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A NOTE ON THE CUBICAL DIMENSION OF NEW CLASSES OF BINARY TREES

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Abstract. The cubical dimension of a graph G is the smallest dimension of a hypercube into which G is embeddable as a subgraph. The conjecture of Havel (1984) claims that the cubical dimension of every balanced binary tree with 2^n vertices, $n \geqslant 1$, is n. The 2-rooted complete binary tree of depth n is obtained from two copies of the complete binary tree of depth n by adding an edge linking their respective roots. In this paper, we determine the cubical dimension of trees obtained by subdividing twice a 2-rooted complete binary tree and prove that every such balanced tree satisfies the conjecture of Havel.

Keywords: cubical dimension; embedding; Havel's conjecture; hypercube; tree

MSC 2010: 05C05, 05C60

1. Introduction

For a given graph G, $V(G)$ and $E(G)$ denote, respectively, the set of vertices and the set of edges of G. The *hypercube of dimension* n, denoted Q_n, is the graph whose 2^n vertices are boolean vectors of length n, such that two vertices are adjacent if and only if they differ in exactly one coordinate.

An *embedding* of the graph G in the hypercube Q_n is a one-to-one mapping of $V(G)$ into $V(Q_n)$ that preserves adjacency of vertices. In the case when $V(G) = V(Q_n)$, we say that the embedding is *total*. In a general way, the study of an embedding of G into Q_n turns to see if G is isomorphic to a subgraph of Q_n.

This problem is well-known and treated in graph theory. Many efforts have been made for finding conditions (necessary and/or sufficient) under which a graph G is

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a subgraph of Q_n. In particular, the problem of embedding trees in the hypercube has attracted much attention, since trees are widely used in many domains such as computer science, operations research, or combinatorial optimization for instance.

A tree T is a connected graph without cycles. A *binary tree* is a tree in which every vertex has at most two sons. We say that a tree T is *balanced* if, in the bipartition of $V(T)$, both parts have the same cardinality. A tree T is *cubical* if, for some integer n, there is an embedding of T into the hypercube Q_n. If T is cubical then the least positive integer n for which T can be embedded into the hypercube Q_n is called the *cubical dimension* of T, denoted by $\mathrm{cd}(T)$. Firsov [6] showed that all trees are cubical. Hence, for a tree T, the question we consider is to determine the cubical dimension of T. Wagner and Corneil [14] showed that the problem of deciding if a tree T is embeddable into the hypercube Q_n is NP-complete. Binary trees and their embeddings into hypercubes have been studied in [1]–[5], [7], [9], [11], [13]. A longstanding conjecture of Havel claims the following:

Conjecture 1.1 (Havel [8]). *Every balanced binary tree with 2^n vertices, $n \geqslant 1$, is a subgraph of Q_n.*

Several partial results have been obtained in support of this conjecture [3], [4], [12]. In this paper, we introduce some new classes of balanced binary trees which satisfy the conjecture of Havel.

The hypercube Q_n is bipartite, balanced and n-regular; it has 2^n vertices and $n2^{n-1}$ edges. A tree T is said to be C_n-valuated if we can label every edge of T with an integer from the set $\{1, 2, \ldots, n\}$ in such a way that for every path P in T, there exists an integer $k \in \{1, 2, \ldots, n\}$ such that an odd number of edges in P are labeled by k. Havel and Morávek proved the following:

Theorem 1.2 (Havel and Morávek [10]). *A tree T is embeddable in Q_n if and only if there exists a C_n-valuation of T.*

Intuitively speaking, every edge with label k will be mapped to an edge of Q_n in the k-th dimension. The fact that for every path there exists an integer appearing an odd number of times ensures that all vertices are mapped to distinct vertices in Q_n.

The following result was proved by Havel:

Theorem 1.3 (Havel [8]). *Every balanced binary tree of order 2^n having two vertices of degree 3 is embeddable into the hypercube Q_n.*

Havel and Liebl [9] and Nebeský [12] studied embeddings of the classes of binary trees D_n, $\hat{\hat{D}}_n$, $\hat{D}_n$ and $\check{D}_n$, defined, respectively, as follows:

(1) The tree D_1 is the complete bipartite graph $K_{1,2}$ whose root is the unique vertex of degree 2. For $n \geqslant 2$, D_n is the tree obtained from two disjoint copies T and T' of D_{n-1} and a new vertex v by adding an edge from v to the root of T and another edge from v to the root of T'. The new vertex v is the root of D_n. The tree D_n is thus the *complete binary tree of depth* n, with $2^{n+1} - 1$ vertices. Moreover, we have $\mathrm{cd}(D_1) = 2$ and, for $n \geqslant 2$, $\mathrm{cd}(D_n) = n + 2$, see [9].

(2) For $n \geqslant 1$, the 2-*rooted complete binary tree* $\hat{\hat{D}}_n$ is obtained from two disjoint copies of D_n by adding an edge linking the roots of these copies. This new edge will be referred to as the *axial edge* of $\hat{\hat{D}}_n$. The tree $\hat{\hat{D}}_n$ has $2^{n+2} - 2$ vertices and $\mathrm{cd}(\hat{\hat{D}}_n) = n + 2$, see [12].

(3) For $n \geqslant 1$, the tree $\hat{D}_n$ is obtained from $\hat{\hat{D}}_n$ by inserting two new vertices of degree 2 into the axial edge of $\hat{\hat{D}}_n$. The edge joining these two new vertices will be referred to as the *central edge* of $\hat{D}_n$. The tree $\hat{D}_n$ is balanced, has 2^{n+2} vertices and $\mathrm{cd}(\hat{D}_n) = n + 2$, see [12].

(4) For $n \geqslant 1$, the tree $\check{D}_n$ is obtained from $\hat{\hat{D}}_n$ by inserting two new vertices of degree 2 into some end-edge of $\hat{\hat{D}}_n$ (that is an edge incident to a leaf of $\hat{\hat{D}}_n$). The tree $\check{D}_n$ has 2^{n+2} vertices and $\mathrm{cd}(\check{D}_n) = n + 2$, see [12].

Let T be a tree with root r. We define the *level* of an edge uv in T as $\max\{d(r,u), d(r,v)\}$, where $d(x,y)$ denotes the distance from vertex x to vertex y in T. The level of an edge uv of the 2-rooted complete binary tree $\hat{\hat{D}}_n$ is defined as 0 if the edge uv is the axial edge of $\hat{\hat{D}}_n$, or as the level of uv in the corresponding copy of D_n otherwise. An edge uv is an *ancestor* of an edge xy if they both lie on a path linking the root of the tree to some leaf and the level of uv is smaller than the level of xy. Recall that *subdividing* an edge uv consists in replacing the edge uv by two new edges ux and xv where x is a new vertex of degree 2.

In this paper, we will determine the cubical dimension of trees obtained by subdividing twice the 2-rooted complete binary tree $\hat{\hat{D}}_n$. Such a tree has 2^{n+2} vertices and is of one of the following types:

- ▷ Type (A): the tree A_n^k is obtained by subdividing twice an edge of level k, $0 \leqslant k \leqslant n$, in $\hat{\hat{D}}_n$, $n \geqslant 1$.
- ▷ Type (B): the tree B_n^k is obtained by subdividing two distinct edges of the same level k, $1 \leqslant k \leqslant n$, in $\hat{\hat{D}}_n$, not belonging to the same copy of D_n.
- ▷ Type (C): the tree C_n^k is obtained by subdividing two distinct edges of the same level k, $1 \leqslant k \leqslant n$, in $\hat{\hat{D}}_n$, both belonging to the same copy of D_n.
- ▷ Type (D): the tree $D_n^{k,l}$ is obtained by subdividing two distinct edges of distinct levels k and l, $0 \leqslant k < l \leqslant n$, in $\hat{\hat{D}}_n$, not belonging to the same copy of D_n.

- ▷ Type (E): the tree $E_n^{k,l}$ is obtained by subdividing two distinct edges of distinct levels k and l, $1 \leqslant k < l \leqslant n$, in $\hat{\hat{D}}_n$, both belonging to the same copy of D_n, such that none of these edges is the ancestor of the other.
- ▷ Type (F): the tree $F_n^{k,l}$ is obtained by subdividing two distinct edges of distinct levels k and l, $1 \leqslant k < l \leqslant n$, in $\hat{\hat{D}}_n$, both belonging to the same copy of D_n, such that the edge of level k is the ancestor of the edge of level l.

The above-defined types of trees are illustrated in Figure 1, where vertices created by edge subdivisions are drawn as squares.

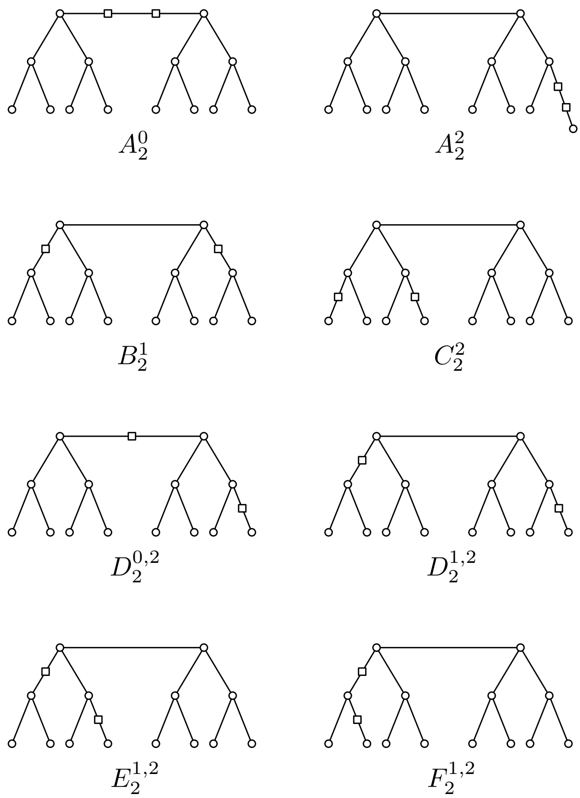


Figure 1. Sample twice subdivided 2-rooted complete binary trees.

Our main result is the following:

Theorem 1.4. *Let T be a tree obtained by subdividing twice the* 2*-rooted complete binary tree $\hat{\hat{D}}_n$. Then we have:*

- ▷ $\mathrm{cd}(T) = n + 2$ *if T is of type* (A) *or* (B),
- ▷ $\mathrm{cd}(T) = n + 3$ *if T is of type* (C), (D), (E) *or* (F).

It is worth noting that such a tree is balanced if and only if it is of type (A) or (B). Our result shows that trees of type (A) or (B) satisfy the conjecture of Havel. We thus generalize the results of Nebeský who considered trees obtained by subdividing twice either the axial edge or an end-edge of $\hat{\check{D}}_n$ (above-defined as $\hat{D}_n$ and $\check{D}_n$, respectively).

2. Proof of the main result

The proof will follow from a series of lemmas, each considering one particular type of tree.

Since every C_n-valuation of a tree T is also a C_n-valuation of any (connected) subtree of T, we easily get the following:

Observation 2.1. If T_1 is a subtree of T_2, then $\mathrm{cd}(T_1) \leqslant \mathrm{cd}(T_2)$.

The following result will be useful in proving structural decompositions of trees of type (A) and (B). Let T_1 and T_2 be two trees, u_1v_1 an edge of T_1, and $u_2x_2y_2v_2$ an induced path of T_2 (both x_2 and y_2 are vertices of degree 2). We define the *$\bowtie$-gluing of T_1 and T_2 along $\{u_1v_1, u_2v_2\}$*, denoted $T_1 \bowtie_{u_1v_1,u_2v_2} T_2$, as the tree obtained by subdividing twice the edge u_1v_1 of T_1, creating the induced path $u_1x_1y_1v_1$, and identifying the two edges x_1y_1 and x_2y_2 (see Figure 2). We then have:

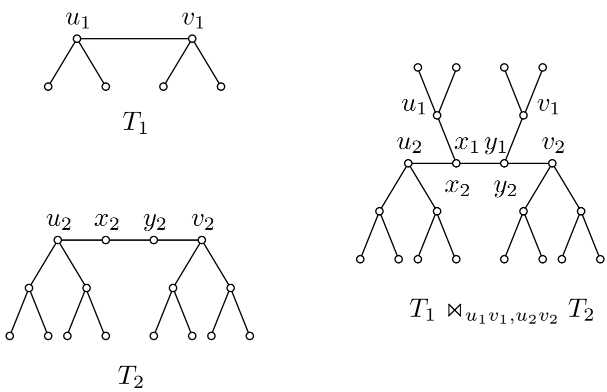


Figure 2. A sample $\bowtie$-gluing of two trees: $T_1 \bowtie_{u_1v_1,u_2v_2} T_2$.

Lemma 2.2. *Let T_1 and T_2 be two trees, u_1v_1 an edge of T_1, and $u_2x_2y_2v_2$ an induced path of T_2. If $\mathrm{cd}(T_1) = \mathrm{cd}(T_2) = k$, then $\mathrm{cd}(T_1 \bowtie_{u_1v_1,u_2v_2} T_2) \leqslant k+1$.*

Proof. Since $\mathrm{cd}(T_1) = \mathrm{cd}(T_2) = k$, there exist a C_k-valuation γ_1 of T_1 and a C_k-valuation γ_2 of T_2. Without loss of generality, we may assume that $\gamma_1(u_1v_1) = \gamma_2(x_2y_2)$. We can then construct a valuation γ of $T = T_1 \bowtie_{u_1v_1,u_2v_2} T_2$ by setting

▹ $\gamma(u_1x_1) = \gamma(y_1v_1) = k+1$,

▹ $\gamma(z_1t_1) = \gamma_1(z_1t_1)$ for every edge z_1t_1 from T_1, and

▹ $\gamma(z_2t_2) = \gamma_2(z_2t_2)$ for every edge z_2t_2 from T_2.

To see that γ is indeed a C_{k+1}-valuation of T, let P be any path in T. If P does not contain the label $k+1$, then P is also a path in T_1 or T_2 and the required property follows from the fact that both γ_1 and γ_2 are C_k-valuations. If the label $k+1$ appears only once in P then $k+1$ appears an odd number of times and we are done. Finally, if the label $k+1$ appears twice in P then the path P' obtained from P by contracting the two edges with label $k+1$ is a path in T_1 and the required property follows from the fact that γ_1 is a C_k-valuation of T_1. □

We now turn to the proof of our main result, considering each type of trees separately.

Lemma 2.3. *For every n and k, $0 \leqslant k \leqslant n$, $n \geqslant 1$, $\mathrm{cd}(A_n^k) = n+2$.*

Proof. Since the tree A_n^k has 2^{n+2} vertices, it cannot be embedded in Q_{n+1} and, therefore, $\mathrm{cd}(A_n^k) \geqslant n+2$. We now prove by induction on k that $\mathrm{cd}(A_n^k) = n+2$ for every $n \geqslant 1$.

The result clearly holds for $k = 0$ since $A_n^0 = \hat{D}_n$ and $\mathrm{cd}(\hat{D}_n) = n+2$ for every $n \geqslant 1$, see [12].

Assume now that $\mathrm{cd}(A_n^{k-1}) = n+2$ for every $n \geqslant 1$, and consider the tree A_n^k. As depicted in Figure 3, it is not difficult to observe that, for every $k \geqslant 1$, A_n^k is the $\bowtie$-gluing of A_{n-1}^{k-1} and $\hat{D}_{n-1}$ along the axial edge of A_{n-1}^{k-1} and the "axial path" of $\hat{D}_{n-1}$. Since $\mathrm{cd}(\hat{D}_{n-1}) = n+1$ by [12] and $\mathrm{cd}(A_{n-1}^{k-1}) = n+1$ by induction hypothesis, we get $\mathrm{cd}(A_n^k) = n+2$ by Lemma 2.2. □

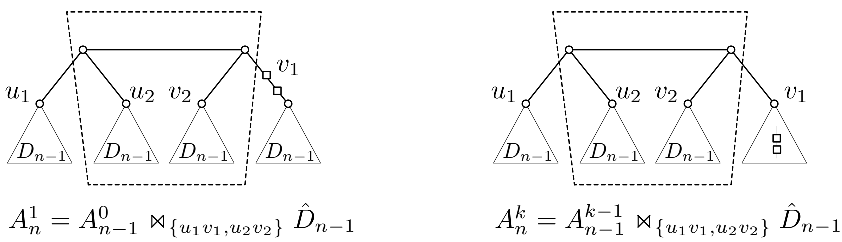


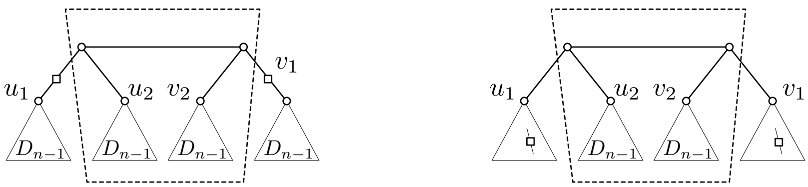
Figure 3. Structural decomposition of A_n^1 and of A_n^k, $k > 1$.

Lemma 2.4. *For every n and k, $1 \leqslant k \leqslant n$, $\mathrm{cd}(B_n^k) = n + 2$.*

P r o o f. The proof is quite similar to the proof of Lemma 2.3. Since the tree B_n^k has 2^{n+2} vertices, it cannot be embedded in Q_{n+1} and, therefore, $\mathrm{cd}(B_n^k) \geqslant n + 2$. We now prove by induction on k that $\mathrm{cd}(B_n^k) = n + 2$ for every $n \geqslant 1$.

As depicted in Figure 4, B_n^1 is the $\bowtie$-gluing of $\hat{D}_{n-1}$ and itself along its central edge and axial path. Since $\mathrm{cd}(\hat{D}_{n-1}) = n + 1$ by [12] we get $\mathrm{cd}(B_n^1) = n + 2$ by Lemma 2.2.

Assume now that $\mathrm{cd}(B_n^{k-1}) = n + 2$ for every $n \geqslant 1$, and consider the tree B_n^k. As depicted in Figure 4, B_n^k is the $\bowtie$-gluing of B_{n-1}^{k-1} and $\hat{D}_{n-1}$ along the axial edge of B_{n-1}^{k-1} and the axial path of $\hat{D}_{n-1}$. Since $\mathrm{cd}(\hat{D}_{n-1}) = n + 1$ by [12] and $\mathrm{cd}(B_{n-1}^{k-1}) = n + 1$ by induction hypothesis, we get $\mathrm{cd}(B_n^k) = n + 2$ by Lemma 2.2. $\square$



$B_n^1 = \hat{D}_{n-1} \bowtie_{\{u_1v_1, u_2v_2\}} \hat{D}_{n-1}$ $\qquad$ $B_n^k = B_{n-1}^{k-1} \bowtie_{\{u_1v_1, u_2v_2\}} \hat{D}_{n-1}$

Figure 4. Structural decomposition of B_n^1 and of B_n^k, $k > 1$.

Lemma 2.5. *For every n and k, $1 \leqslant k \leqslant n$, $\mathrm{cd}(C_n^k) = n + 3$.*

P r o o f. Since the tree C_n^k has 2^{n+2} vertices and is not balanced, it cannot be embedded in Q_{n+2} and, therefore, $\mathrm{cd}(C_n^k) \geqslant n + 3$.

The fact that $\mathrm{cd}(C_n^k) = n+3$ for every n and k, $1 \leqslant k \leqslant n$, then simply follows from Observation 2.1 since C_n^k is a subtree of $\hat{\hat{D}}_{n+1}$ (see Figure 5) and $\mathrm{cd}(\hat{\hat{D}}_{n+1}) = n + 3$ by [12]. $\square$

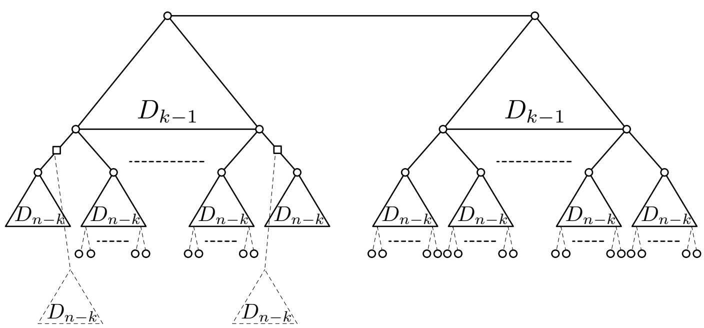


Figure 5. C_n^k is a subtree of $\hat{\hat{D}}_{n+1}$ (dashed lines depict elements of $\hat{\hat{D}}_{n+1} \setminus C_n^k$).

Lemma 2.6. *For every n, k and l, $0 \leqslant k < l \leqslant n$, $\operatorname{cd}(D_n^{k,l}) = n + 3$.*

P r o o f. Since the tree $D_n^{k,l}$ has 2^{n+2} vertices and is not balanced, it cannot be embedded in Q_{n+2} and, therefore, $\operatorname{cd}(D_n^{k,l}) \geqslant n + 3$.

The fact that $\operatorname{cd}(D_n^{k,l}) = n + 3$ for every n, k and l, $0 \leqslant k < l \leqslant n$, then follows from Observation 2.1 since $D_n^{k,l}$ is a subtree of $\hat{\hat{D}}_{n+1}$ (see Figure 6 for $k = 0$ and Figure 7 for $k > 0$) and $\operatorname{cd}(\hat{\hat{D}}_{n+1}) = n + 3$ by [12]. $\square$

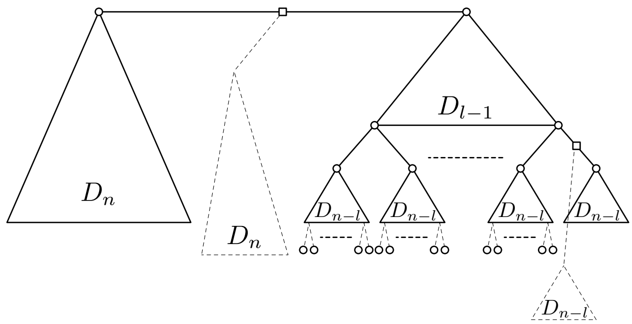


Figure 6. $D_n^{0,l}$ is a subtree of $\hat{\hat{D}}_{n+1}$ (dashed lines depict elements of $\hat{\hat{D}}_{n+1} \setminus D_n^{0,l}$).

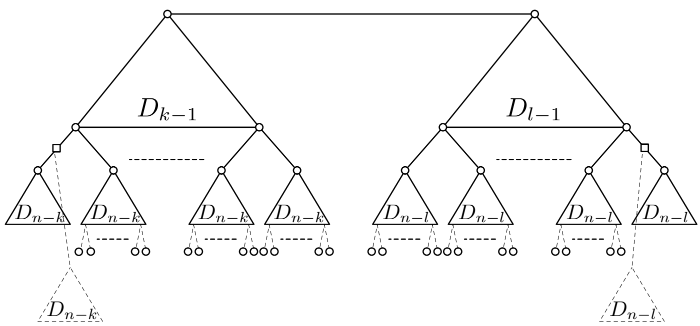


Figure 7. $D_n^{k,l}$ is a subtree of $\hat{\hat{D}}_{n+1}$ (dashed lines depict elements of $\hat{\hat{D}}_{n+1} \setminus D_n^{k,l}$).

Lemma 2.7. *For every n, k and l, $1 \leqslant k < l \leqslant n$, $\operatorname{cd}(E_n^{k,l}) = n + 3$.*

P r o o f. Since the tree $E_n^{k,l}$ has 2^{n+2} vertices and is not balanced, it cannot be embedded in Q_{n+2} and, therefore, $\operatorname{cd}(E_n^{k,l}) \geqslant n + 3$.

The fact that $\operatorname{cd}(E_n^{k,l}) = n+3$ for every n, k and l, $1 \leqslant k < l \leqslant n$, then follows from Observation 2.1 since $E_n^{k,l}$ is a subtree of $\hat{\hat{D}}_{n+1}$ (see Figure 8) and $\operatorname{cd}(\hat{\hat{D}}_{n+1}) = n+3$ by [12]. $\square$

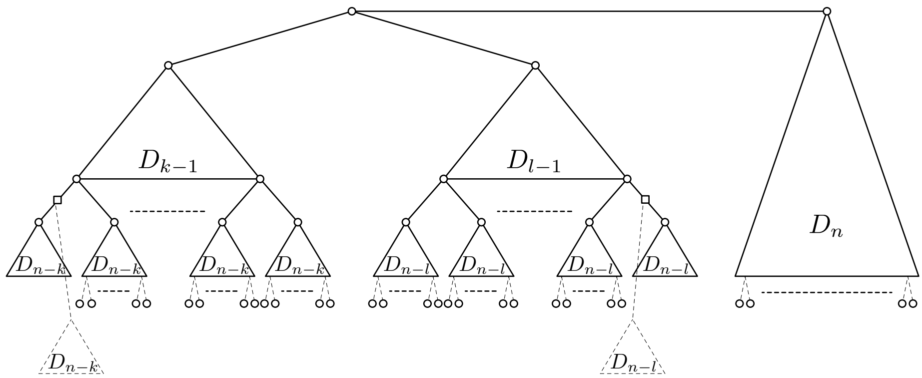


Figure 8. $E_n^{k,l}$ is a subtree of $\hat{\hat{D}}_{n+1}$ (dashed lines depict elements of $\hat{\hat{D}}_{n+1} \setminus E_n^{k,l}$).

Lemma 2.8. *For every n, k and l, $1 \leqslant k < l \leqslant n$, $\mathrm{cd}(F_n^{k,l}) = n+3$.*

P r o o f. Since the tree $F_n^{k,l}$ has 2^{n+2} vertices and is not balanced, it cannot be embedded in Q_{n+2} and, therefore, $\mathrm{cd}(F_n^{k,l}) \geqslant n+3$. We now prove that $\mathrm{cd}(F_n^{k,l}) = n+3$ by constructing a C_{n+3}-valuation of $F_n^{k,l}$.

Let $u_k v_k$ and $u_l v_l$ denote the two edges of $\hat{\hat{D}}_n$ that have been subdivided, with levels k and l, respectively, and let x_k and x_l denote the corresponding two created vertices of degree 2. Since $\mathrm{cd}(\hat{\hat{D}}_n) = n+2$, there exists a C_{n+2}-valuation of $\hat{\hat{D}}_n$, say γ_0. We define the valuation γ of $F_n^{k,l}$ as follows:

- ▷ $\gamma(u_k x_k) = \gamma_0(u_k v_k)$, $\gamma(u_l x_l) = \gamma_0(u_l v_l)$,
- ▷ $\gamma(x_k v_k) = \gamma(x_l v_l) = n+3$,
- ▷ $\gamma(uv) = \gamma_0(uv)$ for every edge $uv \notin \{u_k x_k, x_k v_k, u_l x_l, x_l v_l\}$.

To see that γ is indeed a C_{n+3}-valuation of $F_n^{k,l}$, let P be any path in $F_n^{k,l}$. If P does not contain the label $n+3$, then P has the same labeling as a path in $\hat{\hat{D}}_n$ and the required property follows from the fact that γ_0 is a C_{n+2}-valuation. If the label $n+3$ appears only once in P then $n+3$ appears an odd number of times and we are done. Finally, if the label $n+3$ appears twice in P then the path P' obtained from P by contracting the two edges with label $n+3$ is a path in $\hat{\hat{D}}_n$ and the required property follows from the fact that γ_0 is a C_{n+2}-valuation of $\hat{\hat{D}}_n$. □

The proof of Theorem 1.4 now clearly follows from Lemmas 2.3 through 2.8.

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