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GENERALIZED 3-EDGE-CONNECTIVITY OF CARTESIAN PRODUCT GRAPHS

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Abstract. The generalized k-connectivity $\kappa_k(G)$ of a graph G was introduced by Chartrand et al. in 1984. As a natural counterpart of this concept, Li et al. in 2011 introduced the concept of generalized k-edge-connectivity which is defined as $\lambda_k(G) = \min\{\lambda(S)\colon S \subseteq V(G)$ and $|S| = k\}$, where $\lambda(S)$ denotes the maximum number l of pairwise edge-disjoint trees $T_1, T_2, \ldots, T_l$ in G such that $S \subseteq V(T_i)$ for $1 \leqslant i \leqslant l$. In this paper we prove that for any two connected graphs G and H we have $\lambda_3(G \,\square\, H) \geqslant \lambda_3(G) + \lambda_3(H)$, where $G \,\square\, H$ is the Cartesian product of G and H. Moreover, the bound is sharp. We also obtain the precise values for the generalized 3-edge-connectivity of the Cartesian product of some special graph classes.

Keywords: generalized connectivity; generalized edge-connectivity; Cartesian product

MSC 2010: 05C40, 05C76

1. Introduction

All graphs in this paper are undirected, finite and simple. We follow the notation and terminology of [1] for those not defined in this paper. Connectivity is one of the most important concepts in graph theory and its applications, both in a combinatorial sense and an algorithmic sense. In theoretical computer science, connectivity is a basic measure of reliability of networks. By the well-known Menger's theorem, the (vertex) *connectivity* of a graph $G = (V(G), E(G))$, denoted $\kappa(G)$, can be defined as the minimum $\kappa(\{u, v\})$ over all 2-subsets $\{u, v\}$ of $V(G)$, where $\kappa(\{u, v\})$ denotes the maximum number of internally disjoint $u-v$ paths in G. In [2], Chartrand et al. introduced the following generalized connectivity. Let G be a graph of order $n \geqslant 2$

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and let k be an integer with $2 \leqslant k \leqslant n$. For a set S of k vertices of G, let $\kappa(S)$ denote the maximum number l of edge-disjoint trees $T_1, T_2, \ldots, T_l$ in G such that $V(T_i) \cap V(T_j) = S$ for $1 \leqslant i < j \leqslant l$. (Note that these trees are vertex-disjoint in $G \setminus S$.) A collection $\{T_1, T_2, \ldots, T_l\}$ of trees in G with this property is called a *set of internally disjoint trees connecting* S. The *generalized* k*-connectivity* of G is then defined as

$$\kappa_k(G) = \min\{\kappa(S);\ S \subseteq V(G) \text{ and } |S| = k\}.$$

Thus $\kappa_2(G) = \kappa(G)$ and $\kappa_k(G) = 0$ when G is disconnected. As a natural counterpart of the generalized connectivity, recently Li et al. [9] introduced the following concept of generalized edge-connectivity. Let $\lambda(S)$ denote the maximum number l of pairwise edge-disjoint trees $T_1, T_2, \ldots, T_l$ in G such that $S \subseteq V(T_i)$ for $1 \leqslant i \leqslant l$. The *generalized* k*-edge-connectivity* of G is defined as

$$\lambda_k(G) = \min\{\lambda(S);\ S \subseteq V(G) \text{ and } |S| = k\}.$$

Thus $\lambda_2(G) = \lambda(G)$ is the usual edge-connectivity, and $\lambda_k(G) = 0$ when G is disconnected. Clearly, we have $\kappa_k(G) \leqslant \lambda_k(G)$.

The generalized connectivity and edge-connectivity are also called the *tree connectivities*. In addition to being a natural combinatorial measure, the tree connectivity can be motivated by its interesting interpretation in practice. For example, suppose that G represents a network. If one wants to connect a set S of nodes of G with $|S| \geqslant 3$, then a tree has to be used to connect them. This kind of tree with minimum order for connecting a set of nodes is usually called a *Steiner tree*, and popularly used in the physical design of VLSI [13]. Usually, one wants to consider how reliable (or tough) a network can be for the connection of a set of vertices. Then the number of totally independent ways to connect them is a measure for this purpose. The tree connectivities can serve for measuring the capability of a network G to connect any k vertices in G. The reader is referred to a recent survey [8] on the state-of-the-art of research on tree connectivities.

Products of graphs occur naturally in discrete mathematics as tools in combinatorial constructions, they give rise to important classes of graphs and deep structural problems. The Cartesian product is one of the most important graph products and plays a key role in design and analysis of networks. Many researchers have investigated the (edge) connectivity of the Cartesian product graphs in the past several decades [3], [4], [6], [12], [14], [5], [15]. Specially, the exact formula for $\kappa(G \,\square\, H)$ was obtained.

Theorem 1.1 ([11], [14]). *Let G and H be graphs on at least two vertices. Then* $\kappa(G \,\square\, H) = \min\{\kappa(G)|H|, \kappa(H)|G|, \delta(G) + \delta(H)\}$.

This theorem was first stated by Liouville [11]. However, his proof never appeared. In the meantime, several partial results were obtained until Špacapan [14] provided the proof. Theorem 1.1 in particular implies the following result of Sabidussi [12]:

Theorem 1.2 ([12]). *Let G and H be connected graphs. Then $\kappa(G \,\square\, H) \geqslant \kappa(G) + \kappa(H)$.*

Li, Li and Sun [7] investigated the generalized 3-connectivity of the Cartesian product graphs and get the following result which could be seen as an extension of Theorem 1.2.

Theorem 1.3 ([7]). *Let G and H be connected graphs such that $\kappa_3(G) \geqslant \kappa_3(H)$. Then we have:*

(i) *If $\kappa_3(G) < \kappa(G)$, then $\kappa_3(G \,\square\, H) \geqslant \kappa_3(G) + \kappa_3(H)$. Moreover, the bound is sharp.*

(ii) *If $\kappa_3(G) = \kappa(G)$, then $\kappa_3(G \,\square\, H) \geqslant \kappa_3(G) + \kappa_3(H) - 1$. Moreover, the bound is sharp.*

In this paper, we continue the research on the generalized connectivity of the Cartesian product graphs and get the following result for the generalized 3-edge-connectivity of Cartesian product.

Theorem 1.4. *Let G and H be two connected graphs. Then we have $\lambda_3(G \,\square\, H) \geqslant \lambda_3(G) + \lambda_3(H)$. Moreover, the bound is sharp.*

The proof of Theorem 1.4 consists of Lemmas 3.1, 3.2 and 3.3. In order to prove these lemmas we need a few preliminary results that will be given in the next section. In Section 4, we also obtain the precise values for the generalized 3-edge-connectivity of the Cartesian product of some special graph classes (Propositions 4.1 and 4.2).

2. Preliminaries

The *Cartesian product* of two graphs G and H, denoted $G \,\square\, H$, is defined to have the vertex set $V(G) \times V(H)$ such that (u, u') and (v, v') are adjacent if and only if either $u = v$ and $u'v' \in E(H)$, or $u' = v'$ and $uv \in E(G)$. Note that this product is commutative, that is, $G \,\square\, H = H \,\square\, G$.

We use the following useful notion which was introduced in [4], [5]. The mappings $p_G\colon (u, v) \mapsto u$ and $p_H\colon (u, v) \mapsto v$ from $V(G \,\square\, H)$ into $V(G)$ and $V(H)$ are weak homomorphisms from $G \,\square\, H$ onto the factors G and H, respectively. They are called *projections* in the literature.

Let G and H be two graphs with $V(G) = \{u_i;\ 1 \leqslant i \leqslant n\}$ and $V(H) = \{v_j;\ 1 \leqslant j \leqslant m\}$. We use $G(v_j)$ to denote the subgraph of $G \,\square\, H$ induced by the vertex set $\{(u_i, v_j);\ 1 \leqslant i \leqslant n\}$ where $1 \leqslant j \leqslant m$, and use $H(u_i)$ to denote the subgraph of $G \,\square\, H$ induced by the vertex set $\{(u_i, v_j);\ 1 \leqslant j \leqslant m\}$ where $1 \leqslant i \leqslant n$. Clearly, we have $G(v_j) \cong G$ and $H(u_i) \cong H$. For example, as shown in Figure 1, $G(v_j) \cong G$ for $1 \leqslant j \leqslant 4$ and $H(u_i) \cong H$ for $1 \leqslant i \leqslant 3$. For $1 \leqslant j_1 \neq j_2 \leqslant m$, the vertices (u_i, v_{j_1}) and (u_i, v_{j_2}) belong to the same graph $H(u_i)$ where $u_i \in V(G)$, we call (u_i, v_{j_2}) the *vertex corresponding to* (u_i, v_{j_1}) in $G(v_{j_2})$; for $1 \leqslant i_1 \neq i_2 \leqslant n$, we call (u_{i_2}, v_j) the vertex corresponding to (u_{i_1}, v_j) in $H(u_{i_2})$ [7]. Similarly, we can define the path and the tree corresponding to some path and tree, respectively. For example, in graph (c) of Figure 1, let P_1 and P_2 be the paths whose edges are labelled 1 and 2 in $H(u_1)$ and $H(u_2)$, respectively. Then P_2 is called the path corresponding to P_1 in $H(u_2)$. Clearly, P_1 and P_2 correspond to the path v_1, v_2, v_3, v_4 in graph H.

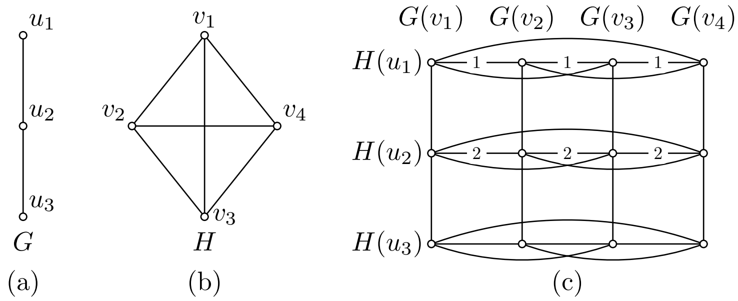

Figure 1. Graphs G, H and their Cartesian product.

Lemma 2.1 ([10]). *Let G be a connected graph of order n. If there exist two adjacent vertices of degree $\delta(G)$, then $\lambda_k(G) \leqslant \delta(G) - 1$ for every integer k with $3 \leqslant k \leqslant n$, and, moreover, the bound is sharp.*

Theorem 2.2 ([9]). *For every two integers n and k with $2 \leqslant k \leqslant n$, $\lambda_k(K_n) = n - \lceil k/2 \rceil$.*

Note that in the sequel we assume that every tree T which connects S is *minimal*, that is, the subgraph which is obtained by deleting any set of vertices or edges of T will not connect S. This assumption will not affect our results.

3. Proof of Theorem 1.4

For a set S, we use $|S|$ to denote its size. Let $\lambda_3(G)=k$, $\lambda_3(H)=l$. Without loss of generality, we assume that $k\geqslant l$. For any set $S=\{x,y,z\}$ of three vertices in $V(G\,\square\,H)$ where $x\in V(G(v_i))$, $y\in V(G(v_j))$, $z\in V(G(v_k))$, we need to find at least $k+l$ edge-disjoint trees connecting S.

Lemma 3.1. *In the case that i, j, k are distinct, we can construct at least $k+l$ edge-disjoint trees connecting S.*

Proof. Without loss of generality, we assume that $x\in V(G(v_1))\cap V(H(u_1))$, $y\in V(G(v_2))$, $z\in V(G(v_3))$. Furthermore, let y', z' be the vertices corresponding to y, z in $G(v_1)$, x', z'' be the vertices corresponding to x, z in $G(v_2)$ and x'', y'' be the vertices corresponding to x, y in $G(v_3)$.

Case 1. $|\{p_G(x),p_G(y),p_G(z)\}|=1$. Now we know that x, y', z' are the same vertex in $G(v_1)$. Let x_1 be a neighbor of x in $G(v_1)$. Without loss of generality, we assume that $x_1\in H(u_2)$. Let x_1' and x_1'' be the corresponding vertices of x_1 in $G(v_2)$ and $G(v_3)$, respectively. Clearly, $yx_1'\in E(G(v_2))$, $zx_1''\in E(G(v_3))$. Let T_1 be the tree obtained from T_1' and edges xx_1, yx_1', zx_1'', where T_1' is a tree connecting $\{x_1,x_1',x_1''\}$ in $H(u_2)$ (see Figure 2). Since x has at least k neighbors in $G(v_1)$, we can find k such trees. Thus, we get at least $k+l$ trees connecting S since there are l edge-disjoint trees connecting S in $H(u_1)$. It is easy to show that any two of these trees are edge-disjoint.

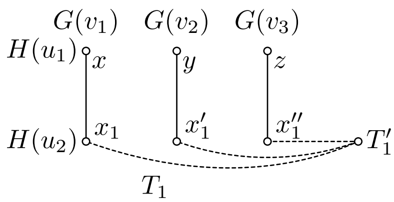

Figure 2. Graph of Case 1.

Case 2. $|\{p_G(x),p_G(y),p_G(z)\}|=3$. Now we know that x,y',z' are three distinct vertices in $G(v_1)$. Without loss of generality, we assume that $y'\in V(H(u_2))$, $z'\in V(H(u_3))$. As $\lambda_3(G(v_1))=k$, there are k edge-disjoint trees connecting $\{x,y',z'\}$ in $G(v_1)$, say T_j' where $1\leqslant j\leqslant k$. Let $\{T_i;\ 1\leqslant i\leqslant l\}$ be a set of l edge-disjoint trees connecting $\{v_1,v_2,v_3\}$ in H since $\lambda_3(H)=l$. Let $k_0,k_1,\ldots,k_l$ be integers such that $0=k_0<k_1<\ldots<k_l=k$ since $k\geqslant l$.

Subcase 2.1. $xy',xz'\notin E(G(v_1))$. We need the following claim.

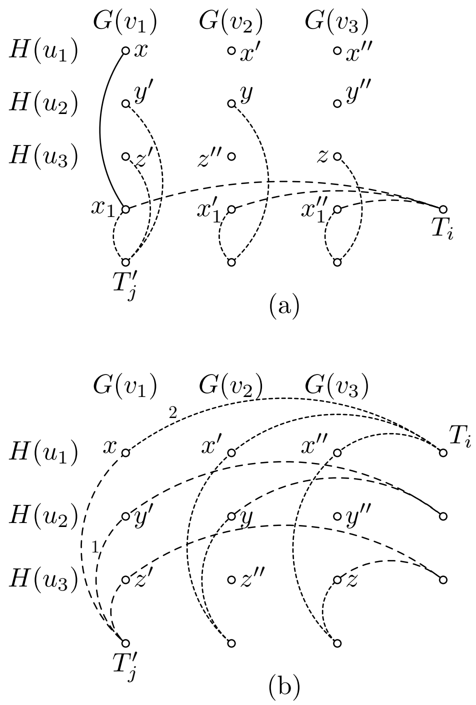

Figure 3. Graphs in the proof of Claim 1.

Claim 1. If $xy', xz' \notin E(G(v_1))$, then there are $k_i - k_{i-1} + 1$ edge-disjoint trees connecting S in $\left(\bigcup\limits_{j=k_{i-1}+1}^{k_i} T'_j\right) \square\, T_i$ for each $1 \leqslant i \leqslant l$.

Proof of Claim 1. For each $1 \leqslant i \leqslant l$ and $k_{i-1}+1 \leqslant j \leqslant k_i - 1$, we can construct a tree connecting S in the graph $T'_j \,\square\, T_i$ as shown in graph (a) of Figure 3. Note that $x_1 \in V(T'_j)$ is a neighbor of x in $G(v_1)$, and x'_1, x''_1 are vertices corresponding to x_1 in $G(v_2), G(v_3)$, respectively. For simplicity, we also use T_i and T'_j to denote the tree connecting $\{x_1, x'_1, x''_1\}$ and $\{x, y', z'\}$, respectively, (see graph (a) of Figure 3). For the case $j = k_i$, in the graph $T'_{k_i} \,\square\, T_i$ we can construct two trees connecting S as shown in graph (b) of Figure 3. Thus, for each $1 \leqslant i \leqslant l$, in the graph $\left(\bigcup\limits_{j=k_{i-1}+1}^{k_i} T'_j\right) \square$ T_i we can get $k_i - k_{i-1} + 1$ trees and it is not hard to show that any two of these trees are edge-disjoint so that the claim holds.

Now for the case that $xy', xz' \notin E(G(v_1))$, we can find $\sum\limits_{i=1}^{l}(k_i - k_{i-1} + 1) = k + l$ trees by Claim 1 and any two of these trees are edge-disjoint by the definition of the Cartesian product.

Subcase 2.2. $xy', xz' \in E(G(v_1))$. If both xy' and xz' belong to the same tree, named $\overline{T} \in \{T'_j;\ 1 \leqslant j \leqslant k\}$, then we reorder these trees so that $T'_{k_1} = \overline{T}$. With an argument similar to that of Subcase 2.1, we can construct $k + l$ edge-disjoint trees connecting S.

If xy' and xz' belong to distinct trees, say $\overline{T}$ and $\widetilde{T}$, respectively. Clearly, both of them are paths by the assumption of the note in the end of Section 2. If $l \geqslant 2$, then we reorder the elements of $\{T'_j;\ 1 \leqslant j \leqslant k\}$ so that $T'_{k_1} = \overline{T}$ and $T'_{k_2} = \widetilde{T}$. With an argument similar to that of Subcase 2.1, we can get $k+l$ edge-disjoint trees connecting S. Otherwise, we have $l = 1$ and reorder the elements of $\{T'_j;\ 1 \leqslant j \leqslant k\}$ so that $T'_{k-1} = \overline{T}$ and $T'_k = \widetilde{T}$. For $1 \leqslant j \leqslant k-2$, we can construct $k-2$ trees connecting S by a method similar to that of graph (a) in Figure 3. For $j = k-1$, we can use T'_{k-1} and T_1 to construct two trees connecting S by a method similar to that of graph (b) in Figure 3. If $\deg_{G(v_1)}(x) = \deg_{G(v_1)}(z') = \delta(G)$, we know $\deg_{G(v_1)}(x) = \deg_{G(v_1)}(z') \geqslant k+1$ by Lemma 2.1; otherwise, we have $\deg_{G(v_1)}(x)$ or $\deg_{G(v_1)}(z') \geqslant \delta(G) + 1 \geqslant \lambda_3(G) + 1 = k + 1$. Without loss of generality, we assume that $\deg_{G(v_1)}(x) \geqslant k+1$, which means that there exists a vertex, say x_{k+1}, such that $xx_{k+1} \notin E(T'_j)$ for $1 \leqslant j \leqslant k$. Then we can use xx_{k+1}, T'_k and T_1 to construct a new tree connecting S as shown in Figure 4. Thus, we get $k+l$ trees in total for the case $l = 1$ and it is not hard to show that any two of these trees are edge-disjoint.

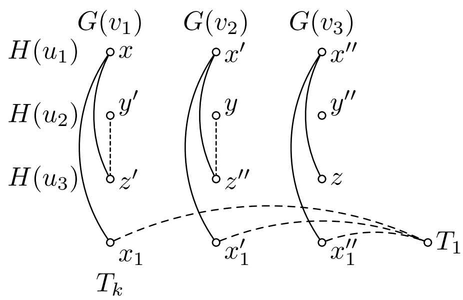

Figure 4. The tree constructed from xx_{k+1}, T'_k and T_1.

The remaining subcase is that exactly one of xy', xz' belongs to $E(G(v_1))$. Without loss of generality, we assume $xy' \in E(G(v_1))$.

Subcase 2.3. $xy' \in E(G(v_1))$. Let xy' belong to the tree $\overline{T} \in \{T'_j;\ 1 \leqslant j \leqslant k\}$, then we reorder these trees so that $T'_{k_1} = \overline{T}$. By an argument similar to that of Subcase 2.1, we can construct $k + l$ edge-disjoint trees connecting S.

Case 3. $|\{p_G(x), p_G(y), p_G(z)\}| = 2$. Now we have that two of x, y', z' are the same vertex in $G(v_1)$. Since $\lambda(G(v_1)) \geqslant k$, there exist k edge-disjoint $x - y'$ paths P_i in $G(v_1)$ where $1 \leqslant i \leqslant k$. By an argument similar to that of Case 2, we can construct at least $k + l$ edge-disjoint trees connecting S. $\square$

Lemma 3.2. *In the case that exactly two of i, j, k are the same, we can construct at least $k+l$ edge-disjoint trees connecting S.*

Proof. Without loss of generality, we assume that $i=j=1$ and $k=2$. Furthermore, we assume that $x \in V(G(v_1)) \cap V(H(u_1))$ and $y \in V(G(v_1)) \cap V(H(u_2))$. In the case that $z \in V(G(v_2)) \cap V(H(u_i))$ where $i \neq 1,2$, we can construct $k+l$ edge-disjoint trees connecting S since it is similar to Case 3 of Lemma 3.1. So it suffices to consider the case $i=1$ (the case $i=2$ is similar).

As $\lambda(G) \geqslant \lambda_3(G) = k$ and $\lambda(H) \geqslant \lambda_3(H) = l$, there are at least k paths $P_i\colon x = a_{i,0}, a_{i,1}, \ldots, a_{i,t_i} = y$ connecting x and y in graph $G(v_1)$ and l paths $Q_j\colon x = b_{j,0}, b_{j,1}, \ldots, b_{j,t_j} = z$ connecting x and z in graph $H(u_1)$ where $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$. We set $P_k := x, y$ if $xy \in E(G(v_1))$ and $Q_l := x, z$ if $xz \in E(H(u_1))$. For $1 \leqslant i \leqslant k-1$, we could construct a tree $T_i := P_i \cup Q_1^i \cup \{a'_{i,1}, z\}$ where $a'_{i,1}$ is the vertex corresponding to $a_{i,1}$ in graph $G(v_2)$ and Q_1^i is the $a_{i,1} - a'_{i,1}$ path corresponding to Q_1 in graph $H(a_{i,1})$ (see the lines labelled by T_i in Figure 5). Similarly, for $1 \leqslant j \leqslant l-1$, we construct a tree $T'_j := Q_j \cup P_1^j \cup \{b'_{j,1}, y\}$ where $b'_{j,1}$ is the vertex corresponding to $b_{j,1}$ in the graph $H(u_2)$ and P_1^j is the $b_{j,1} - b'_{j,1}$ path corresponding to P_1 in the graph $G(b_{j,1})$ (see the lines labelled by T'_j in Figure 5).

If $xy \notin E(G(v_1))$, we can construct a tree T_k from P_k which is similar to T_i where $1 \leqslant i \leqslant k-1$. In the case that $xz \notin E(H(u_1))$, we can also construct a tree T'_l from Q_l which is similar to T'_j where $1 \leqslant j \leqslant l-1$. Thus, we find $k+l$ edge-disjoint trees connecting S in total. In the case that $xz \in E(H(u_1))$, if $\deg_{H(u_1)}(x) = \deg_{H(u_1)}(z) = \delta(H(u_1))$, then we have $l = \lambda_3(H(u_1)) \leqslant \delta(H(u_1)) - 1$ by Lemma 2.1, it means that x has a neighbor, say $b_{l+1,1}$, which is distinct from $b_{j,1}$ in the graph $H(u_1)$, where $1 \leqslant j \leqslant l$. We can construct a tree $T'_l := \{xz\} \cup \{xb_{l+1,1}\} \cup P_1^{l+1} \cup \{yb'_{l+1,1}\}$, where $b'_{l+1,1}$ is the vertex corresponding to $b_{l+1,1}$ in the graph $H(u_2)$ and P_1^{l+1} is the path corresponding to P_1 in $G(b_{l+1,1})$ (see the lines labelled by T'_l in Figure 5). Thus, there are $k+l$ edge-disjoint trees connecting S in total. Otherwise, we have $\deg_{H(u_1)}(x) > \delta(H(u_1))$ or $\deg_{H(u_1)}(z) > \delta(H(u_1))$, then using a similar argument, we can also construct a tree T'_l.

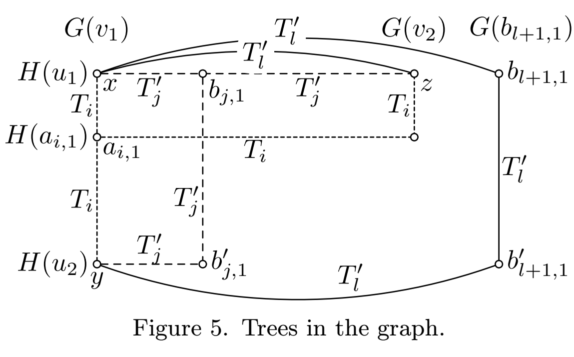

Figure 5. Trees in the graph.

In the case $xy \in E(G(v_1))$, using an argument similar to the above, we can construct $k+l$ edge-disjoint trees connecting S. $\square$

The final case that i, j, k are the same is similar to Case 1 of Lemma 3.1, so the following result holds.

Lemma 3.3. *For the case that i, j, k are the same, we can construct at least $k+l$ edge-disjoint trees connecting S.*

P r o o f of Theorem 1.4. By Lemmas 3.1, 3.2 and 3.3, we have $\lambda_3(G \,\square\, H) \geqslant \lambda_3(G) + \lambda_3(H)$. The following example and Proposition 4.1 imply that the bound is sharp.

We use the graph of Example 3.1 in [7]: Let $K_{2n} (n \geqslant 2)$ be a complete graph with vertex set $V(K_{2n}) = \{u_i;\ 1 \leqslant i \leqslant 2n\}$ and G a graph obtained from K_{2n} by adding a new vertex u and edge set $\{uu_i;\ 1 \leqslant i \leqslant n\}$.

For any vertex set $S = \{x, y, z\} \subseteq V(G)$, if $u \notin S$, we clearly get at least $2n - \lceil \frac{3}{2} \rceil = 2n - 2 \geqslant n$ edge-disjoint trees connecting S by Theorem 2.2.

Otherwise, we have $u \in S$ and assume $x = u$. If $y, z \in \{u_i;\ 1 \leqslant i \leqslant n\}$, without loss of generality we assume that $y = u_1$, $z = u_2$. Let T_1 and T_2 be the paths u, u_1, u_{n+1}, u_2 and u, u_2, u_{n+2}, u_1, respectively, and let T_i be the tree obtained from uu_i and the path u_1, u_i, u_2, where $3 \leqslant i \leqslant n$. If $y, z \in \{u_i;\ n+1 \leqslant i \leqslant 2n\}$, without loss of generality we assume that $y = u_{n+1}$, $z = u_{n+2}$. Let T_i be the tree obtained from the edges uu_i, u_iu_{n+1}, u_iu_{n+2} for $n+1 \leqslant i \leqslant 2n$. If $y \in \{u_i;\ 1 \leqslant i \leqslant n\}$ and $z \in \{u_i;\ n+1 \leqslant i \leqslant 2n\}$, without loss of generality we assume that $y = u_1$, $z = u_{n+1}$. Let T_1 be the path u, u_1, u_{n+1} and T_i the tree obtained from the edges uu_i, u_iu_1, u_iu_{n+1} for $2 \leqslant i \leqslant n$. It is easy to show that any two trees are edge-disjoint in each case.

Thus, we have $\lambda_3(G) \geqslant n$. Since $\lambda_3(G) \leqslant \delta(G) = n$, we have $\lambda_3(G) = n$. So $\lambda_3(G \,\square\, C_m) \geqslant \lambda_3(G) + \lambda_3(C_m) = n+1$. As there are two adjacent vertices with minimum degree $n+2$ in the graph $G \,\square\, C_m$, we have $\lambda_3(G \,\square\, C_m) \leqslant \delta(G \,\square\, C_m) - 1 = n+1$ by Lemma 2.1. Thus, $\lambda_3(G \,\square\, C_m) = \lambda_3(G) + \lambda_3(C_m)$. $\square$

4. Results on special graph classes

For $m \geqslant 3$, the *wheel graph* W_n is defined as a graph constructed by joining a new vertex to every vertex of a cycle C_m. The following result concerns the Cartesian products of connected graphs with $\delta(G) = 1$ and some special graph classes.

Proposition 4.1. *Let G be a connected graph with $\delta(G) = 1$ and order $n \geqslant 3$.*

(a) *If H is a connected graph with $\delta(G) = 1$ and order $m \geqslant 3$, then $\lambda_3(G \square H) = 2$.*

(b) *If H is a cycle, then $\lambda_3(G \square H) = 2$.*

(c) *If H is a wheel graph, then $\lambda_3(G \square H) = 3$.*

(d) *If H is a complete graph with order $m \geqslant 3$, then $\lambda_3(G \square H) = m - 1$.*

Proof. We first verify (a). If H is a connected graph with $\delta(G) = 1$ and order $m \geqslant 3$, then $\lambda_3(G \square H) \leqslant \delta(G \square H) = 2$. By Theorem 1.4, we have $\lambda_3(G \square H) = 2$.

We then verify (b). If H is a cycle, we have $\lambda_3(G \square H) \geqslant \lambda_3(G) + \lambda_3(H) = 2$ by Theorem 1.4. Since $\delta(G \square H) = 3$ and there are two adjacent vertices of degree 3 in $G \square C_m$, we have $\lambda_3(G \square C_m) \leqslant 2$ by Lemma 2.1. Thus, (b) holds.

It is easy to show that $\lambda_3(H) = 2$ if H is a wheel graph. If H is a complete graph with order $m \geqslant 3$, then $\lambda_3(H) = m - 2$ by Theorem 2.2. By an argument similar to that of (b), we can prove (c) and (d). $\square$

Note that since any nontrivial tree is a connected graph with minimum degree 1, Proposition 4.1 also determines the precise value for $\lambda_3(G \square H)$, where G is a tree of order at least 3 and H is a tree, a cycle, a wheel graph or a complete graph of order at least 3.

We know $Q_r \cong K_2 \square K_2 \square \ldots \square K_2$ is the *r-hypercube*, where r is the number of K_2. In [7], Li, Li and Sun derive that the precise value of $\kappa_3(Q_r)$ equals $r - 1$. In fact, we can get a similar result for $\lambda_3(Q_r)$.

Proposition 4.2. $\lambda_3(Q_r) = r - 1$.

Proof. Since Q_r is a r-regular graph, by Lemma 2.1 we have $\lambda_3(Q_r) \leqslant r - 1$. We also have $\lambda_3(Q_r) \geqslant \kappa_3(Q_r) = r - 1$. Thus, the result holds. $\square$

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References

[1] *J. A. Bondy, U. S. R. Murty*: Graph Theory. Graduate Texts in Mathematics 244, Springer, Berlin, 2008. zbl MR

[2] *G. Chartrand, S. F. Kappor, L. Lesniak, D. R. Lick*: Generalized connectivity in graphs. Bull. Bombay Math. Colloq. *2* (1984), 1–6.

[3] *W.-S. Chiue, B.-S. Shieh*: On connectivity of the Cartesian product of two graphs. Appl. Math. Comput. *102* (1999), 129–137. zbl MR

[4] *W. Imrich, S. Klavžar*: Product Graphs. Structure and Recognition. Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley, New York, 2000. zbl MR

[5] *W. Imrich, S. Klavžar, D. F. Rall*: Topics in Graph Theory. Graphs and Their Cartesian Product. A K Peters, Wellesley, 2008. zbl MR

[6] *S. Klavžar, S. Špacapan*: On the edge-connectivity of Cartesian product graphs. Asian-Eur. J. Math. *1* (2008), 93–98. zbl MR
[7] *H. Li, X. Li, Y. Sun*: The generalized 3-connectivity of Cartesian product graphs. Discrete Math. Theor. Comput. Sci. (electronic only) *14* (2012), 43–54. zbl MR
[8] *X. Li, Y. Mao*: A survey on the generalized connectivity of graphs. ArXiv:1207.1838v2 [math.CO].
[9] *X. Li, Y. Mao, Y. Sun*: On the generalized (edge-)connectivity of graphs. Australas. J. Comb. (electronic only) *58* (2014), 304–319. zbl MR
[10] *X. Li, Y. Mao, L. Wang*: Graphs with large generalized 3-edge-connectivity. ArXiv:1201.3699v1 [math.CO].
[11] *B. Liouville*: Sur la connectivité des produits de graphes. C. R. Acad. Sci., Paris, Sér. A *286* (1978), 363–365. (In French.) zbl MR
[12] *G. Sabidussi*: Graphs with given group and given graph-theoretical properties. Can. J. Math. *9* (1957), 515–525. zbl MR
[13] *N. A. Sherwani*: Algorithms for VLSI Physical Design Automation. Kluwer Academic Publishers, Boston, 1999. zbl
[14] *S. Špacapan*: Connectivity of Cartesian products of graphs. Appl. Math. Lett. *21* (2008), 682–685. zbl MR
[15] *J.-M. Xu, C. Yang*: Connectivity of Cartesian product graphs. Discrete Math. *306* (2006), 159–165. zbl MR

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