. Algebra *26* (1998), 2949–2957. zbl MR

[5] *P. Grzeszczuk, E. R. Puczyłowski*: On Goldie and dual Goldie dimensions. J. Pure Appl. Algebra *31* (1984), 47–54. zbl MR

[6] *K. Hanada, Y. Kuratomi, K. Oshiro*: On direct sums of extending modules and internal exchange property. J. Algebra *250* (2002), 115–133. zbl MR

[7] *A. Harmanci, P. F. Smith*: Finite direct sums of CS-modules. Houston J. Math. *19* (1993), 523–532. zbl MR

[8] *M. A. Kamal, B. J. Müller*: Extending modules over commutative domains. Osaka J. Math. *25* (1988), 531–538. zbl MR

[9] *M. A. Kamal, B. J. Müller*: The structure of extending modules over Noetherian rings. Osaka J. Math. *25* (1988), 539–551. zbl MR

[10] *M. A. Kamal, B. J. Müller*: Torsion free extending modules. Osaka J. Math. *25* (1988), 825–832. zbl MR

[11] *T. Y. Lam*: Lectures on Modules and Rings. Graduate Texts in Mathematics 189, Springer, New York, 1999. zbl MR

[12] *S. H. Mohamed, B. J. Müller*: Ojective modules. Commun. Algebra *30* (2002), 1817–1827. zbl MR

Authors' address: Shriram K. Nimbhorkar, Rupal C. Shroff, Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad 431 004, India, e-mail: `sknimbhorkar@gmail.com`, `rupal_shroff84@yahoo.co.in`.