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Connectedness of some rings of quotients of $C(X)$ with the m-topology

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Abstract. In this article we define the m-topology on some rings of quotients of $C(X)$. Using this, we equip the classical ring of quotients $q(X)$ of $C(X)$ with the m-topology and we show that $C(X)$ with the r-topology is in fact a subspace of $q(X)$ with the m-topology. Characterization of the components of rings of quotients of $C(X)$ is given and using this, it turns out that $q(X)$ with the m-topology is connected if and only if X is a pseudocompact almost P-space, if and only if $C(X)$ with r-topology is connected. We also observe that the maximal ring of quotients $Q(X)$ of $C(X)$ with the m-topology is connected if and only if X is finite. Finally for each point x, we introduce a natural ring of quotients of $C(X)/O_x$ which is connected with the m-topology.

Keywords: r-topology; m-topology; almost P-space; pseudocompact space; component; classical ring of quotients of $C(X)$

Classification: Primary 54C35; Secondary 54C40

1. Introduction

In this article, X stands for a completely regular Hausdorff space, $C(X)$ for the ring of all real valued continuous functions on X and $C^*(X)$ denotes the subring of $C(X)$ consisting of bounded functions. If $C(X) = C^*(X)$, we say that X is pseudocompact. For each $f \in C(X)$, $Z(f) = \{x \in X : f(x) = 0\}$ is called the zero-set of f. $X \setminus Z(f)$ is called the cozero-set of f, denoted by $\mathrm{coz}(f)$ and $\mathrm{cl}_X\mathrm{coz}(f)$ is called the support of f. Regular functions in $C(X)$ are non-zero divisors, and units in $C(X)$ are invertible functions. It is easy to see that $r \in C(X)$ is regular if and only if $\mathrm{int}_X Z(r) = \emptyset$ (or equivalently, $\mathrm{coz}(r)$ is dense in X) and $u \in C(X)$ is a unit if and only if $Z(f) = \emptyset$. The set of all units and the set of all regular elements of $C(X)$ are denoted by $U(X)$ and $r(X)$ respectively and we refer the reader to [4], [6], [16] and [17] for undefined terms and notations.

The m-topology on $C(X)$ was first introduced in [10] by taking sets of the form

$$B(f,u) = \{g \in C(X) : |f(x) - g(x)| < u(x), \forall x \in X\}, \quad u \in U^+(X)$$

as a base for neighborhood system at f for each $f \in C(X)$, where $U^+(X)$ is the set of all positive units in $C(X)$. The ring $C(X)$ equipped with the m-topology is

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denoted by $C_m(X)$ which is a Hausdorff topological ring. Next, the m-topology on $C(X)$ is also studied in [1], [7], [12] and [14]. In [8] the authors have defined a finer topology, namely the r-topology on $C(X)$, based on nonnegative regular elements of $C(X)$ instead of positive units. In that article, the r-topology on $C(X)$ is defined by taking sets of the form

$$R(f,r) = \{g \in C(X) : |f(x) - g(x)| < r(x), \forall x \in coz(r)\}, \quad r \in r^+(X),\ f \in C(X)$$

as a base for the topology, where $r^+(X)$ is the set of all nonnegative regular elements of $C(X)$. $C(X)$ endowed with the r-topology is denoted by $C_r(X)$ which is a topological ring. It is easy to see that the r-topology coincides with the m-topology on $C(X)$ if and only if X is an almost P-space (a space in which every nonempty G_δ-set has a nonempty interior), see also Theorem 1.8 in [8].

Now, as in [14], we let $\mathfrak{F}$ be a filter of subsets of a topological space X and $P(X,\mathfrak{F})$ be the set of all real-valued continuous functions with domains in $\mathfrak{F}$. If $f, g \in P(X,\mathfrak{F})$, we define an equivalence relation $\sim$ on $P(X,\mathfrak{F})$ by $f \sim g$ if and only if f agrees with g on a member of $\mathfrak{F}$ or equivalently, $\{x \in D(f) \cap D(g) : f(x) = g(x)\} \in \mathfrak{F}$, where $D(h)$ means the domain of h. If for each f and g in $P(X,\mathfrak{F})$, we define $f+g$ and $f \cdot g$ pointwise in $D(f) \cap D(g)$, then $P(X,\mathfrak{F})/\sim$ or briefly $P(X,\mathfrak{F})$ is clearly a ring with identity. In addition, for each $f, g \in P(X,\mathfrak{F})$, we define the partial ordering $f \leq g$ if and only if $f(x) \leq g(x)$ for all x in some member of $\mathfrak{F}$ contained in $D(f) \cap D(g)$ and $f \wedge g$ by $(f \wedge g)(x) = \inf\{f(x), g(x)\}$ for each x in some member of $\mathfrak{F}$ contained in $D(f) \cap D(g)$. Thus $P(X,\mathfrak{F})$ becomes a lattice ordered ring.

Now we define a topology on $P(X,\mathfrak{F})$ whose base is the collection of all sets of the form

$$N(f,e) = \{g \in P(X,\mathfrak{F}) : |f-g| < e \text{ on some } F \in \mathfrak{F}\}$$

where $f \in P(X,\mathfrak{F})$ and $e \in P^+(X,\mathfrak{F}) = \{f \in P(X,\mathfrak{F}) : f > 0\}$ are arbitrary, see also [14] (note that $f > 0$ means that $f(x) > 0$ for all $x \in F$, for some $F \in \mathfrak{F}$).

Whenever $\mathfrak{B}$ is a base for the filter $\mathfrak{F}$, then $P(X,\mathfrak{F}) = P(X,\mathfrak{B})$. In fact if $f \in P(X,\mathfrak{B})$, then clearly $f \in P(X,\mathfrak{F})$ and if $f \in P(X,\mathfrak{F})$, then $f \in C(F)$ for some $F \in \mathfrak{F}$. But F contains some $B \in \mathfrak{B}$ and since $f|_B$ agrees with f on $B \in \mathfrak{B}$, we have $f|_B = f$, i.e., $f \sim f|_B \in P(X,\mathfrak{B})$. Now, if for each $f \in P(X,\mathfrak{F})$, we let the set $N'(f,e) = \{g \in P(X,\mathfrak{B}) : |f-g| < e \text{ on some } B \in \mathfrak{B}\}$, then clearly the topology on $P(X,\mathfrak{F})$ whose base is the set $\{N^{'}(f,e) : f \in P(X,\mathfrak{F}) \text{ and } e \in P^+(X,\mathfrak{B})\}$, coincides with the latter topology on $P(X,\mathfrak{F})$. We call this topology the m-topology, and $P(X,\mathfrak{F})$ equipped with the m-topology is denoted by $P_m(X,\mathfrak{F})$.

It is not hard to see that

$$+\Big(N(f,\frac{e}{2}) \times N(g,\frac{e}{2})\Big) \subseteq N(f+g,e)$$
$$\cdot\,(N(f,d) \times N(g,d)) \subseteq N(fg,e)$$

where $1 > d \in P^+(X, \mathfrak{F})$ and $(1 + |f| + |g|)d < e$. This means that $P_m(X, \mathfrak{F})$ is a topological ring.

It is well-known that whenever $\mathfrak{F}$ is considered as a filter with the base consisting of all open dense subsets of X or whenever the collection of all dense cozero-sets is considered as a base for the filter $\mathfrak{F}$, then $P(X, \mathfrak{F})$ is in fact the maximal ring of quotients of $C(X)$ and the classical ring of quotients of $C(X)$ respectively, see [5]. These rings are denoted by $Q(X)$ and $q(X)$ respectively and we denote these rings with the m-topology by $Q_m(X)$ and $q_m(X)$. We show that $C(X)$ with r-topology ($C_r(X)$) is in fact a subspace of $q_m(X)$ and using this, we characterize the spaces X for which $C_r(X)$ is connected.

Proposition 1.1. *If $\mathfrak{F}$ is a filter of subsets of X with a base consisting of dense sets, then $P_m(X, \mathfrak{F})$ is Hausdorff. In particular, $Q_m(X)$ and $q_m(X)$ are Hausdorff.*

Proof: Let $f, g \in P_m(X, \mathfrak{F})$ and $f \neq g$. This implies that there exists $x_0 \in D = D(f) \cap D(g)$ such that $f(x_0) \neq g(x_0)$. If we take $|f(x_0) - g(x_0)| = \alpha > 0$, then we have $N(f, \frac{\alpha}{4}) \cap N(g, \frac{\alpha}{4}) = \emptyset$. In fact, if $h \in N(f, \frac{\alpha}{4}) \cap N(g, \frac{\alpha}{4})$, then $|h - f| < \frac{\alpha}{4}$ and $|h - g| < \frac{\alpha}{4}$ on dense open sets D_1 and D_2 respectively and therefore $|f - g| < \frac{\alpha}{2}$ on dense open set $D_1 \cap D_2$. But $D_1 \cap D_2$ is dense in D, so $|f - g| \leq \frac{\alpha}{2}$ on D. Since $x_0 \in D$, we have $\alpha = |f(x_0) - g(x_0)| \leq \frac{\alpha}{2}$, a contradiction. $\square$

Remark 1.2. The converse of the above proposition is not true, i.e., whenever $P_m(X, \mathfrak{F})$ is Hausdorff, it is not necessary that $\mathfrak{F}$ has a base with dense elements. For instance, if $a \in X$, $A \subseteq X$ and we take $\mathcal{F}_a = \{F \subseteq X : a \in F\}$ and $\mathcal{F}_A = \{F \subseteq X : A \subseteq F\}$, then $P_m(X, \mathcal{F}_a)$ and $P_m(X, \mathcal{F}_A)$ are both Hausdorff but elements of $\mathcal{F}_a$ and $\mathcal{F}_A$ are not necessarily dense. In fact $P_m(X, \mathcal{F}_a) = \mathbb{R}$ with usual topology and $P_m(X, \mathcal{F}_A) = C_m(A)$ which are Hausdorff. If we also consider a fixed z-ultrafilter $\mathcal{A}_p$ for some $p \in X$, again we have $P_m(X, \mathcal{A}_p) = \mathbb{R}$.

In the above remark, $\mathcal{F}_a$ and $\mathcal{A}_p$ are in fact a fixed ultrafilter and a fixed z-ultrafilter respectively. We observe in the following result that not only for a fixed ultrafilter but for each ultrafilter $\mathfrak{F}$, fixed or free, $P_m(X, \mathfrak{F})$ is Hausdorff.

Proposition 1.3. *If $\mathfrak{F}$ is an ultrafilter of subsets of X, then $P_m(X, \mathfrak{F})$ is Hausdorff.*

Proof: Let $f, g \in P_m(X, \mathfrak{F})$ and $f \neq g$. Hence for every $F \in \mathfrak{F}$, there exists $x \in F$ such that $f(x) \neq g(x)$. This implies that the set $A = \{x \in D(f) \cap D(g) : f(x) \neq g(x)\}$ is a subset of X which intersects each member of $\mathfrak{F}$. But $\mathfrak{F}$ is an ultrafilter, then $A \in \mathfrak{F}$ and therefore $|f - g|$ is unit on A, consequently $|f - g| \in P^+(X, \mathfrak{F})$. Now $N(f, \frac{|f-g|}{2})$ and $N(g, \frac{|f-g|}{2})$ are disjoint open balls containing f and g respectively, i.e., $P_m(X, \mathfrak{F})$ is Hausdorff. $\square$

Using above propositions, we observe that for a filter $\mathfrak{F}$ of subsets of X with a base consisting dense sets and also for an ultrafilter $\mathfrak{F}$ of subsets of X, $P_m(X, \mathfrak{F})$ is completely regular because, it is a topological ring.

An element $f \in P(X, \mathfrak{F})$ is called bounded if f is bounded on some member of $\mathfrak{F}$ contained in $D(f)$. We denote by $P^*(X, \mathfrak{F})$, the set of all bounded elements of

$P(X,\mathfrak{F})$. For example, if $X = \mathbb{R}$ and we consider the Frechet filter $\mathfrak{F}$ with base $\{[a,\infty) : a \in \mathbb{R}\}$, then the identity function $i \in P(X,\mathfrak{F})$ is not bounded. In fact, in this case $P^*(X,\mathfrak{F}) = \{f \in C(\mathbb{R}) : f|_{[a,\infty)}$ is bounded for some $a \in \mathbb{R}\}$. $P^*(X,\mathfrak{F})$ may coincide with $P(X,\mathfrak{F})$. For instance if $\mathfrak{U}$ is the filter of neighborhoods of 0 in $\mathbb{R}$, then $P^*(\mathbb{R},\mathfrak{U}) = P(\mathbb{R},\mathfrak{U})$ for, if $f \in C(G)$ for some $G \in \mathfrak{U}$, there exists $\varepsilon > 0$ such that $[-\varepsilon,\varepsilon] \subseteq G$ and since f is bounded on $[-\varepsilon,\varepsilon]$, $f \in P^*(X,\mathfrak{U})$. Clearly, $P^*(X,\mathfrak{F})$ is a subring of $P(X,\mathfrak{F})$. We also observe in the following result that $P^*(X,\mathfrak{F})$ is a closed-open subset of $P_m(X,\mathfrak{F})$.

Proposition 1.4. *$P^*(X,\mathfrak{F})$ is a closed-open subset of $P_m(X,\mathfrak{F})$. In particular, $Q^*(X)$ ($q^*(X)$) is a closed-open subset of $Q_m(X)$ ($q_m(X)$).*

PROOF: It is evident that whenever $f \in P^*(X,\mathfrak{F})$, then $N(f,1) \subseteq P^*(X,\mathfrak{F})$ and whenever $f \notin P^*(X,\mathfrak{F})$, then $N(f,1) \cap P^*(X,\mathfrak{F}) = \emptyset$. □

Remark 1.5. Whenever $P(X,\mathfrak{F}) = P^*(X,\mathfrak{F})$, then the members of $\mathfrak{F}$ are not necessarily pseudocompact. For example, if $\mathfrak{U}$ is the filter of neighborhoods of 0 on $\mathbb{R}$, then clearly $(-\varepsilon,\varepsilon) \in \mathfrak{U}$ is not pseudocompact but we have already observed that $P(\mathbb{R},\mathfrak{U}) = P^*(\mathbb{R},\mathfrak{U})$. In the case where $\mathfrak{F}$ is a filter with dense elements, if $f \in \mathfrak{F}$ is bounded, then f is bounded on some dense subset of X contained in $D(f)$ which implies that f is bounded on $D(f)$. Hence, in this case, $P(X,\mathfrak{F}) = P^*(X,\mathfrak{F})$ if and only if each member of $\mathfrak{F}$ is pseudocompact.

2. $P_{\mathfrak{a}}$-spaces

For every infinite cardinal number $\mathfrak{a}$, we define

$$r_{\mathfrak{a}}(X) = \{g \in r(X) : |Z(g)| \leq \mathfrak{a}\}.$$

We also define $r_f(X) = \{g \in r(X) : |Z(g)| < \infty\}$. Corresponding to each infinite cardinal number $\mathfrak{a}$, we consider a filter $\mathfrak{F}_{\mathfrak{a}}$ on X with base $\mathfrak{B}_{\mathfrak{a}} = \{X \setminus Z(g) : g \in r_{\mathfrak{a}}(X)\}$ and we denote by $\mathfrak{F}_f$, a filter on X with base $\mathfrak{B}_f = \{X \setminus Z(g) : g \in r_f(X)\}$. Whenever $|X| = \mathfrak{b}$, then clearly $r_{\mathfrak{b}}(X) = r(X)$ and for any infinite cardinal numbers $\mathfrak{a} \leq \mathfrak{d} \leq \mathfrak{b}$, we have the following chain

$$U(X) \subseteq r_f(X) \subseteq r_{\mathfrak{a}}(X) \subseteq r_{\mathfrak{d}}(X) \subseteq r_{\mathfrak{b}}(X) = r(X).$$

For each infinite cardinal number $\mathfrak{a}$, we have a ring $P(X,\mathfrak{F}_{\mathfrak{a}})$ which is a ring of quotients of $C(X)$ since, $C(X) \subseteq P(X,\mathfrak{F}_{\mathfrak{a}}) \subseteq q(X) \subseteq Q(X)$. As we defined before, we consider the m-topology on $P(X,\mathfrak{F}_{\mathfrak{a}})$ by taking sets of the form $N_{\mathfrak{a}}(f,e) = \{g \in P(X,\mathfrak{F}_{\mathfrak{a}}) : |f-g| < e$ on some $F \in \mathfrak{F}_{\mathfrak{a}}\}$ as its base, where $f \in P(X,\mathfrak{F}_{\mathfrak{a}})$ and $e \in P^+(X,\mathfrak{F}_{\mathfrak{a}})$. $P(X,\mathfrak{F}_{\mathfrak{a}})$ with the m-topology is denoted by $P_m(X,\mathfrak{F}_{\mathfrak{a}})$. For an infinite cardinal number $\mathfrak{a}$, we also define a topology, namely $r_{\mathfrak{a}}$-topology, on $C(X)$ by taking sets of the form

$$R_{\mathfrak{a}}(f,u) = \{g \in C(X) : |f(x) - g(x)| < u(x), \forall x \in \operatorname{coz}(u)\}, u \in r_{\mathfrak{a}}^+(X), f \in C(X)$$

as a base for the topology, where $r_{\mathfrak{a}}^+(X)$ is the set of all nonnegative elements of $r_{\mathfrak{a}}(X)$. We denote $C(X)$ with $r_{\mathfrak{a}}$-topology by $C_{r_{\mathfrak{a}}}(X)$. It is easy to see that

$C_{r_{\mathfrak{a}}}(X)$ is a topological ring and whenever $\mathfrak{a} \geq |X|$, then the $r_{\mathfrak{a}}$-topology coincides with the r-topology and $P_m(X, \mathfrak{F}_{\mathfrak{a}}) = q_m(X)$.

We call X a $P_{\mathfrak{a}}$-space if each member of $\mathfrak{B}_{\mathfrak{a}}$ is C-embedded in X. In fact a space X is a $P_{\mathfrak{a}}$-space if every dense cozero-set $X \setminus Z(f)$ with $|Z(f)| \leq \mathfrak{a}$ is C-embedded in X. We also call a space X a P_f-space if every cofinite dense cozero-set is C-embedded in X. Note that a similar notation namely "F_α-space" is used in the literature to refer to various sets and C^*-embedding, see [2] and [13] for instance. To give some characterizations of $P_{\mathfrak{a}}$-spaces, we need the following lemmas.

Lemma 2.1. *If $\mathfrak{F}$ is a filter of dense subsets of X, then $P(X, \mathfrak{F}) = C(X)$ if and only if each member of $\mathfrak{F}$ is C-embedded in X.*

Proof: If $P(X, \mathfrak{F}) = C(X)$ and $F \in \mathfrak{F}$, then for each $f \in C(F)$, there exists $g \in C(X)$ such that $f \sim g$, i.e., $f = g$ on some $E \in \mathfrak{F}$ contained in F. Since E is dense in F, we have $f = g$ on F which means that g is an extension of f in $C(X)$, i.e., F is C-embedded. Conversely, suppose that every element of $\mathfrak{F}$ is C-embedded. If $f \in P(X, \mathfrak{F})$, then $f \in C(F)$ for some $F \in \mathfrak{F}$ and hence by our hypothesis, there exists $g \in C(X)$ such that $f = g$ on F. This shows that $f \sim g \in C(X)$, i.e., $P(X, \mathfrak{F}) = C(X)$. □

Lemma 2.2. *If $\mathfrak{a} < \mathfrak{b}$ are two infinite cardinal numbers, then the following statements are equivalent.*

(1) $r_{\mathfrak{a}}(X) = r_{\mathfrak{b}}(X)$.
(2) $C_{r_{\mathfrak{a}}}(X) = C_{r_{\mathfrak{b}}}(X)$.
(3) *If $f \in C(X)$ and $\mathfrak{a} < |Z(f)| \leq \mathfrak{b}$, then $\mathrm{int}_X Z(f) \neq \emptyset$.*

Proof: Clearly (1) implies (2). Now let (2) holds, we show that $r_{\mathfrak{b}}(X) \subseteq r_{\mathfrak{a}}(X)$ ($r_{\mathfrak{a}}(X) \subseteq r_{\mathfrak{b}}(X)$ is evident). Suppose, on the contrary, that $r_{\mathfrak{b}}(X) \nsubseteq r_{\mathfrak{a}}(X)$, hence there exists $r \in r_{\mathfrak{b}}(X) \setminus r_{\mathfrak{a}}(X)$. Since $R_{\mathfrak{b}}(0, |r|)$ is a neighborhood of 0 in $C_{r_{\mathfrak{b}}}(X)$ and hence in $C_{r_{\mathfrak{a}}}(X)$, there exists $t \in r_{\mathfrak{a}}^+(X)$ such that $R_{\mathfrak{a}}(0, t) \subseteq R_{\mathfrak{b}}(0, |r|)$. But $\frac{t}{2} \in R_{\mathfrak{a}}(0, t) \subseteq R_{\mathfrak{b}}(0, |r|)$ implies that $\frac{t(x)}{2} < |r(x)|$ for each $x \in \mathrm{coz}(r)$. Since $\mathrm{coz}(r)$ is dense in X, we have $\frac{t(x)}{2} \leq |r(x)|$ for each $x \in X$, hence $Z(r) \subseteq Z(t)$. Therefore $|Z(r)| \leq |Z(t)|$. But $r \in r_{\mathfrak{b}}(X)$ implies that $|Z(r)| \leq \mathfrak{b}$, $\mathrm{int}_X Z(r) = \emptyset$ and $r \notin r_{\mathfrak{a}}(X)$ implies that $|Z(r)| > \mathfrak{a}$ for $\mathrm{int}_X Z(r) = \emptyset$. Hence $\mathfrak{a} < |Z(r)| \leq \mathfrak{b}$ which contradicts $|Z(r)| \leq |Z(t)| \leq \mathfrak{a}$. Hence $r_{\mathfrak{b}}(X) \subseteq r_{\mathfrak{a}}(X)$ and we are through. Now we show that (1) and (3) are also equivalent. Let (1) apply and $\mathfrak{a} < |Z(f)| \leq \mathfrak{b}$. If $\mathrm{int}_X Z(f) = \emptyset$, then $f \in r_{\mathfrak{b}}(X) = r_{\mathfrak{a}}(X)$ which implies that $|Z(f)| \leq \mathfrak{a}$, a contradiction. Conversely suppose that (3) holds and $r \in r_{\mathfrak{b}}(X)$. Then $|Z(r)| \leq \mathfrak{b}$ and $\mathrm{int}_X Z(r) = \emptyset$. If $r \notin r_{\mathfrak{a}}(X)$, then $|Z(r)| > \mathfrak{a}$ for $\mathrm{int}_X Z(r) = \emptyset$. Therefore $\mathfrak{a} < |Z(r)| \leq \mathfrak{b}$ and $\mathrm{int}_X Z(r) = \emptyset$ which contradicts part (3). □

Corollary 2.3. *For an infinite cardinal number $\mathfrak{a}$, the following statements hold.*

(1) *$U(X) = r_{\mathfrak{a}}(X)$ if and only if every nonempty zero-set with cardinal number less than or equal to $\mathfrak{a}$ has a nonempty interior.*

(2) $r_{\mathfrak{a}}(X) = r(X)$ *if and only if every zero-set with cardinal number greater than* $\mathfrak{a}$ *has a nonempty interior.*

Proposition 2.4. *For an infinite cardinal number* $\mathfrak{a}$, *the following statements are equivalent.*

(1) $U(X) = r_{\mathfrak{a}}(X)$.
(2) *Every nonempty* G_δ*-set with cardinal number less than or equal to* $\mathfrak{a}$ *has a nonempty interior.*
(3) *Every nonempty zero-set with cardinal number less than or equal to* $\mathfrak{a}$ *is regular closed.*
(4) $C_m(X) = C_{r_{\mathfrak{a}}}(X)$.
(5) X *is a* $P_{\mathfrak{a}}$*-space.*
(6) $C(X) = P(X, \mathfrak{F}_{\mathfrak{a}})$.

Proof: Equivalence of parts (1), (2) and (3) is evident by Corollary 2.3. Part (1) clearly implies part (4) and by Lemma 2.2, part (4) also implies part (1). Parts (1) and (5) are also equivalent. In fact, if $U(X) = r_{\mathfrak{a}}(X)$, then using part (1) of Corollary 2.3, X does not contain a dense cozero-set whose complement has cardinal number less than or equal to $\mathfrak{a}$ except X itself, so X is a $P_{\mathfrak{a}}$-space. Conversely, let X be a $P_{\mathfrak{a}}$-space and $g \in r_{\mathfrak{a}}(X) \setminus U(X)$. Hence $\frac{1}{g} \in C(\mathrm{coz}(g))$ and $Z(g) \neq \emptyset$. This implies that $\frac{1}{g}$ has no extension in $C(X)$, a contradiction. Finally parts (5) and (6) are equivalent by Lemma 2.1. □

By the following result, every compact F-space is a $P_{\aleph_0}$-space.

Proposition 2.5. *The following statements are equivalent.*

(1) X *is a* $P_{\aleph_0}$*-space.*
(2) X *is a* P_f*-space.*
(3) *Every cofinite cozero-set in* X *is* C*-embedded.*
(4) *Every* G_δ*-point in* X *is an isolated point.*
(5) *Every countable* G_δ*-set in* X *consists entirely of isolated points.*

Proof: Clearly (1) implies (2) and (2) implies (3). Part (3) also implies (4), in fact whenever $\{x\}$ is a G_δ-set, then it is a zero-set, say $Z(f)$. If x is not an isolated point, then $\frac{1}{f} \in C(X \setminus Z(f))$ has no extension in $C(X)$, which contradicts part (3). Now, suppose part (4) holds and G is a countable G_δ-set, say $G = \{a_1, a_2, \dots, a_n, \dots\}$. We claim that each a_i is an isolated point. Fix $a_i \in G$. For each $j \in \mathbb{N} \setminus \{i\}$, take an open set U_j containing a_i but not a_j. Clearly $\{a_i\} = \bigcap_{j\in\mathbb{N}\setminus\{i\}} U_j \cap G$ is a G_δ-point and hence a_i is an isolated point, by our hypothesis. Finally, if part (5) holds, by equivalence of parts (2) and (5) of Proposition 2.4, X is a $P_{\aleph_0}$-space. □

Corollary 2.6. *Every cocountable* (*cofinite*) *dense cozero-set in* X *is* C*-embedded if and only if every cocountable* (*cofinite*) *cozero-set in* X *is* C*-embedded.*

Proof: If every cocountable dense cozero-set in X is C-embedded, then X is a $P_{\aleph_0}$-space. Now let $\mathrm{coz}(f)$ be cocountable (not necessarily dense), then $Z(f)$ is a countable G_δ-set and hence each element of $Z(f)$ should be an isolated point by Proposition 2.5. Therefore $Z(f)$ is open, i.e., $\mathrm{coz}(f)$ is a closed-open set and hence it is C-embedded. □

Using Proposition 2.5 and Corollary 2.6, the following result is now evident.

Corollary 2.7. *Every cofinite cozero-set in X is C-embedded if and only if every cocountable cozero-set in X is C-embedded if and only if every G_δ-point in X is an isolated point.*

Example 2.8. By Proposition 2.5, every sequentially compact quasi F-space is a $P_{\aleph_0}$-space. Because, every G_δ-point in a sequentially compact space has a countable base and every point in a quasi F-space with a countable base is an isolated point, see Proposition 5.5 in [3]. Thus, if X is a quasi F-space, then βX is also a quasi F-space, by Theorem 5.1 in [3], so it is a $P_{\aleph_0}$-space. In particular, $\beta\Sigma$ is a $P_{\aleph_0}$-space, whereas it is not an almost P-space, see 4M in [6] for structure of Σ. More generally, for an infinite cardinal number $\mathfrak{a}$, let X be a $P_{\mathfrak{a}}$-space which is not an almost P-space. The largest cardinal number α exists such that $U(X) = r_\alpha(X)$. Now consider a cardinal number β with $\alpha < \beta \leq |X|$, then $U(X) \neq r_\beta(X)$ and this shows that X is a P_α-space but not a P_β-space. By part (4) of Proposition 2.5, we also note that X is a $P_{\aleph_0}$-space if and only if υX is.

Whenever a space X is a $P_{|X|}$-space, then $U(X) = r(X)$, by Proposition 2.4 and hence X will be an almost P-space. The converse is also true, i.e., every almost P-space X is a $P_{\mathfrak{a}}$-space for each infinite cardinal number $\mathfrak{a}$. Using Proposition 2.4, whenever $\beta > \alpha$ and X is a P_β-space, then it is also a P_α-space, but we already observed that the converse is not true. For another example, take $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$. By Theorem 3.3 in [15], the cardinality of each infinite zero-set in $\mathbb{N}^*$ is $2^{\mathfrak{c}}$ (note, every zero-set in $\mathbb{N}^*$ is closed in $\beta\mathbb{N}$ for $\mathbb{N}^*$ is closed in $\beta\mathbb{N}$). On the other hand, every nonempty zero-set in $\mathbb{N}$ has a nonempty interior, by Corollary 3.27 in [15], i.e., $\mathbb{N}^*$ is an almost P-space. Since every finite subset of $\mathbb{N}^*$ has an empty interior, the cardinality of every nonempty zero-set in $\mathbb{N}^*$ is $2^{\mathfrak{c}}$. Now consider the space $X = \mathbb{R} \times \mathbb{N}^*$ and let Z be a zero-set in $\mathbb{N}^*$. Then $\{0\} \times Z$ is a zero-set in X and $|\{0\} \times Z| = 2^{\mathfrak{c}}$. Since $\mathrm{int}_X(\{0\} \times Z) = \emptyset$, we have $U(X) \neq r_{2^{\mathfrak{c}}}(X)$ which means that X is not a $P_{2^{\mathfrak{c}}}$-space. But X has no zero-sets with $\mathfrak{c}$ points and this implies that $U(X) = r_{\mathfrak{c}}(X)$, i.e., X is a $P_{\mathfrak{c}}$-space.

In the above examples, for the space $X = \mathbb{R} \times \mathbb{N}^*$, we observed that $U(X) = r_{\mathfrak{c}}(X) \subsetneqq r_{2^{\mathfrak{c}}}(X) = r(X)$, i.e., X is a $P_{\mathfrak{c}}$-space but not a $P_{2^{\mathfrak{c}}}$-space. The space $\mathbb{R}$ of real numbers is not even P_f-space and moreover, we have the ascending chain $U(\mathbb{R}) \subsetneqq r_f(\mathbb{R}) \subsetneqq r_{\aleph_0}(\mathbb{R}) \subsetneqq r_{\aleph_1}(\mathbb{R}) = r(\mathbb{R})$ by Lemma 2.2. In fact nonempty finite subsets of $\mathbb{R}$ are zero-sets with empty interior, the set of natural numbers $\mathbb{N}$ is an infinite countable zero-set in $\mathbb{R}$ with empty interior, and the Cantor set in $\mathbb{R}$ is an uncountable zero-set with empty interior. In the following example, using an appropriate space X, we extend this ascending chain with any arbitrary length.

Example 2.9. For each infinite cardinal number α, we are going to construct an ascending chain with length α of rings of quotients of $C(X)$ contained in the classical ring of quotients $q(X)$ of $C(X)$. To see this, let S be an ascending chain of cardinal numbers with length α and for each $\mathfrak{a} \in S$, take a set $A_{\mathfrak{a}}$ with cardinality $\mathfrak{a}$. Put $T = \bigcup_{\mathfrak{a}\in S} A_{\mathfrak{a}}$. Let $Y = \mathbb{N} \cup \{\sigma\}$ be a one-point compactification of $\mathbb{N}$, $Y_t = \mathbb{N} \cup \{\sigma_t\}$ for each $t \in T$, be a copy of Y and consider the free union $X = \dot{\bigcup}_{t\in T} Y_t$. Clearly $N_t = \{\sigma_t : t \in T\}$ is the set of all non-almost P-points of X which is a zero-set in X. Assume that $|N_t| = \delta$. Hence every zero-set with empty interior has at most the cardinality δ (in the other words, if $|Z(f)| > \delta$, then $\mathrm{int}_X Z(f) \neq \emptyset$) and this means that $r_\delta(X) = r(X)$. Since every subset of N_t is also a zero-set in X for each $\mathfrak{a} \in S$, we may have a zero-set in X with cardinality $\mathfrak{a}$ whose interior is empty (for example $\{\sigma_a : a \in A_{\mathfrak{a}}\}$). This implies that for $\mathfrak{a}, \mathfrak{b} \in S$ and $\mathfrak{a} < \mathfrak{b} < \delta$, we have $r_{\mathfrak{a}}(X) \subsetneqq r_{\mathfrak{b}}(X) \subsetneqq r_\delta(X) = r(X)$. Therefore $\{r_{\mathfrak{a}}(X) : \mathfrak{a} \in S\}$ is an ascending chain with length α. Now using this, $P(X, \mathfrak{F}_{\mathfrak{a}}) \subsetneqq P(X, \mathfrak{F}_{\mathfrak{b}}) \subsetneqq P(X, \mathfrak{F}_\delta) = q(X)$ and hence $\{P(X, \mathfrak{F}_{\mathfrak{a}}) : \mathfrak{a} \in S\}$ is an ascending chain with length α of rings of quotients of $C(X)$ contained in $q(X)$. Moreover, $\{C_{r_{\mathfrak{a}}}(X) : \mathfrak{a} \in S\}$ is also an ascending chain with length α.

3. Connectedness of ring of quotients $P_m(X, \mathfrak{F}_{\mathfrak{a}})$ of $C(X)$

In [1], the authors have shown that the component of 0 in $C_m(X)$ is the ideal $C_\psi(X)$ consisting of all functions in $C(X)$ with pseudocompact support, see [11] for more details about the ideal $C_\psi(X)$. The set

$$\{f \in C(X) : fg \ \text{ is bounded for each } \ g \in C(X)\}$$

is a different characterization of $C_\psi(X)$ which is given in [9]. This is equivalent to saying that $C_\psi(X)$ is the set of all functions $f \in C(X)$ such that fe is bounded for each $e \in U^+(X)$. In imitation of [1], it seems that the component of 0 in $P_m(X, \mathfrak{F})$ should be of the form

$$\begin{aligned} P_\psi(X, \mathfrak{F}) &= \{f \in P(X, \mathfrak{F}) : ft \ \text{ is bounded}, \forall\, t \in P(X, \mathfrak{F})\} \\ &= \{f \in P(X, \mathfrak{F}) : fe \ \text{ is bounded}, \forall\, e \in P^+(X, \mathfrak{F})\}. \end{aligned}$$

We show that $P_\psi(X, \mathfrak{F})$ is indeed the component of 0 in $P_m(X, \mathfrak{F})$.

By the definition of $P_\psi(X, \mathfrak{F})$, we have $P_\psi(X, \mathfrak{F}) \subseteq P^*(X, \mathfrak{F})$. Moreover, for a filter $\mathfrak{F}$ with dense elements, the ideal $P_\psi(X, \mathfrak{F})$ is in fact the set of functions in $P_m(X, \mathfrak{F})$ with pseudocompact support, in the sense that $f \in P_\psi(X, \mathfrak{F})$ if and only if $f|_A \in C_\psi(A)$, for all $A \in \mathfrak{F}$ contained in $D(f)$. In fact if $f \in P_\psi(X, \mathfrak{F})$ and $D(f) \supseteq A \in \mathfrak{F}$, then for each $e \in C(A)$, fe is bounded on some $B \in \mathfrak{F}$ and $B \subseteq A \cap D(f) = A$. But B is dense in A, so fe is bounded on A and hence $f|_A e$ is bounded which means that $f|_A \in C_\psi(A)$. Conversely, suppose that $f \in P(X, \mathfrak{F})$ such that $f|_A \in C_\psi(A)$, for all $A \in \mathfrak{F}$ contained in $D(f) \in \mathfrak{F}$. If $t \in P(X, \mathfrak{F})$, then $f|_{D(f)\cap D(t)} \in C_\psi(D(f) \cap D(t))$. So ft is bounded on $D(f) \cap D(t)$ and this means that ft is bounded, i.e., $f \in P_\psi(X, \mathfrak{F})$.

Using the characterization of the component of 0 in $P_m(X,\mathfrak{F})$, we may characterize the connectedness of some rings of quotients of $C(X)$. First, it is easy to see that $P_\psi(X,\mathfrak{F})$ is an ideal of $P(X,\mathfrak{F})$, we cite this result without proof.

Proposition 3.1. *For a filter $\mathfrak{F}$ on X, $P_\psi(X,\mathfrak{F})$ is an ideal of $P(X,\mathfrak{F})$.*

To characterize the components of $P_m(X,\mathfrak{F})$, we need the following basic lemma.

Lemma 3.2. *If $f \in P(X,\mathfrak{F})$, then the function $\varphi_f : \mathbb{R} \to P_m(X,\mathfrak{F})$ defined by $\varphi_f(r) = rf$ for each $r \in \mathbb{R}$, is continuous if and only if $f \in P_\psi(X,\mathfrak{F})$.*

Proof: Let $f \in P_\psi(X,\mathfrak{F})$, $e \in P^+(X,\mathfrak{F})$, and $r \in \mathbb{R}$. Hence $\frac{1}{e} \in P^+(X,\mathfrak{F})$ and therefore, $\frac{f}{e}$ is bounded. Suppose that $|\frac{f}{e}| \leq M$ for some integer M. Now, $\varphi_f(r - \frac{1}{M}, r + \frac{1}{M}) \subseteq N(rf, e)$. In fact, whenever $s \in \mathbb{R}$ and $|s - r| < \frac{1}{M}$, then $|sf - rf| = |s - r||f| < \frac{1}{M}|f| \leq e$. Conversely, let $f \in P(X,\mathfrak{F})$ and φ_f is continuous, so it is continuous at 0. Hence for each $g \in P(X,\mathfrak{F})$, there exists a positive real number ε such that $\varphi_f(-\varepsilon,\varepsilon) \subseteq N(0, \frac{1}{1+|g|})$. This implies that $|\frac{\varepsilon}{2}f| < \frac{1}{1+|g|}$ on some $D \in \mathfrak{F}$ contained in $D(f) \cap D(g)$. Thus $|fg| \leq |f(1+|g|)| < \frac{2}{\varepsilon}$ on $D \in \mathfrak{F}$, i.e., fg is bounded and therefore $f \in P_\psi(X,\mathfrak{F})$. □

Proposition 3.3. *The ideal $P_\psi(X,\mathfrak{F})$ is the component of 0 in $P_m(X,\mathfrak{F})$.*

Proof: First by Lemma 3.2, the function $\varphi_f : \mathbb{R} \to P_m(X,\mathfrak{F})$ is continuous for each $f \in P_\psi(X,\mathfrak{F})$. Hence $\varphi_f(\mathbb{R})$ is connected for each $f \in P_\psi(X,\mathfrak{F})$. But $P_\psi(X,\mathfrak{F}) = \bigcup_{f \in P_\psi(X,\mathfrak{F})} \varphi_f(\mathbb{R})$ implies that $P_\psi(X,\mathfrak{F})$ is connected (note that $\bigcap_{f \in P_\psi(X,\mathfrak{F})} \varphi_f(\mathbb{R}) \neq \emptyset$, in fact it contains 0). Next suppose that J is the component of 0 in $P_m(X,\mathfrak{F})$, so J is an ideal of $P(X,\mathfrak{F})$. We show that $J \subseteq P_\psi(X,\mathfrak{F})$. Let there exist $f \in J \setminus P_\psi(X,\mathfrak{F})$. Hence there is $t \in P(X,\mathfrak{F})$ such that ft is not bounded, i.e., $ft \notin P^*(X,\mathfrak{F})$. Now consider the sets $J \cap P^*(X,\mathfrak{F})$ and $J \setminus P^*(X,\mathfrak{F})$. By Proposition 1.4, these two sets are open in J and since $0 \in J \cap P^*(X,\mathfrak{F})$ and $ft \in J \setminus P^*(X,\mathfrak{F})$, they are nonempty disjoint open subsets of J. This implies that J is disconnected, a contradiction. Therefore, $J \subseteq P_\psi(X,\mathfrak{F})$ and this means that $P_\psi(X,\mathfrak{F})$ is the largest connected ideal containing 0, so $P_\psi(X,\mathfrak{F})$ is the component of 0 in $P_m(X,\mathfrak{F})$. □

Remark 3.4. The ideal $P_\psi(X,\mathfrak{F})$ is the quasicomponent of 0 in $P_m(X,\mathfrak{F})$ as well. We recall that the intersection of all closed-open subsets of a space X containing $x \in X$ is called the quasicomponent of x. It is well known that quasicomponent of x contains the component of x. Hence whenever the set K is considered as the quasicomponent of 0 in $P_m(X,\mathfrak{F})$, then $P_\psi(X,\mathfrak{F}) \subseteq K$. We show that $K \subseteq P_\psi(X,\mathfrak{F})$ also holds. First, for each $e \in P^+(X,\mathfrak{F})$, take $A_e = \{f \in P(X,\mathfrak{F}) : ef \text{ is bounded }\}$. Next each A_e is a closed-open subset of $P(X,\mathfrak{F})$, in fact for every $f \in A_e$, we have $N(f, \frac{1}{e}) \subseteq A_e$ and for each $f \notin A_e$, we have $N(f, \frac{1}{e}) \cap A_e = \emptyset$. But $K \subseteq \bigcap_{e \in P^+(X,\mathfrak{F})} A_e = P_\psi(X,\mathfrak{F})$ and we are through.

Part (b) of the following corollary is also given in Corollary 3.3 of [1].

Corollary 3.5. (a) *The ideal* $q_\psi(X) = \{f \in q(X) : ef$ *is bounded,* $\forall\, e \in q(X)\}$ *is the component of* 0 *in* $q_m(X)$ *and is the quasicomponent of* 0 *in* $q_m(X)$ *as well.*

(b) *If* $\mathfrak{F} = \{X\}$, *then* $P_\psi(X, \mathfrak{F}) = C_\psi(X)$ *is the component and the quasi-component of* 0 *in* $P_m(X, \mathfrak{F}) = C_m(X)$.

To prove the main theorem of this section, we also need the following result which shows that $C_{r_\mathfrak{a}}(X)$ is in fact a subspace of $P_m(X, \mathfrak{F}_\mathfrak{a})$.

Proposition 3.6. *For any infinite cardinal number* $\mathfrak{a}$, *the identity function* $i : C_{r_\mathfrak{a}}(X) \to P_m(X, \mathfrak{F}_\mathfrak{a})$ *carries* $C_{r_\mathfrak{a}}(X)$ *homeomorphically onto* $C(X)$ *as a subspace of* $P_m(X, \mathfrak{F}_\mathfrak{a})$.

Proof: Let $f \in C_{r_\mathfrak{a}}(X)$ and $e \in P^+(X, \mathfrak{F}_\mathfrak{a})$, then $0 < e \in C(\mathrm{coz}(r))$ for some $r \in r_\mathfrak{a}^+(X)$. Without loss of generality, we let e be bounded, i.e., $e \in C^*(\mathrm{coz}(r))$. Now define

$$s(x) = \begin{cases} (1 \wedge r)(x)e(x), & x \in X \setminus Z(r), \\ 0, & x \in Z(r). \end{cases}$$

Clearly $s \geq 0$ and $\mathrm{coz}(s) = \mathrm{coz}(r)$. To see that $s \in r_\mathfrak{a}^+(X)$, we must show that s is continuous. To see this, it is enough to show that s is continuous at each $x \in Z(r)$. Since $e \in C^*(\mathrm{coz}(r))$, there exists a positive number k such that $|e(y)| < k$, for all $y \in X \setminus Z(r)$. On the other hand, r is continuous at $x \in Z(r)$, so given $\epsilon > 0$, there exists a neighborhood G of x in X such that $|r(y)| < \frac{\epsilon}{k}$, for all $y \in G$. Therefore $|s(y) - s(x)| = |s(y)| \leq |e(y)r(y)| < k|r(y)| < \epsilon$ for all $y \in G \cap \mathrm{coz}(r)$, i.e., s is continuous at x.

We claim that $i(R_\mathfrak{a}(f, s)) \subseteq N_\mathfrak{a}(f, e)$ and this means that i is continuous. In fact whenever $g \in i(R_\mathfrak{a}(f, s))$, then $g \in R_\mathfrak{a}(f, s)$ which implies that $|g - f| < s$ on $\mathrm{coz}(s) = \mathrm{coz}(r)$. But $s < e$ implies that $|g - f| < e$ on $\mathrm{coz}(r) = D(e)$, i.e., $g \in N_\mathfrak{a}(f, e)$. Now we show that $i : C_{r_\mathfrak{a}}(X) \to C(X)$ is open, where $C(X)$ is the subspace of $P_m(X, \mathfrak{F}_\mathfrak{a})$. To see this, it is enough to prove that $i(R_\mathfrak{a}(f, r))$ is open in $C(X)$ as a subspace of $P_m(X, \mathfrak{F}_\mathfrak{a})$ for each $f \in C(X)$ and $r \in r_\mathfrak{a}^+(X)$. Let $g \in i(R_\mathfrak{a}(f, r))$, then $g \in R_\mathfrak{a}(f, r)$ which implies that $|g - f| < r$ on $\mathrm{coz}(r)$. Since $\mathrm{coz}(r - |f - g|) \supseteq \mathrm{coz}(r)$, we have $N_\mathfrak{a}(f, r - |f - g|) \cap C(X) \subseteq i(R_\mathfrak{a}(f, r))$. In fact if $h \in N_\mathfrak{a}(f, r - |f - g|) \cap C(X)$, then $|f - h| < r - |f - g|$ on $\mathrm{coz}(r - |f - g|) \cap D(f) \cap D(h) = \mathrm{coz}(r - |f - g|)$. Consequently, $|g - f| < r$ on $\mathrm{coz}(r - |f - g|)$ and hence on $\mathrm{coz}(r)$, i.e., $h \in i(R_\mathfrak{a}(f, r))$ (note that $r - |f - g| \in r_\mathfrak{a}^+(X)$ for $\mathrm{coz}(r - |f - g|)$ contains $\mathrm{coz}(r)$ which is dense). $\square$

As a consequence of Proposition 3.6, we have the following result which states that $C_r(X)$ is a subspace of $q_m(X)$.

Corollary 3.7. *The identity function* $i : C_r(X) \to q_m(X)$ *carries* $C_r(X)$ *homeomorphically onto* $C(X)$ *as a subspace of* $q_m(X)$.

Theorem 3.8. *For each infinite cardinal number* $\mathfrak{a}$, *the following statements are equivalent.*

(a) $C_{r_{\mathfrak{a}}}(X)$ *is connected.*
(b) $P_m(X,\mathfrak{F}_{\mathfrak{a}})$ *is connected.*
(c) X *is a pseudocompact* $P_{\mathfrak{a}}$*-space.*

Proof: If $C_{r_{\mathfrak{a}}}(X)$ is connected, then $C(X)$ as a subspace of $P_m(X,\mathfrak{F}_{\mathfrak{a}})$ is connected by Proposition 3.6. Since $0 \in C(X)$, we have $C(X) \subseteq P_\psi(X,\mathfrak{F}_{\mathfrak{a}})$ for $P_\psi(X,\mathfrak{F}_{\mathfrak{a}})$ is the largest connected set in $P_m(X,\mathfrak{F}_{\mathfrak{a}})$ containing 0. Hence $1 \in P_\psi(X,\mathfrak{F}_{\mathfrak{a}})$, i.e., $P_\psi(X,\mathfrak{F}_{\mathfrak{a}}) = P_m(X,\mathfrak{F}_{\mathfrak{a}})$, so $P_m(X,\mathfrak{F}_{\mathfrak{a}})$ is connected.

Now suppose that $P_m(X,\mathfrak{F}_{\mathfrak{a}})$ is connected, then $P_\psi(X,\mathfrak{F}_{\mathfrak{a}}) = P(X,\mathfrak{F}_{\mathfrak{a}})$. But $P_\psi(X,\mathfrak{F}_{\mathfrak{a}}) \subseteq P^*(X,\mathfrak{F}_{\mathfrak{a}})$ implies that $P(X,\mathfrak{F}_{\mathfrak{a}}) = P^*(X,\mathfrak{F}_{\mathfrak{a}})$ and hence X is a pseudocompact space by Remark 1.5. To see that X is a $P_{\mathfrak{a}}$-space, let $r \in r_{\mathfrak{a}}(X)$, hence $\frac{1}{r} \in C(\operatorname{coz}(r))$. Therefore $\frac{1}{r} \in P(X,\mathfrak{F}_{\mathfrak{a}}) = P^*(X,\mathfrak{F}_{\mathfrak{a}})$ which implies that $\frac{1}{r}$ should be bounded, so r is unit, i.e., $U(X) = r_{\mathfrak{a}}(X)$. Now by Proposition 2.4, X is a $P_{\mathfrak{a}}$-space.

Finally, suppose that X is a pseudocompact $P_{\mathfrak{a}}$-space. Since X is a $P_{\mathfrak{a}}$-space, $C_{r_{\mathfrak{a}}}(X) = C_m(X)$ by Proposition 2.4. Now pseudocompactness of X implies that $C_m(X)$ is connected by Proposition 3.12 in [1] and this shows that $C_{r_{\mathfrak{a}}}(X)$ is also connected. □

Corollary 3.9. *If $\alpha \leq \beta$ are two cardinal numbers and $P_m(X,\mathfrak{F}_\beta)$ ($C_{r_\beta}(X)$) is connected, then $P_m(X,\mathfrak{F}_\alpha)$ ($C_{r_\alpha}(X)$) is also connected.*

The following proposition states another characterization for connectedness of $P_m(X,\mathfrak{F})$, where $\mathfrak{F}$ is an arbitrary filter on X. Using this proposition and Remark 1.5, whenever every member of a filter $\mathfrak{F}$ is dense, then $P_m(X,\mathfrak{F})$ is connected if and only if each member of $\mathfrak{F}$ is pseudocompact.

Proposition 3.10. *For each filter $\mathfrak{F}$, $P_m(X,\mathfrak{F})$ is connected if and only if $P(X,\mathfrak{F}) = P^*(X,\mathfrak{F})$.*

Proof: If $P_m(X,\mathfrak{F})$ is connected, then clearly $P(X,\mathfrak{F}) = P^*(X,\mathfrak{F})$, by Proposition 1.4. Conversely, let $P(X,\mathfrak{F}) = P^*(X,\mathfrak{F})$ and $f \in P(X,\mathfrak{F}) = P^*(X,\mathfrak{F})$. Since each $t \in P(X,\mathfrak{F})$ is bounded, ft is bounded, i.e., $f \in P_\psi(X,\mathfrak{F})$. This means that $P(X,\mathfrak{F}) = P_\psi(X,\mathfrak{F})$ and hence $P_m(X,\mathfrak{F})$ is connected. □

Corollary 3.11. *The following statements are equivalent.*
(a) $C_r(X)$ *is connected.*
(b) $q_m(X)$ *is connected.*
(c) X *is a pseudocompact almost* P*-space.*
(d) $q(X) = q^*(X) = C^*(X)$.
(e) *Every dense cozero-set in* X *is pseudocompact.*

By the following lemma and proposition, we observe that whenever $C_m(X)$ is totally disconnected (equivalently, if $\beta X \setminus \upsilon X$ is dense in βX, see [1]), then $P_m(X,\mathfrak{F}_{\mathfrak{a}})$ for each infinite cardinal number $\mathfrak{a}$, is totally disconnected. In particular, whenever $C_m(X)$ is totally disconnected, then so is $q_m(X)$.

Lemma 3.12. *Every $f \in P_\psi(X,\mathfrak{F}_{\mathfrak{a}})$ has an extension in $C_\psi(X) \subseteq C(X)$.*

Proof: Let $f \in P_\psi(X, \mathfrak{F}_\mathfrak{a})$. Then $f \in C(\mathrm{coz}(r))$ for some $r \in r_\mathfrak{a}(X)$. We define

$$\hat{f}(x) = \begin{cases} f(x), & x \in X \setminus Z(r), \\ 0, & x \in Z(r). \end{cases}$$

Since $f \in P_\psi(X, \mathfrak{F}_\mathfrak{a})$, $f\frac{1}{|r|}$ is bounded on $\mathrm{coz}(r)$ and hence there exists a positive integer M such that $|\frac{f}{r}| \leq M$ on $\mathrm{coz}(r)$. To prove that $\hat{f}$ is continuous, it is enough to show that $\hat{f}$ is continuous at each $x \in Z(r)$. Let $x_0 \in Z(r)$. Since r is continuous at x_0, given $\varepsilon > 0$, there exists an open set G containing x_0 such that $|r(x)| < \frac{\varepsilon}{M}$ for each $x \in G$. Now, for each $x \in \mathrm{coz}(r) \cap G$, we have $|\hat{f}(x) - \hat{f}(x_0)| = |\hat{f}(x)| = |f(x)| = |\frac{f(x)}{r(x)}||r(x)| < M\frac{\varepsilon}{M} = \varepsilon$, this means that $\hat{f}$ is continuous at x_0. On the other hand, if u is a positive unit in $C(X)$, clearly $fu|_{\mathrm{coz}(r)}$ is bounded and consequently $\hat{f}u$ is bounded, i.e. $\hat{f} \in C_\psi(X)$. □

Proposition 3.13. *For each infinite cardinal number $\mathfrak{a}$, $P_\psi(X, \mathfrak{F}_\mathfrak{a}) \subseteq C_\psi(X) \subseteq P(X, \mathfrak{F}_\mathfrak{a})$.*

Proof: If $f \in P_\psi(X, \mathfrak{F}_\mathfrak{a})$, then $f \in C(\mathrm{coz}(r))$ for some $r \in r_\mathfrak{a}(X)$. Now define $\hat{f}$ as in the proof of above lemma. Since $\hat{f}|_{\mathrm{coz}(r)} = f$, we have $\hat{f} \sim f$, hence $f \sim \hat{f} \in C_\psi(X)$. □

Corollary 3.14. *Let $C_m(X)$ be totally disconnected. Then $P_m(X, \mathfrak{F}_\mathfrak{a})$ for each infinite cardinal number $\mathfrak{a}$, is totally disconnected. In particular, $q_m(X)$ is also totally disconnected.*

By the following result, whenever X is infinite, then the maximal ring of quotients $Q(X)$ of $C(X)$ with the m-topology is never connected.

Proposition 3.15. *$Q_m(X)$ is connected if and only if X is finite.*

Proof: If X is finite, then the only open dense subset of X is X itself, hence $Q_m(X) = C_m(X) = \mathbb{R}^n$ for some positive integer n and clearly it is connected. Conversely, let $Q_m(X)$ be connected, then by Proposition 3.10, $Q^*(X) = Q(X)$ and by 3.6 in [5], X should be finite. □

We conclude the article by introducing a ring of quotients of $C(X)/O_p$, $p \in X$ which is connected with the m-topology (Note, $O_p = \{f \in C(X) : p \in \mathrm{int}_X Z(f)\}$). If μ_p is a neighborhood system at p, we show in the following propositions that the ring $P(X, \mu_p)$ is a ring of quotients of the ring $C(X)/O_p$ and $P_m(X, \mu_p)$ is connected.

Proposition 3.16. *Let μ_p be a neighborhood system at $p \in X$. Then $P(X, \mu_p)$ is a ring of quotients of $C(X)/O_p$.*

Proof: Let $\varphi : C(X)/O_p \to P(X, \mu_p)$ be defined by $\varphi(f + O_p) = [f]$, where $f \in C(X)$ and $[f]$ is the equivalent class of f in $P(X, \mu_p)$. We show that φ is an embedding. Clearly φ is a homomorphism. For injectivity of φ, let $\varphi(f + O_p) = \varphi(g + O_p)$, where $f, g \in C(X)$. Hence $[f] = [g]$ implies that $f \sim g$, i.e.,

$f = g$ on some $B \in \mu_p$ or equivalently, $B \subseteq Z(f-g)$. Hence $f - g \in O_p$, i.e., $f + O_p = g + O_p$ and φ is injective. This means that $C(X)/O_p \subseteq P(X, \mu_p)$. Now suppose that $0 \neq f \in P(X, \mu_p)$, hence $f \in C(B)$, for some $B \in \mu_p$. Since $f \neq 0$, $p \notin \mathrm{int}_B Z_B(f) = \mathrm{int}_X Z_B(f)$, where $Z_B(f)$ is the zeros of f in B, so there exists $g \in C(X)$ such that $X \setminus B \subseteq \mathrm{int}_X Z(g)$ and $g(p) = 1$. We define

$$\widehat{fg}(x) = \begin{cases} f(x)g(x), & x \notin \mathrm{int}_X Z(g), \\ 0, & x \in \mathrm{int}_X Z(g). \end{cases}$$

We show that $\widehat{fg} \in C(X)$. To see this, it is enough to show that $\widehat{fg}$ is continuous at each $x \notin \mathrm{int}_X Z(g)$. But if $x \notin \mathrm{int}_X Z(g)$, then $x \in B$ and $\widehat{fg}|_B = fg|_B$ which is continuous on the open set B and hence at $x \in B$. Moreover $p \notin \mathrm{int}_X Z(\widehat{fg})$ for, if $p \in \mathrm{int}_X Z(\widehat{fg})$, then there exists open set V containing p such that $V \subseteq Z_B(f) \cup Z(g)$. Therefore $V \cap g^{-1}((\frac{1}{2}, \frac{3}{2})) \subseteq Z_B(f) \cup Z(g)$ implies that $V \cap g^{-1}((\frac{1}{2}, \frac{3}{2})) \subseteq Z_B(f)$ for $V \cap g^{-1}((\frac{1}{2}, \frac{3}{2})) \cap Z(g) = \emptyset$. But $V \cap g^{-1}((\frac{1}{2}, \frac{3}{2}))$ is an open set containing p, hence $p \in \mathrm{int}_X Z(f)$, a contradiction. Therefore $\widehat{fg} \notin O_p$, i.e., $\widehat{fg} + O_p \neq 0$ and $\widehat{fg} + O_p \in C(X)/O_p$, so $P(X, \mu_p)$ is a ring of quotients of $C(X)/O_p$. □

Proposition 3.17. *The ring of quotients $P(X, \mu_p)$ of $C(X)/O_p$ with the m-topology is connected.*

Proof: By Proposition 3.10, we must show that $P(X, \mu_p) = P^*(X, \mu_p)$. Let $f \in P(X, \mu_p)$, hence $f \in C(B)$ for some open set $B \in \mu_p$. Take $H = \{x \in B : |f(x) - f(p)| < 1\}$. Since H is open in B containing p and B is open in X, $H \in \mu_p$. But f is bounded on $H \in \mu_p$ and this implies that f is bounded. □

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