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GEODESIC MAPPING ONTO KÄHLERIAN SPACES
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Abstract. In the present paper a generalized Kählerian space $\mathbb{G}_1\mathbb{K}_N$ of the first kind is considered as a generalized Riemannian space $\mathbb{G}\mathbb{R}_N$ with almost complex structure F_i^h that is covariantly constant with respect to the first kind of covariant derivative.

Using a non-symmetric metric tensor we find necessary and sufficient conditions for geodesic mappings $f: \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}_1\mathbb{K}_N$ with respect to the four kinds of covariant derivatives. These conditions have the form of a closed system of partial differential equations in covariant derivatives with respect to unknown components of the metric tensor and the complex structure of the Kählerian space $\mathbb{G}_1\mathbb{K}_N$.

Keywords: geodesic mapping; equitorsion geodesic mapping; generalized Kählerian space

MSC 2010: 53B05, 53B35

1. INTRODUCTION

Geodesic mappings of Kählerian manifolds have been studied by many authors. We continue the general idea by introducing the notion of generalized Kählerian spaces of the first kind $\mathbb{G}_1\mathbb{K}_N$, which generalize Kählerian spaces in the spirit of Einstein's Unified Field Theory and Moffat's non-symmetrical gravitational theory. This paper is devoted to the study of geodesic mappings of *generalized Riemannian spaces* to *generalized Kählerian spaces of the first kind* $\mathbb{G}_1\mathbb{K}_N$,

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The main results of the paper: New explicit formulas of geodesic mappings onto $\mathbb{G}\mathbb{K}_N$ are given in Subsection 3.1, new explicit formulas of equitorsion geodesic mappings onto $\mathbb{G}\mathbb{K}_N$ in Subsection 3.2.

In a similar way we can consider generalized Kählerian spaces of the second, the third and the fourth kind.

2. GENERALIZED KÄHLERIAN SPACES OF THE FIRST KIND

2.1. Generalized Riemannian spaces. A *generalized Riemannian space* $\mathbb{G}\mathbb{R}_N$ in the sense of Eisenhart's definition [5] is a differentiable N -dimensional manifold, equipped with a non-symmetric metric tensor g_{ij} (i.e. $g_{ij} \neq g_{ji}$). The symmetric and the antisymmetric parts of g_{ij} are

$$g_{\underline{ij}} = \frac{1}{2}(g_{ij} + g_{ji}) = \frac{1}{2}g_{(ij)}, \quad g_{\underset{\vee}{ij}} = \frac{1}{2}(g_{ij} - g_{ji}) = \frac{1}{2}g_{[ij]}.$$

The lowering and the rising of indices are defined by the tensors $g_{\underline{ij}}$ and $g^{\overset{i}{j}}$, respectively, where $g^{\overset{i}{j}}$ is defined by the equation

$$(2.1) \quad g_{\underline{ij}}g^{\overset{j}{k}} = \delta_i^k$$

(δ_i^k is the Kronecker symbol). From (2.1) we have that the matrix $(g^{\overset{i}{j}})$ is inverse to $(g_{\underline{ij}})$, wherefrom it is necessary that $g = \det(g_{\underline{ij}}) \neq 0$. Connection coefficients of this space are generalized Christoffel symbols of the second kind, where

$$\Gamma_{jk}^i = g^{\overset{ip}{p}}\Gamma_{p.jk}, \quad \Gamma_{i.jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}), \quad g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k}.$$

Generally $\Gamma_{jk}^i \neq \Gamma_{kj}^i$. Therefore, one can define the symmetric and the anti-symmetric part of Γ_{jk}^i , respectively, by

$$\Gamma_{\underline{jk}}^i = \frac{1}{2}(\Gamma_{jk}^i + \Gamma_{kj}^i) = \frac{1}{2}\Gamma_{(jk)}^i, \quad \Gamma_{\underset{\vee}{jk}}^i = \frac{1}{2}(\Gamma_{jk}^i - \Gamma_{kj}^i) = \frac{1}{2}\Gamma_{[jk]}^i.$$

The quantity $\Gamma_{\underset{\vee}{jk}}^i$ is the *torsion tensor* of the spaces $\mathbb{G}\mathbb{R}_N$.

The use of a non-symmetric metric tensor and a non-symmetric connection became especially topical after the appearance of the papers of A. Einstein [2]–[4] related to the attempt to formulate a Unified Field Theory (UFT). We remark that in UFT the symmetric part $g_{\underline{ij}}$ of g_{ij} is related to gravitation, and the antisymmetric one $g_{\underset{\vee}{ij}}$ to electromagnetism. More recently the ideas of a non-symmetric metric tensor

appears in Moffat's non-symmetric gravitational theory [17]. In Moffat's theory the antisymmetric part represents a Proca field (massive Maxwell field) which is part of the gravitational interaction, contributing to the rotation of galaxies.

Based on the non-symmetry of the connection in a generalized Riemannian space one can define four kinds of covariant derivatives. For example, for a tensor a_j^i in $\mathbb{G}\mathbb{R}_N$ we have

$$\begin{aligned} a_{j|_1}^i &= a_{j,m}^i + \Gamma_{pm}^i a_j^p - \Gamma_{jm}^p a_p^i, & a_{j|_2}^i &= a_{j,m}^i + \Gamma_{mp}^i a_j^p - \Gamma_{mj}^p a_p^i, \\ a_{j|_3}^i &= a_{j,m}^i + \Gamma_{pm}^i a_j^p - \Gamma_{mj}^p a_p^i, & a_{j|_4}^i &= a_{j,m}^i + \Gamma_{mp}^i a_j^p - \Gamma_{jm}^p a_p^i. \end{aligned}$$

By applying four kinds of covariant derivatives of tensors, it is possible to construct several Ricci type identities. In these identities 12 curvature tensors appear as well as 15 quantities, which are not tensors, named "curvature pseudotensors" by S. M. Minčić [12], [13]. In the case of the space $\mathbb{G}\mathbb{R}_N$ we have five independent curvature tensors.

2.2. Generalized Kählerian space of the first kind. Kählerian spaces and their mappings were investigated by many authors, for example T. Otsuki and Y. Tasiro [18], [25], K. Yano [26], J. Mikeš, V. V. Domashev [1], [6], [7], [8], [9], [10], [11], [22], M. Prvanović [19], N. Pušić [21], S. S. Pujar [20], M. S. Stanković et al. [16], [24], and many others.

An N -dimensional Riemannian space with metric tensor g_{ij} is a *Kählerian space* \mathbb{K}_N if there exists an almost complex structure F_j^i such that

$$\begin{aligned} F_p^h F_i^p &= -\delta_i^h, \\ g_{pq} F_i^p F_j^q &= g_{ij}, & g^{ij} &= g^{pq} F_p^i F_q^j, \\ F_{i;j}^h &= 0, \end{aligned}$$

where ";" denotes the covariant derivative with respect to the metric tensor g_{ij} .

Definition 2.1. A generalized N -dimensional Riemannian space with non-symmetric metric tensor g_{ij} is a *generalized Kählerian space of the first kind* $\mathbb{G}\mathbb{K}_N$ if there exists an almost complex structure F_j^i such that

$$\begin{aligned} (2.2) \quad F_p^h F_i^p &= -\delta_i^h, \\ (2.3) \quad g_{\underline{p}\underline{q}} F_i^{\underline{p}} F_j^{\underline{q}} &= g_{\underline{ij}}, & g^{\underline{ij}} &= g^{\underline{pq}} F_p^i F_q^j, \\ F_{i|\underline{j}}^h &= 0, & F_{i;j}^h &= 0, \end{aligned}$$

where " $\underline{\quad}$ " denotes the covariant derivative of the first kind with respect to the connection Γ_{jk}^i ($\Gamma_{jk}^i \neq \Gamma_{kj}^i$) and ";" denotes the covariant derivative with respect to the symmetric part of the metric tensor $\Gamma_{\underline{jk}}^i$.

From (2.3), using (2.2), we get $F_{ij} = -F_{ji}$ and $F^{ij} = -F^{ji}$, where we denote $F_{ji} = F_j^p g_{pi}$, $F^{ji} = F_p^j g^{pi}$.

The following theorem holds.

Theorem 2.1 ([23]). *For the almost complex structure F_j^i of $\mathbb{G}\mathbb{K}_N$ the relations*

$$F_{i|j}^h = 0, \quad F_{i|j}^h = 2F_p^h \Gamma_{ij}^p, \quad F_{i|j}^h = 2F_i^p \Gamma_{jp}^h$$

are valid, where Γ_{ij}^h is the torsion tensor.

3. GEODESIC MAPPING

3.1. Geodesic mapping between generalized Kählerian spaces of the first kind. In this part we consider geodesic mappings $f: \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\overline{\mathbb{K}}_N$.

Definition 3.1. A diffeomorphism $f: \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\overline{\mathbb{K}}_N$ is *geodesic*, if geodesics of the space $\mathbb{G}\mathbb{R}_N$ are mapped to geodesics of the space $\mathbb{G}\overline{\mathbb{K}}_N$.

At the corresponding points M and \overline{M} we can put

$$(3.1) \quad \overline{\Gamma}_{jk}^i = \Gamma_{jk}^i + P_{jk}^i \quad (i, j, k = 1, \dots, N),$$

where P_{jk}^i is the *deformation tensor* of the connection Γ of $\mathbb{G}\mathbb{R}_N$ corresponding to the mapping $f: \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\overline{\mathbb{K}}_N$.

Theorem 3.1 ([14]). *A necessary and sufficient condition for the mapping $f: \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\overline{\mathbb{K}}_N$ to be geodesic is that the deformation tensor P_{jk}^i from (3.1) has the form*

$$(3.2) \quad P_{jk}^i = \delta_j^i \psi_k + \delta_k^i \psi_j + \xi_{jk}^i,$$

where

$$\psi_i = \frac{1}{N+1} (\overline{\Gamma}_{i\alpha}^\alpha - \Gamma_{i\alpha}^\alpha), \quad \xi_{jk}^i = P_{jk}^i = \frac{1}{2} (P_{jk}^i - P_{kj}^i).$$

We remark that in $\mathbb{G}\mathbb{K}_N$ the following equations are valid:

$$\Gamma_{i\alpha}^\alpha = 0, \quad \xi_{i\alpha}^\alpha = 0, \quad F_\alpha^\alpha = 0.$$

In [11] Mikeš et al. proved necessary and sufficient conditions for geodesic mappings of a Riemannian space onto a Kählerian space.

Theorem 3.2. *The Riemannian space \mathbb{R}_N admits a nontrivial geodesic mapping onto the Kählerian space $\overline{\mathbb{K}}_N$ with metric \overline{g}_{ij} and complex structure \overline{F}_i^h satisfying*

$$\overline{g}_{ij} = \overline{g}_{ji}, \quad \det(\overline{g}_{ij}) \neq 0, \quad \overline{F}_i^p \overline{g}_{pj} + \overline{F}_j^p \overline{g}_{pi} = 0, \quad \overline{F}_p^h \overline{F}_i^p = -\delta_i^h,$$

if and only if, in the common coordinate system x with respect to the mapping, the conditions

- a) $\overline{g}_{ij;k} = 2\psi_k \overline{g}_{ij} + \psi_i \overline{g}_{jk} + \psi_j \overline{g}_{ik};$
- b) $\overline{F}_{i;k}^h = \overline{F}_k^h \psi_i - \delta_k^h \overline{F}_i^\alpha \psi_\alpha$

hold, where $\psi_i \neq 0$.

Our idea is to find the corresponding equations with respect to the four kinds of covariant derivative.

In all the following theorems concerning mappings from a generalized Riemannian space onto a generalized Kählerian space, \overline{g}_{ij} and \overline{F}_i^j denote the metric and the almost complex structure of $\mathbb{G}\overline{\mathbb{K}}_N$, respectively, satisfying

$$(3.3) \quad \overline{g}_{ij} \neq \overline{g}_{ji}, \quad \det(\overline{g}_{ij}) \neq 0, \quad \overline{F}_i^p \overline{g}_{pj} + \overline{F}_j^p \overline{g}_{pi} = 0, \quad \overline{F}_p^h \overline{F}_i^p = -\delta_i^h.$$

Theorem 3.3. *The generalized Riemannian space $\mathbb{G}\mathbb{R}_N$ admits a nontrivial geodesic mapping onto the generalized Kählerian space $\mathbb{G}\overline{\mathbb{K}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

- $$(3.4) \quad \begin{aligned} \text{a)} \quad \overline{g}_{ij|k} &= \overline{g}_{ij|_1k} + 2\psi_k \overline{g}_{ij} + \psi_i \overline{g}_{jk} + \psi_j \overline{g}_{ik} + \xi_{ik}^\alpha \overline{g}_{\alpha j} + \xi_{jk}^\alpha \overline{g}_{i\alpha}; \\ \text{b)} \quad \overline{F}_{i|k}^h &= \overline{F}_k^h \psi_i - \delta_k^h \overline{F}_i^\alpha \psi_\alpha - \xi_{\alpha k}^h \overline{F}_i^\alpha + \xi_{ik}^\alpha \overline{F}_\alpha^h, \end{aligned}$$

hold with respect to the first kind of covariant derivatives, where $\psi_i \neq 0$.

Proof. Equation (3.4) a) guarantees the existence of a geodesic mapping from the generalized Riemannian space $\mathbb{G}\mathbb{R}_N$ onto the generalized Riemannian space $\mathbb{G}\overline{\mathbb{R}}_N$ with metric tensor \overline{g}_{ij} with respect to the first kind of covariant derivatives (see [15]).

Formula (3.4) b) implies that the structure \overline{F}_i^h in $\mathbb{G}\overline{\mathbb{R}}_N$ is covariantly constant with respect to the first kind of covariant derivative. The algebraic conditions (3.3) guarantee that \overline{g}_{ij} and \overline{F}_i^h are the metric tensor and the structure of $\mathbb{G}\overline{\mathbb{K}}_N$, respectively.

The deformation tensor is determined by equation (3.2), i.e.,

$$(3.5) \quad \overline{\Gamma}_{ij}^h - \Gamma_{ij}^h = \psi_i \delta_j^h + \psi_j \delta_i^h + \xi_{ij}^h.$$

For the structure \overline{F} , we have the following equations:

$$(3.6) \quad \overline{F}_{i|k}^h = \overline{F}_{i,k}^h + \Gamma_{pk}^h \overline{F}_i^p - \Gamma_{ik}^p \overline{F}_p^h, \quad \overline{F}_{i|k}^h = \overline{F}_{i,k}^h + \Gamma_{kp}^h \overline{F}_i^p - \Gamma_{ki}^p \overline{F}_p^h.$$

Replacing Γ_{ij}^h from (3.5) in (3.6), we get

$$\begin{aligned} \overline{F}_{i|k}^h &= \overline{F}_{i,k}^h + (\overline{\Gamma}_{pk}^h - \psi_p \delta_k^h - \psi_k \delta_p^h - \xi_{pk}^h) \overline{F}_i^p - (\overline{\Gamma}_{ik}^p - \psi_i \delta_k^p - \psi_k \delta_i^p - \xi_{ik}^p) \overline{F}_p^h \\ &= \overline{F}_{i,k}^h + \overline{\Gamma}_{pk}^h \overline{F}_i^p - \psi_p \delta_k^h \overline{F}_i^p - \psi_k \delta_p^h \overline{F}_i^p - \xi_{pk}^h \overline{F}_i^p - \overline{\Gamma}_{ik}^p \overline{F}_p^h \\ &\quad + \psi_i \delta_k^p \overline{F}_p^h + \psi_k \delta_i^p \overline{F}_p^h + \xi_{ik}^p \overline{F}_p^h \\ &= \overline{F}_{i|k}^h - \psi_p \delta_k^h \overline{F}_i^p - \psi_k \delta_p^h \overline{F}_i^p - \xi_{pk}^h \overline{F}_i^p + \psi_i \delta_k^p \overline{F}_p^h + \psi_k \delta_i^p \overline{F}_p^h + \xi_{ik}^p \overline{F}_p^h \\ &= \overline{F}_{i|k}^h - \psi_p \delta_k^h \overline{F}_i^p - \psi_k \overline{F}_i^h - \xi_{pk}^h \overline{F}_i^p + \psi_i \overline{F}_k^h + \psi_k \overline{F}_i^h + \xi_{ik}^p \overline{F}_p^h \\ &= \underbrace{\overline{F}_{i|k}^h}_0 - \psi_p \delta_k^h \overline{F}_i^p + \psi_i \overline{F}_k^h - \xi_{pk}^h \overline{F}_i^p + \xi_{ik}^p \overline{F}_p^h, \end{aligned}$$

where “|”, and “ $\overline{|}$ ” are covariant derivatives in $\mathbb{G}\mathbb{R}_N$ and $\mathbb{G}\overline{\mathbb{K}}_N$, respectively. \square

Theorem 3.4. *The generalized Riemannian space $\mathbb{G}\mathbb{R}_N$ admits a nontrivial geodesic mapping onto the generalized Kählerian space $\mathbb{G}\overline{\mathbb{K}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

- a) $\overline{g}_{ij|k} = \overline{g}_{ij\overline{\nabla}_2 k} + 2\psi_k \overline{g}_{ij} + \psi_i \overline{g}_{jk} + \psi_j \overline{g}_{ik} + \xi_{ki}^\alpha \overline{g}_{\alpha j} + \xi_{kj}^\alpha \overline{g}_{i\alpha}$;
- b) $\overline{F}_{i|k}^h = \overline{F}_k^h \psi_i - \delta_k^h \overline{F}_i^\alpha \psi_\alpha - \xi_{k\alpha}^h \overline{F}_i^\alpha + \xi_{ki}^\alpha \overline{F}_\alpha^h$,

hold with respect to the second kind of covariant derivatives, where $\psi_i \neq 0$.

Proof. For the second kind of covariant derivatives in $\mathbb{G}\mathbb{R}_N$, we have

$$\begin{aligned} \overline{F}_{i|k}^h &= \overline{F}_{i,k}^h + (\overline{\Gamma}_{kp}^h - \psi_k \delta_p^h - \psi_p \delta_k^h - \xi_{kp}^h) \overline{F}_i^p - (\overline{\Gamma}_{ki}^p - \psi_k \delta_i^p - \psi_i \delta_k^p - \xi_{ki}^p) \overline{F}_p^h \\ &= \overline{F}_{i,k}^h + \overline{\Gamma}_{kp}^h \overline{F}_i^p - \psi_k \delta_p^h \overline{F}_i^p - \psi_p \delta_k^h \overline{F}_i^p - \xi_{kp}^h \overline{F}_i^p - \overline{\Gamma}_{ki}^p \overline{F}_p^h \\ &\quad + \psi_k \delta_i^p \overline{F}_p^h + \psi_i \delta_k^p \overline{F}_p^h + \xi_{ki}^p \overline{F}_p^h \\ &= \overline{F}_{i|k}^h - \psi_k \delta_p^h \overline{F}_i^p - \psi_p \delta_k^h \overline{F}_i^p - \xi_{kp}^h \overline{F}_i^p + \psi_k \delta_i^p \overline{F}_p^h + \psi_i \delta_k^p \overline{F}_p^h + \xi_{ki}^p \overline{F}_p^h \\ &= \overline{F}_{i|k}^h - \psi_k \overline{F}_i^h - \psi_p \delta_k^h \overline{F}_i^p - \xi_{kp}^h \overline{F}_i^p + \psi_k \overline{F}_i^h + \psi_i \overline{F}_k^h + \xi_{ki}^p \overline{F}_p^h \\ &= \psi_i \overline{F}_k^h - \psi_p \delta_k^h \overline{F}_i^p - \xi_{kp}^h \overline{F}_i^p + \xi_{ki}^p \overline{F}_p^h. \end{aligned}$$

\square

In a similar way, we can prove the corresponding theorems for the third and the fourth kind of covariant derivative:

Theorem 3.5. *The generalized Riemannian space $\mathbb{G}\mathbb{R}_N$ admits a nontrivial geodesic mapping onto the generalized Kählerian space $\mathbb{G}\overline{\mathbb{K}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

$$\begin{aligned} \text{a)} \quad \overline{g}_{ij}|_k &= \overline{g}_{ij}\overline{|}_k + 2\psi_k\overline{g}_{ij} + \psi_i\overline{g}_{jk} + \psi_j\overline{g}_{ik} + \xi_{ik}^\alpha\overline{g}_{\alpha j} + \xi_{kj}^\alpha\overline{g}_{i\alpha}; \\ \text{b)} \quad \overline{F}_{i|k}^h &= \overline{F}_{i|k}^h - \psi_p\delta_k^h\overline{F}_i^p + \psi_i\overline{F}_k^h - \xi_{pk}^h\overline{F}_i^p + \xi_{ki}^p\overline{F}_p^h, \end{aligned}$$

hold with respect to the third kind of covariant derivatives, where $\psi_i \neq 0$.

Theorem 3.6. *The generalized Riemannian space $\mathbb{G}\mathbb{R}_N$ admits a nontrivial geodesic mapping onto the generalized Kählerian space $\mathbb{G}\overline{\mathbb{K}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

$$\begin{aligned} \text{a)} \quad \overline{g}_{ij}|_k &= \overline{g}_{ij}\overline{|}_k + 2\psi_k\overline{g}_{ij} + \psi_i\overline{g}_{jk} + \psi_j\overline{g}_{ik} + \xi_{ki}^\alpha\overline{g}_{\alpha j} + \xi_{jk}^\alpha\overline{g}_{i\alpha}; \\ \text{b)} \quad \overline{F}_{i|k}^h &= \overline{F}_{i|k}^h - \psi_p\delta_k^h\overline{F}_i^p + \psi_i\overline{F}_k^h - \xi_{kp}^h\overline{F}_i^p + \xi_{ik}^p\overline{F}_p^h, \end{aligned}$$

hold with respect to the fourth kind of covariant derivatives, where $\psi_i \neq 0$.

3.2. Equitorsion geodesic mapping. Equitorsion mappings play an important role in the theories of geodesic, conformal and holomorphically projective transformations between two spaces of non-symmetric affine connection.

Definition 3.2 ([14]). A mapping $f: \mathbb{G}\mathbb{R}_N \rightarrow \mathbb{G}\overline{\mathbb{K}}_N$ is an *equitorsion geodesic mapping* if the torsion tensors of the spaces $\mathbb{G}\mathbb{R}_N$ and $\mathbb{G}\overline{\mathbb{K}}_N$ are equal. Then from (3.1), (3.2) and (3.5):

$$\overline{\Gamma}_{ij}^h - \Gamma_{ij}^h = \xi_{ij}^h = 0,$$

where \overline{ij} denotes an antisymmetrization with respect to i, j .

In the case of these mappings, the previous Theorems 3.3–3.6 become:

Theorem 3.7. *The generalized Riemannian space $\mathbb{G}\mathbb{R}_N$ admits a nontrivial equitorsion geodesic mapping onto the generalized Kählerian space $\mathbb{G}\overline{\mathbb{K}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

$$\begin{aligned} \text{a)} \quad \overline{g}_{ij}|_k &= 2\psi_k\overline{g}_{ij} + \psi_i\overline{g}_{jk} + \psi_j\overline{g}_{ik}; \\ \text{b)} \quad \overline{F}_{i|k}^h &= \overline{F}_k^h\psi_i - \delta_k^h\overline{F}_i^p\psi_p, \end{aligned}$$

hold with respect to the first kind of covariant derivatives, where $\psi_i \neq 0$.

Theorem 3.8. *The generalized Riemannian space $\mathbb{G}\mathbb{R}_N$ admits a nontrivial equitortion geodesic mapping onto the generalized Kählerian space $\mathbb{G}\overline{\mathbb{K}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

$$\begin{aligned} \text{a) } \overline{g}_{ij}|_k &= 2\psi_k \overline{g}_{ij} + \psi_i \overline{g}_{jk} + \psi_j \overline{g}_{ik}; \\ \text{b) } \overline{F}^h_{i|k} &= \overline{F}^h_k \psi_i - \delta_k^h \overline{F}^p_i \psi_p, \end{aligned}$$

hold with respect to the second kind of covariant derivatives, where $\psi_i \neq 0$.

Theorem 3.9. *The generalized Riemannian space $\mathbb{G}\mathbb{R}_N$ admits a nontrivial equitortion geodesic mapping onto the generalized Kählerian space $\mathbb{G}\overline{\mathbb{K}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

$$\begin{aligned} \text{a) } \overline{g}_{ij}|_k &= 2\psi_k \overline{g}_{ij} + \psi_i \overline{g}_{jk} + \psi_j \overline{g}_{ik}; \\ \text{b) } \overline{F}^h_{i|k} &= \overline{F}^h_{i|k} - \psi_p \delta_k^h \overline{F}^p_i + \psi_i \overline{F}^h_k, \end{aligned}$$

hold with respect to the third kind of covariant derivatives, where $\psi_i \neq 0$.

Theorem 3.10. *The generalized Riemannian space $\mathbb{G}\mathbb{R}_N$ admits a nontrivial equitortion geodesic mapping onto the generalized Kählerian space $\mathbb{G}\overline{\mathbb{K}}_N$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

$$\begin{aligned} \text{a) } \overline{g}_{ij}|_k &= 2\psi_k \overline{g}_{ij} + \psi_i \overline{g}_{jk} + \psi_j \overline{g}_{ik}; \\ \text{b) } \overline{F}^h_{i|k} &= \overline{F}^h_{i|k} - \psi_p \delta_k^h \overline{F}^p_i + \psi_i \overline{F}^h_k, \end{aligned}$$

hold with respect to the fourth kind of covariant derivatives, where $\psi_i \neq 0$.

4. CONCLUSION

We have shown that the notions of geodesic and equitortion geodesic mappings from Riemannian to Kählerian spaces can be generalized to the case of a non-symmetric metric, and we have given necessary and sufficient conditions for nontrivial such mappings.

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