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CHARACTERIZING PURE, CRYPTIC AND CLIFFORD INVERSE SEMIGROUPS

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Abstract. An inverse semigroup S is pure if $e = e^2$, $a \in S$, $e < a$ implies $a^2 = a$; it is cryptic if Green's relation $\mathcal{H}$ on S is a congruence; it is a Clifford semigroup if it is a semillatice of groups.

We characterize the pure ones by the absence of certain subsemigroups and a homomorphism from a concrete semigroup, and determine minimal nonpure varieties. Next we characterize the cryptic ones in terms of their group elements and also by a homomorphism of a semigroup constructed in the paper. We also characterize groups and Clifford semigroups in a similar way by means of divisors.

The paper also contains characterizations of completely semisimple inverse and of combinatorial inverse semigroups in a similar manner. It ends with a description of minimal non-$\mathcal{V}$ varieties, for varieties $\mathcal{V}$ of inverse semigroups considered.

Keywords: inverse semigroup; pure inverse semigroup; cryptic inverse semigroup; Clifford semigroup; group-closed inverse semigroup; pure variety; completely semisimple inverse semigroup; combinatorial inverse semigroup; variety

MSC 2010: 20M07, 20M20

1. Notation, terminology and summary

Let S be an inverse semigroup with the set E_S of idempotents. We say that S is *pure* (or *E-unitary*) if $a \in S$, $e \in E_S$ and $e < a$ imply that $a \in E_S$; S is *cryptic* if Green's relation $\mathcal{H}$ on S is a congruence; it is a *Clifford semigroup* (or *semillatice of groups*) if its idempotents are central. For a property $\mathcal{P}$, a variety $\mathcal{V}$ of inverse semigroups is a *$\mathcal{P}$-variety* if every member of $\mathcal{V}$ has the property $\mathcal{P}$. For $\mathcal{P}$ we consider: pure, cryptic, Clifford, strict, completely semisimple and combinatorial inverse semigroups.

In Theorem 2.1 we characterize pure inverse semigroups by means of: forbidden subsemigroups and homomorphisms from the free monogenic inverse semigroup with a zero adjoined into the semigroup. Minimal nonpure varieties form the subject of Theorem 2.2. In Theorem 2.3 we determine all maximal pure varieties of inverse semigroups. A construction of a special inverse semigroup leads to Theorem 3.3 which provides an isomorphic copy of a semigroup needed in Theorem 4.1. The latter contains a multiple characterization of cryptic inverse semigroups. Theorem 5.2 includes a similar characterization of Clifford semigroups. We conclude in Theorem 5.5 with a complete list of minimal non-$\mathcal{V}$ varieties for several familiar varieties of inverse semigroups.

We adhere to the notation and terminology of the book [3]. *Throughout the paper S stands for an arbitrary inverse semigroup unless specified otherwise.*

2. Pure inverse semigroups

Recall the following notation from [3]: the bicyclic semigroup C page 113, M_n page 418, the free monogenic inverse semigroup I_x page 394. Also, $\mathbb{Z}$ denotes the additive group of integers, $\mathbb{Z}_n = \mathbb{Z}/(n)$ for $n > 1$, and S^0 the semigroup S with a zero adjoined. For $a \in S$, $\langle a \rangle$ denotes the subsemigroup of S generated by a.

Theorem 2.1. *The following conditions on S are equivalent.*

(i) *S is pure.*

(ii) *S has no subsemigroups isomorphic to*

$$C^0,\ \mathbb{Z}^0,\ \mathbb{Z}_p^0 \quad \text{for a prime } p, \quad M_n \quad \text{for } n > 1. \tag{2.1}$$

(iii) *If $\chi\colon I_x^0 \to S$ is a homomorphism, then the image of χ is a 1- or 2-element semilattice.*

In such a case, the set of semigroups in (2.1) *cannot be replaced by a proper subset.*

Proof. (i) $\Rightarrow$ (ii). Since each semigroup in (2.1) has a zero and is not a semilattice, none of them is pure. Hence S cannot have a subsemigroup isomorphic to one of them.

(ii) $\Rightarrow$ (i). Assume that S is not pure. Then there exist $e \in E_S$ and $a \in S$ such that $e < a$ and $a \notin E_S$. Let T be the inverse subsemigroup of S generated by the set $\{e, a\}$. Since $e < a$, [3], Lemma II.1.6, implies that $e = ea^{-1} = a^{-1}e$ which, by taking inverses, yields $e = ae = ea$. Hence e is the zero of T. We consider two cases.

Case 1: $e \in \langle a \rangle$. Then $T = \langle a \rangle$ which is a monogenic inverse semigroup. Since T has a zero, according to [3], Theorem IX.3.4, T must be isomorphic to M_n for some $n > 1$.

Case 2: $e \notin \langle a \rangle$. Then $T = \{e\} \cup \langle a \rangle$ where e acts as an adjoined zero to $\langle a \rangle$. There are two types of such semigroup $\langle a \rangle$ according to the last cited reference: either the bicyclic semigroup or a cyclic group extended by some M_n. Hence S has a subsemigroup which is either a bicyclic semigroup or a cyclic group each with a zero adjoined.

(ii) $\Leftrightarrow$ (iii). This follows immediately from [3], Theorem IX.3.4.

Since any non trivial finite cyclic group has a cyclic subgroup of prime order, we may omit finite cyclic groups of nonprime order. None of the semigroups in (2.1) has another one as a subsemigroup. Hence we may not omit any from that list and preserve the validity of the theorem. □

In addition to [3], Notations XII.1.4 and XII.5.2, and Definition XII.4.1, we let

$$\mathcal{SA}_n = \mathcal{AG}_n \vee \mathcal{S} \quad \text{for } n \geqslant 1.$$

That is, $\mathcal{SA}_n$ consists of semilattices of abelian groups whose exponent divides n.

Recall that $\mathcal{B}$ denotes the variety of strict combinatorial inverse semigroups. Also, $\mathcal{L}(\mathcal{V})$ denotes the lattice of all subvarieties of a variety $\mathcal{V}$. A variety containing only pure semigroups will be said to be *pure*. Finally, $\langle S\rangle$ denotes the variety of inverse semigroups generated by S.

The next result handles minimal nonpure varieties.

Theorem 2.2. *For any prime p, let $\mathcal{V}_p = \langle \mathbb{Z}_p^0\rangle$.*

(i) $\mathcal{V}_p = \mathcal{SA}_p = [xy = yx^{p+1}]$, $\mathcal{B} = [yxy^{-1} = (yxy^{-1})^2]$.

(ii) *For any prime p, $\mathcal{V}_p$ and $\mathcal{B}$ are minimal nonpure varieties of inverse semigroups.*

(iii) *Every nonpure variety of inverse semigroups contains either $\mathcal{V}_p$ for some prime p or $\mathcal{B}$.*

(iv) *If $\mathcal{SA}_p = \mathcal{SA}_q$ for some primes p and q, then $p = q$. Moreover $\mathcal{V}_p \neq \mathcal{B}$ for any prime p.*

(v) $\mathcal{AG} \vee \mathcal{B} = [(zxz)^{-1}(zyz^{-1}) = (zyz^{-1})(zxz^{-1})] = \langle B(\mathbb{Z}, 2)\rangle$ *is the join of all minimal nonpure varieties of inverse semigroups.*

Proof. (i) The first two equalities follow from [3], Theorem XII.5.4 (ii). The third equality was established in [3], Proposition XII.4.8.

(ii) In the list of semigroups in (2.1) we retain only those semigroups which cannot be obtained as homomorphic images of some others: $\mathbb{Z}_p$ is a homomorphic image of $\mathbb{Z}$ which is a homomorphic image of C, and M_2 ($\cong B_2$) is a homomorphic image of M_n for $n > 2$. There remain $\mathbb{Z}_p^0$ for all primes p and B_2, with varieties generated by them: $\mathcal{V}_p$ and $\mathcal{B}$. By [3], Corollaries XII.4.5 and XII.4.14, we have the diagrams of $\mathcal{L}(\mathcal{SA}_p)$ and $\mathcal{L}(\mathcal{B})$, respectively:

$$\begin{tikzcd} & \mathcal{SA}_p \arrow[dl, dash] \arrow[dr, dash] & \\ \mathcal{S} \arrow[dr, dash] & & \mathcal{A}_p \arrow[dl, dash] \\ & \mathcal{J} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B} \arrow[d, dash] \\ \mathcal{S} \arrow[d, dash] \\ \mathcal{J} \end{tikzcd}$$

which shows that all proper subvarieties of $\mathcal{V}_p$ and $\mathcal{B}$ are pure. Therefore both $\mathcal{V}_p$ and $\mathcal{B}$ are minimal nonpure varieties.

(iii) Let $\mathcal{V}$ be a nonpure variety. Then $\mathcal{V}$ contains a nonpure semigroup S. By Theorem 2.1, S has a subsemigroup isomorphic to one in the list in that theorem. By the argument in the proof of part (ii), we conclude that either $\mathcal{V}$ contains $\mathcal{V}_p$ for some prime p or $\mathcal{B}$. These were proved to be minimal nonpure varieties.

(iv) It is easy to see that for primes p and q, we have

$$\mathbb{Z}_p^0 \in \mathcal{SA}_q \Rightarrow p = q; \quad B_2 \notin \mathcal{SA}_p,$$

which implies the contention.

(v) By parts (ii) and (iii), $\{\mathcal{A}_p;\ p \text{ prime}\} \cup \{\mathcal{B}\}$ is the complete set of minimal nonpure varieties. Clearly $\bigvee_p \mathcal{SA}_p = \mathcal{SA}$ and $\mathcal{SA} \vee \mathcal{B} = \mathcal{AG} \vee \mathcal{B}$. It remains to apply [3], Theorem XII.5.3 (iii). □

In Theorem 2.2, parts (ii) and (iii) imply that $\{\mathcal{V}_p;\ p \text{ prime }\} \cup \{\mathcal{B}\}$ is the complete set of minimal nonpure varieties of inverse semigroups, and part (iv) that there is no repetition in this set.

We now cast a brief look at pure varieties.

Theorem 2.3. *Let $\mathcal{V}$ be a variety of inverse semigroups. Then $\mathcal{V}$ is pure if and only if either $\mathcal{V} \subseteq \mathcal{G}$ or $\mathcal{V} = \mathcal{S}$.*

Proof. *Necessity.* Assume that $\mathcal{V} \nsubseteq \mathcal{G}$. By [3], Proposition XII.4.13 (i), we get $Y_2 \in \mathcal{V}$ and thus $\mathcal{S} \subseteq \mathcal{V}$. If $\mathcal{V} \cap \mathcal{G} \neq \mathcal{J}$, then $\mathcal{V}$ contains a nontrivial group and thus it must contain either a copy of $\mathbb{Z}$ or $\mathbb{Z}_p$ for some prime p. But then $\mathcal{V}$ contains either $Y_2 \times \mathbb{Z}$ or $Y_2 \times \mathbb{Z}_p$ and thus also $\mathbb{Z}^0$ or $\mathbb{Z}_p^0$ since the latter are homomorphic images of the former. Therefore $\mathcal{A}_p \subseteq \mathcal{V}$ and thus $\mathcal{SA}_p \subseteq \mathcal{V}$. This is in contradiction with Theorem 2.2 (ii) since $\mathcal{SA}_p = \mathcal{V}_p$ is not pure and thus $\mathcal{V} \cap \mathcal{G} = \mathcal{J}$. By the same reference, we have $\mathcal{B} \nsubseteq \mathcal{V}$ whence $B_2 \notin \mathcal{V}$. Now [3], Proposition XII.4.13 (ii), yields that $\mathcal{V}$ is a Clifford semigroup variety. But then $\mathcal{V} = \mathcal{S}$.

Sufficiency. Trivial. □

Corollary 2.4. *The set $\{\mathcal{G}, \mathcal{S}\}$ is the complete set of maximal pure varieties of inverse semigroups. Every pure variety of inverse semigroups is contained in a maximal one.*

3. A special inverse semigroup

We construct here an isomorphic copy of the inverse semigroup $\langle a, e \mid aa^{-1} = a^{-1}a > e\rangle$. This will be needed in the next section for characterizing cryptic inverse semigroups among inverse semigroups by means of forbidden subsemigroups. The construction is effected in two steps: the construction of Γ whose elements are tuples of integers, and the quotient Δ of Γ relative to a relation γ.

Let

$$\Gamma = \{(n_1, \dots, n_k);\ k \geqslant 1,\ n_1, n_k \in \mathbb{Z},\ n_2, \dots, n_{k-1} \in \mathbb{Z} \setminus \{0\}\}$$

with the multiplication

$$(m_1, \dots, m_p)(n_1, \dots, n_q) = \begin{cases} (m_1, \dots, m_{p-1}, m_p + n_1, n_2, \dots, n_q) & \text{if } m_p + n_1 \neq 0,\ p > 1,\ q > 1, \\ (m_1, \dots, m_{p-1}, n_2, \dots, n_q) & \text{if } m_p + n_1 = 0,\ p > 1,\ q > 1, \\ (m_1 + n_1, n_2, \dots, n_q) & \text{if } p = 1,\ q > 1, \\ (m_1, \dots, m_{p-1}, m_p + n_1) & \text{if } p > 1,\ q = 1, \\ (m_1 + n_1) & \text{if } p = q = 1 \end{cases}$$

and unitary operation

$$(m_1, \dots, m_p)^{-1} = (-m_1, \dots, -m_p).$$

Lemma 3.1. *Γ is a unitary semigroup generated by $a = (1)$ and $e = (0, 0)$.*

P r o o f. Clearly Γ is closed under multiplication. The associative law requires straightforward verification and may be safely omitted. Next,

$$\begin{aligned} a^m &= (m), \qquad m \in \mathbb{Z}, \\ a^{m_1} e a^{m_2} e \dots e a^{m_p} &= (m_1)(0,0)(m_2)(0,0) \dots (0,0)(m_p) \\ &= (m_1, 0)(m_2)(0,0) \dots (0,0)(m_p) \\ &= (m_1, m_2)(0,0) \dots (0,0)(m_p) \\ &= (m_1, m_2, 0)(m_3)(0,0) \dots (0,0)(m_p) \\ &= (m_1, m_2, m_3)(0,0) \dots (0,0)(m_p) \\ &= \dots = (m_1, m_2, m_3, \dots, m_p) \end{aligned}$$

and the set $\{e, a\}$ generates Γ. $\square$

Let

$$M = (m_1, \ldots, m_p, -m_p, \ldots, -m_1), \quad N = (n_1, \ldots, n_q, -n_q, \ldots, -n_1),$$

and let γ be the unary congruence on Γ generated by the pairs

$$((0, m, -m, m, 0), (0, m, 0)), \quad m \in \mathbb{Z}, \tag{3.1}$$

$$(MN, NM). \tag{3.2}$$

In terms of a and e introduced in Lemma 3.1, we have

$$ea^m ea^{-m} ea^m e \gamma ea^m e, \quad m \in \mathbb{Z}, \tag{3.3}$$

and for $\mathfrak{m} = a^{m_1} e \ldots ea^{m_p} ea^{-m_p} e \ldots ea^{-m_1}$ and $\mathfrak{n} = a^{n_1} e \ldots ea^{n_q} ea^{-n_q} e \ldots ea^{-n_1}$

$$\mathfrak{m}\mathfrak{n}\gamma\mathfrak{n}\mathfrak{m}. \tag{3.4}$$

Let $\Delta = \Gamma/\gamma$. Consult [3], Definition I.4.8.

Lemma 3.2. *Δ is an inverse semigroup.*

Proof. Indeed,

$$\begin{aligned}
&a^{m_1} ea^{m_2} e \ldots ea^{m_p} ea^{-m_p} (a^{-m_p} e \ldots ea^{-m_2} ea^{-m_1}) a^{m_1} ea^{m_2} e \ldots ea^{m_p} \\
&\quad = a^{m_1} ea^{m_2} (ea^{m_3} e \ldots ea^{m_{p-1}} ea^{-m_{p-1}} e \ldots ea^{-m_3} e)(ea^{-m_2} ea^{m_2} e) \ldots ea^{m_p} \\
&\qquad \gamma a^{m_1} (ea^{m_2} ea^{-m_2} ea^{m_2} e)(ea^{m_3} e \ldots ea^{m_{p-1}} ea^{-m_{p-1}} e \ldots ea^{-m_3} e) a^{m_3} e \ldots ea^{m_p}
\end{aligned}$$

by (3.2)

$$\gamma a^{m_1} ea^{m_2} ea^{m_3} (ea^{m_4} e \ldots ea^{m_{p-1}} ea^{-m_{p-1}} e \ldots ea^{-m_4} e)(ea^{-m_3} ea^{m_3} e) a^{m_4} e \ldots ea^{m_p}$$

by (3.1)

$$\gamma a^{m_1} ea^{m_2} (ea^{m_3} ea^{-m_3} ea^{m_3} e)(ea^{m_4} e \ldots ea^{m_{p-1}} ea^{-m_{p-1}} e \ldots ea^{-m_4} e) a^{m_4} e \ldots ea^{m_p}$$

by (3.2)

$$\begin{gathered}
\gamma \text{ etc} \ldots \\
\gamma a^{m_1} e \ldots ea^{m_p}
\end{gathered}$$

and Δ is regular. In addition, elements of the form

$$((a^{m_1} e \ldots ea^{m_p})(a^{-m_p} e \ldots ea^{-m_1}))\gamma$$

are idempotent. For the converse, we consider:

$$
\begin{aligned}
&((a^{m_1}e\dots ea^{m_p})(a^{n_1}e\dots ea^{n_q}))^{-1}\\
&\quad=\begin{cases}(a^{m_1}e\dots ea^{m_p+n_1}e\dots ea^{n_q})^{-1} & \text{if } m_p+n_1\neq 0,\\ a^{m_1}e\dots ea^{m_{p-1}}ea^{n_2}e\dots ea^{n_q})^{-1} & \text{if } m_p+n_1=0\end{cases}\\
&\quad=\begin{cases}a^{-n_q}e\dots ea^{-m_p-n_1}e\dots ea^{-m_1} & \text{if } m_p+n_1\neq 0,\\ a^{-n_q}e\dots ea^{-n_2}ea^{-m_{p-1}}e\dots ea^{-m_1} & \text{if } m_p+n_1=0,\end{cases}\\
&(a^{n_1}e\dots ea^{n_q})^{-1}(a^{m_1}e\dots ea^{m_p})^{-1}=(a^{-n_q}e\dots ea^{-n_1})(a^{-m_p}e\dots ea^{-m_1})\\
&\quad=\begin{cases}a^{-n_q}e\dots ea^{-n_1-m_p}e\dots ea^{-m_1} & \text{if } -n_1-m_p\neq 0,\\ a^{-n_q}e\dots ea^{-n_2}ea^{-m_{p-1}}e\dots ea^{-m_1})^{-1} & \text{if } -n_1-m_p=0\end{cases}
\end{aligned}
$$

which proves that $(st)^{-1}=t^{-1}s^{-1}$ for all $s,t\in\Gamma$. If now $f\in E_\Delta$, we have

$$f=ff^{-1}f=f(ff)^{-1}f=(ff^{-1})(f^{-1}f).$$

If also $g\in E_\Delta$, we have $g=(gg^{-1})(g^{-1}g)$. In view of (3.4), the elements ff^{-1}, $f^{-1}f$, gg^{-1}, $g^{-1}g$ commute, and thus so do f and g. Therefore idempotents of Δ commute, and thus Δ is an inverse semigroup.

We have seen in the first part of the proof that $s=ss^{-1}s$ for any $s\in\Delta$. But then $(s^{-1})^{-1}=s$ implies that $s^{-1}=s^{-1}ss^{-1}$ and s^{-1} is an inverse of s. By uniqueness of inverses, s^{-1} is the inverse of s in Δ. □

We denote by $\langle A\mid R\rangle$ any relatively free inverse semigroup generated, as inverse semigroup, by a set A subject to a set of relations R.

Theorem 3.3. $\langle a,e\mid aa^{-1}=a^{-1}a>e\rangle\cong\Delta$.

Proof. It follows from Lemma 3.1 that Δ is generated by the set $\{a\gamma,e\gamma\}$. Next, $a\gamma=\{a\}$ and $a=(1)$ so that $(1)(-1)=(-1)(1)=(0)$. Secondly, $e=(0,0)$ so $(0)(0,0)=(0,0)(0)=(0,0)$ and $e\leqslant aa^{-1}=a^{-1}a$, whence $e\gamma\leqslant(a\gamma)(a\gamma)^{-1}$. If $(0,0)\gamma(0)$, there would be a sequence of "elementary transitions" between the two elements, but there are none. So $(0,0)\gamma<0\gamma$.

Let S be generated by the set $\{a,e\}$ where $aa^{-1}=a^{-1}a>e$. First let $ea=ae$. Then S is an ideal extension of the cyclic group generated by ae by the cyclic group generated by a with a zero adjoined. Define a mapping φ from Δ to S by

$$a^n\mapsto a^n\quad(n\in\mathbb{Z}),\quad e\mapsto e,\quad x\mapsto a^{m_1+\ldots+m_p}$$

where in x alternate $a^{m_1},\dots,a^{m_p}$ and e. Then φ is evidently a homomorphism of Δ onto S.

Hence we assume that $ae \neq ea$. Elements of S are of the form $a^{k_1}ea^{k_2}e\dots ea^{k_n}$ where $k_1, k_n \in \mathbb{Z}$, $k_2, \dots, k_{n-1} \in \mathbb{Z} \setminus \{0\}$, $n \geqslant 1$. If $k_1 = 0$, then $a^{k_1} = aa^{-1}$ $(= a^{-1}a)$ and $a^{k_1}e = e$; similarly for k_n. Define a mapping

$$\chi\colon\ (k_1, \dots, k_n) \mapsto a^{k_1}e\dots ea^{k_n}.$$

The multiplication in Γ reflects that in S: $(a^{m_1}e\dots ea^{m_p})(a^{n_1}e\dots ea^{n_q})$ equals in separate cases to:

$$\begin{array}{ll}
a^{m_1}e\dots ea^{m_p+n_1}e\dots ea^{n_q} & \text{if } \ m_p + n_1 \neq 0,\ p > 1,\ q > 1, \\
a^{m_1}e\dots ea^{m_{p-1}}ea^{n_2}e\dots ea^{n_q} & \text{if } \ m_p + n_1 = 0,\ p > 1,\ q > 1, \\
a^{m_1}a^{n_1}e\dots ea^{n_q} = a^{m_1+n_1}e\dots ea^{n_q} & \text{if } \ p = 1,\ q > 1, \\
a^{m_1}e\dots ea^{m_p}a^{n_1} = a^{m_1}e\dots ea^{m_p+n_1} & \text{if } \ p > 1,\ q = 1, \\
a^{m_1}a^{n_1} = a^{m_1+n_1} & \text{if } \ p = 1,\ q = 1,
\end{array}$$

which implies that χ is a homomorphism of Γ onto S. Since S is an inverse semigroup, it satisfies relations (3.3) and (3.4) with γ the equality relation. Hence relations (3.1) and (3.2) are satisfied and thus χ factors through Δ which is a homomorphism of Δ onto S. Since S is generated by the set $\{a, e\}$, this homomorphism is unique. $\square$

4. Cryptic inverse semigroups

This section runs somewhat parallel to Section 2 but now for cryptic inverse semigroups. The case of pure inverse semigroups, handled in Section 2, is considerably simpler than the case at hand, which reflects itself not only in shorter arguments but also in completeness of the results obtained.

Denote by G_S the set of all elements of S contained in a subgroup of S. Then S is said to be *group-closed* if G_S is a subsemigroup of S. Note that

$$G_S = \{a \in S;\ aa^{-1} = a^{-1}a\}.$$

We start with a multiple characterization of cryptic and group-closed inverse semigroups.

Theorem 4.1. *The following conditions on S are equivalent.*

(i) *S is cryptic.*
(ii) $a\mathcal{H}b$, $e \in E_S \Rightarrow aea^{-1} = beb^{-1}$.
(iii) $a \in G_S$, $e \in E_S \Rightarrow ae = ea$.
(iv) *S is group-closed.*
(v) *If $\chi\colon\ \Delta \to S$ is a homomorphism, then the image of χ is commutative.*

P r o o f. (i) $\Rightarrow$ (ii). Let $a\mathcal{H}b$ and $e \in E_S$. Then $ae\mathcal{H}be$ whence $ae(ae)^{-1} = be(be)^{-1}$ so that $aea^{-1} = beb^{-1}$.

(ii) $\Rightarrow$ (iii). Let $a \in S$ and $e, f \in E_S$ be such that $a\mathcal{H}f$. The hypothesis implies that $aea^{-1} = fef$ which yields $ae = efa = ea$.

(iii) $\Rightarrow$ (ii). Let $a\mathcal{H}b$ and $e \in E_S$. Then

$$a^{-1}b(a^{-1}b)^{-1} = a^{-1}bb^{-1}a = a^{-1}aa^{-1}a = a^{-1}a,$$
$$(a^{-1}b)^{-1}a^{-1}b = b^{-1}aa^{-1}b = b^{-1}bb^{-1}b = b^{-1}b = a^{-1}a$$

and thus $a^{-1}b\mathcal{H}a^{-1}a$. The hypothesis implies that $a^{-1}be = ea^{-1}b$. Premultiplying by a and postmultiplying by b^{-1}, we get $aa^{-1}beb^{-1} = aea^{-1}bb^{-1}$ whence $beb^{-1} = aea^{-1}$.

(ii) $\Rightarrow$ (i). Let $a\mathcal{H}b$ and $c \in S$. Then

$$ac(ac)^{-1} = a(cc^{-1})a^{-1} = b(cc^{-1})b^{-1} = bc(bc)^{-1},$$
$$(ac)^{-1}ac = c^{-1}a^{-1}ac = c^{-1}b^{-1}bc = (bc)^{-1}bc$$

and thus $ac\mathcal{H}bc$. Further,

$$ca(ca)^{-1} = caa^{-1}c^{-1} = cbb^{-1}c^{-1} = cb(cb)^{-1}$$

and applying the hypothesis to $a^{-1}\mathcal{H}b^{-1}$, we get

$$(ca)^{-1}ca = a^{-1}c^{-1}ca = b^{-1}c^{-1}cb = (cb)^{-1}cb$$

whence $ca\mathcal{H}cb$. Therefore S is cryptic.

(iii) $\Rightarrow$ (iv). By hypothesis, G_S is contained in the centralizer D of idempotents of S. Clearly D is an inverse subsemigroup of S whose idempotents are in its center, so by [3], Theorem II.2.6, D is a Clifford semigroup. Now, D being a union of its subgroups, the same is valid for G_S. Therefore S is group-closed.

(iv) $\Rightarrow$ (iii). Let $a \in G_S$ and $e \in E_S$. The hypothesis implies that $ae \in G_S$ whence $ae(ae)^{-1} = (ae)^{-1}ae$. It follows that $aea^{-1} = ea^{-1}a$ which yields

$$ae = ea^{-1}aa = eaa^{-1}a = ea.$$

(iii) $\Rightarrow$ (v). Recall that Δ is generated by the set $\{a, e\}$. If $\Delta\chi$ were not commutative, letting $b = a\chi$ and $f = e\chi$, we would have $bb^{-1} = b^{-1}b$ and $bf \neq fb$, contradicting the hypothesis.

(v) $\Rightarrow$ (iii). Let $a \in G_S$ and $e \in E_S$. Using a and e as generators of Δ, the identity mapping $a \mapsto a$, $e \mapsto e$, extends to a homomorphism $\chi\colon \Delta \to S$. Since the image of χ is commutative, we obtain $ae = ea$. $\square$

The equivalence of parts (i), (iii) and (iv) forms part of [5], Theorem 4. The implication (i) $\Rightarrow$ (iii) is the content of [1], Proposition 2.1. In view of Theorem 4.1, the equivalence of parts (i) and (ii) in [1], Proposition 2.3, holds for general inverse semigroups. As an obvious consequence of Theorem 4.1, we obtain [2], Corollary 3.7.

Corollary 4.2. *The semigroup S is cryptic if and only if every inverse subsemigroup of S generated by a group element a and an idempotent e such that $aa^{-1} > e$ is commutative (equivalently, is a semilattice of cyclic groups).*

Proof. This follows easily from Theorem 4.1 and its proof. $\square$

Corollary 4.3. *The semigroup S is cryptic and all its subgroups are abelian if and only if any two group elements commute.*

Proof. *Necessity.* Let $a \in H_{ef}$, $b \in H_f$, $e, f \in E_S$. Then $af, eb \in H_{ef}$, and by Theorem 4.1 and the hypothesis, we get

$$ab = (ae)(fb) = (af)(eb) = (eb)(af) = (bf)(ae) = ba.$$

Sufficiency. This follows directly from Theorem 4.1.

In view of Theorems 3.3 and 4.1 and Corollary 4.2, noncommutative homomorphic images represent the complete collection of forbidden inverse subsemigroups relative to the cryptic property. Their precise determination seems a formidable task. We will now exhibit an infinite family of such inverse semigroups.

Let $\mathbb{Z}_1 = \mathbb{Z}$ and consider $\mathbb{Z}_n$ for $n \geqslant 1$ with identity $\bar{0}$. Let $S_0 = B(\bar{0}, \mathbb{Z}_n)$ be a combinatorial Brandt semigroup and let $S_1 = \mathbb{Z}_n^0$ be the group $\mathbb{Z}_n$ with a zero adjoined. Adopting the pattern on the first pages of [4], we construct an ideal extension of S_0 by S_1 directly as follows.

Let $S_n = S_0 \cup \mathbb{Z}_n$ with the multiplication: for $(s, \bar{0}, t) \in S_0$ and $k \in \mathbb{Z}_n$, we define the product as

$$(s, \bar{0}, t)k = (s, \bar{0}, t + k), \quad k(s, \bar{0}, t) = (-k + s, \bar{0}, t), \quad k0 = 0k = 0,$$

and leave the given multiplication in S_0 and $\mathbb{Z}_n$ unchanged. Let $a \in \mathbb{Z}_n$, $a \neq \bar{0}$ and $e = (\bar{0}, \bar{0}, \bar{0})$. Straightforward verification shows that $\{a, e\}$ generates S_n, a is a group element, e is an idempotent, $-a + a > e$ and $ae \neq ea$. Therefore S_n is a forbidden inverse subsemigroup relative to the cryptic property. Since $\mathbb{Z}_n$ is the group of units of S_n, if $m \neq n$, we have $S_m \ncong S_n$ and the family $\{S_n\}_{n \geqslant 1}$ is indeed infinite. $\square$

For any inverse semigroup and $a, e \in S$, we say that (a, e) is a *bad pair* if $a \in G_S$, $e \in E_S$, $aa^{-1} > e$, and $ea \neq ae$. We now cast a closer look at the case $S = S_1$, that is, $S = B(1, \mathbb{Z}) \cup \mathbb{Z}$.

Proposition 4.4. *The following statements hold in $S = S_1$ and $a, e \in S$.*

(i) *The pair (a, e) is bad if and only if $a = m \in \mathbb{Z}$, $m \neq 0$ and $e \in E_{B(0,\mathbb{Z})}$, $e \neq 0$, say $e = (p, 0, p)$ for some $p \in \mathbb{Z}$.*

(ii) *Denote by $[m, p]$ the inverse subsemigroup of S generated by $\{a, e\}$ in part* (i). *Then $[m, p] = B(0, p + \langle m\rangle) \cup \langle m\rangle$.*

(iii) *With the similar notation $[n, q]$, the mapping*

$$(p + mk, 0, p + ml) \mapsto (q + nk, 0, q + nl), \quad mk \mapsto nk$$

is an isomorphism of $[m, p]$ onto $[n, q]$.

(iv) *In particular, for any bad pair in part* (i), *$[m, p] \cong [1, 0]$, the semigroup constructed above.*

Proof. This requires straightforward verification. □

A variety is said to be cryptic if all its members are cryptic. Reilly [4] proved that the inverse semigroups discussed above generate all minimal non cryptic varieties. As an important byproduct, he showed that all cryptic varieties are completely semisimple. As a consequence, in [3] Theorem XII.7.3 (i), the modifier "completely semisimple" may be omitted and [3], Problem XII.7.13 (ii), admits a negative solution.

In the first cited reference, we have the varieties

$$\mathcal{V}_n = [x^{n+1}yy^{-1}x^{-n-1} = x^n yy^{-1}x^{-n}], \quad n \geqslant 1.$$

Combinatorial inverse semigroups in $\mathcal{V}_n$ make up the variety $\mathcal{C}_n = [x^n = x^{n+1}]$, see [3], Notation XII.1.4 (viii). By [3], Proposition XII.1.8, we have $\bigvee_{n=1}^{\infty} \mathcal{C}_n = \mathcal{I}$. Therefore $\bigvee_{n=1}^{\infty} \mathcal{V}_n = \mathcal{I}$ and there exist no maximal cryptic varieties.

It is an intriguing idea that maybe the inverse semigroups generated by bad pairs generate minimal noncryptic varieties.

5. Further cases

What we have done for pure and cryptic inverse semigroups may be attempted for any remarkable class of inverse semigroups. We first exhibit this on the most outstanding class of inverse semigroups-groups. Denote by Y_2 the semilattice $\{0, 1\}$ and by $\mathcal{S}$ the variety of semilattices.

Proposition 5.1. *The following conditions on S are equivalent.*

(i) *S is a group.*

(ii) *S has no subsemigroup isomorphic to Y_2.*
(iii) *Y_2 does not divide S.*
(iv) *If $\chi\colon Y_2 \to S$ is a homomorphism, then the image of χ is trivial.*

Proof. Straightforward. $\square$

In the layers of varieties of inverse semigroups, we are able to go one more step, namely to Clifford semigroups.

Theorem 5.2. *The following conditions on S are equivalent.*
(i) *S is a Clifford semigroup.*
(ii) *S has no subsemigroup isomorphic to any of: free monogenic inverse semigroups of type (k, ∞^+) for $k \geqslant 1$, of type (k, l) for $k, l > 1$, of type (k, ω) for $k > 1$.*
(iii) *B_2 does not divide S.*
(iv) *If $\chi\colon I_x \to S$ is a homomorphism, then the image of χ is a (cyclic) group.*

Proof. (i) $\Rightarrow$ (ii). Let $a \in S$. Then $\langle a \rangle$ is a monogenic inverse semigroup which satisfies the identity $xx^{-1} = x^{-1}x$. In the classification of monogenic inverse semigroups, see [3], Theorem IX.3.11 and Corollary IX.3.12, the semigroup $\langle a \rangle$ is of type $(1, \omega)$, which together with the semigroups listed constitutes the set of all possible monogenic inverse semigroups. From [3], Theorem IX.3.4, it follows at once that the semigroups listed are not Clifford semigroups.

(ii) $\Rightarrow$ (i). From these remarks, it follows that the only monogenic inverse semigroups must be those which satisfy the identity $xx^{-1} = x^{-1}x$. Hence S itself satisfies this identity, which evidently implies that S is the union of its subgroups and is thus a Clifford semigroup.

(i) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (ii). The argument is by contrapositive. In view of the last reference cited, each of the semigroups listed in part (ii) has B_2 as a homomorphic image, by a Rees congruence.

(i) $\Leftrightarrow$ (iv). Also this follows from the classification of monogenic inverse semigroups since the identity $xx^{-1} = x^{-1}x$ is the only one which is satisfied by a homomorphic image of I_x which is a Clifford semigroup. But this is in turn equivalent, being monogenic, to being a (cyclic) group. $\square$

We are not able to prove an analogous statement for strict inverse semigroups since these are determined by an identity in two variables, and we have no construction of 2-generated inverse semigroups.

For completely semisimple semigroups we have the following statement.

Proposition 5.3. *The following conditions on S are equivalent.*
(i) *S is completely semisimple.*

(ii) *S has no bicyclic subsemigroup.*

(iii) *If $\chi\colon C \to S$ is a homomorphism, then χ is not injective.*

Proof. (i) $\Leftrightarrow$ (ii). This is a special case of [3], Corollary IX.4.13.

(ii) $\Leftrightarrow$ (iii). This is a consequence of the well-known fact that for all proper congruences ϱ on a bicyclic semigroup C we have that C/ϱ is a (cyclic) group. Or, one can use [3], Theorem X.1.1, to show that nonisomorphic homomorphic images of C are (cyclic) groups. □

For the combinatorial case, we have the following simple statement.

Proposition 5.4. *The following conditions on S are equivalent.*

(i) *S is combinatorial.*

(ii) *S has no subsemigroups isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_p$ for any prime p.*

(iii) *If $\chi\colon \mathbb{Z} \to S$ is a homomorphism, then the image of χ is trivial.*

Proof. Straightforward. □

A free inverse semigroup is

▹ pure in view of [3], Theorem VIII.1.5,

▹ completely semisimple by [3], Corollary VIII.1.16,

▹ combinatorial by [3], Proposition VIII.1.14,

and by [3], Corollary VIII.1.9, every inverse semigroup is a homomorphic image of a free inverse semigroup. Hence for pure, completely semisimple and combinatorial inverse semigroups there are no forbidden divisors, which excludes the possibility of a characterization by means of this concept.

We now collect a number of similar results, from here and elsewhere.

Theorem 5.5. *In the lattice $\mathcal{L}(\mathcal{I})$ of varieties of inverse semigroups the following statements hold.*

(i) *$\mathcal{S} = \langle Y_2\rangle$ is a minimal nongroup variety and every nongroup variety contains $\mathcal{S}$.*

(ii) *$\mathcal{B} = \langle B_2\rangle$ is a minimal non Clifford variety and every non Clifford variety contains $\mathcal{B}$.*

(iii) *$\langle B_2^1\rangle$ is a minimal nonstrict variety and every nonstrict variety contains $\langle B_2^1\rangle$.*

(iv) *For any prime p, $\mathcal{V}_p$ and $\mathcal{B}$ are minimal nonpure varieties and every nonpure variety contains either $\mathcal{V}_p$ for some prime p or $\mathcal{B}$, where $\mathcal{V}_p = \langle \mathbb{Z}_p^0\rangle$ and $\mathcal{B} = \langle B_2\rangle$.*

(v) *For any prime p, $\mathcal{V}_p$ and $\mathcal{V}_\infty$ are minimal noncryptic varieties and every noncryptic variety contains $\mathcal{V}_p$ for some prime p or $\mathcal{V}_\infty$, where $\mathcal{V}_p$ and $\mathcal{V}_\infty$ are defined in* [4], *page 472.*

(vi) *$\mathcal{C} = \langle C\rangle$ is a minimal non completely semisimple variety and every non completely semisimple variety contains $\mathcal{C}$.*

(vii) *For any prime p, $\mathcal{AG}_p$ is a minimal noncombinatorial variety and every noncombinatorial variety contains $\mathcal{AG}_p$ for some prime p.*

P r o o f. (i)–(iii). See [3], Corollary XII.4.14, and the discussion leading to it.

(iv) See Theorem 2.2.

(v) This is [4], Corollary 6.4.

(vi) This follows from [3], Corollary IX.4.13.

(viii) It is well known that for any prime p, the variety $\mathcal{AG}_p$ is an atom of $\mathcal{L}(\mathcal{I})$. Conversely, let $\mathcal{V}$ be a noncombinatorial variety. Then $\mathcal{V}$ contains a semigroup S which is not combinatorial, and thus contains a nontrivial subgroup G. Let $a \in G$ be different from the identity of G. If a is of infinite order, then $\mathbb{Z} \in \mathcal{V}$ and thus $\mathcal{AG} \subseteq \mathcal{V}$ and hence $\mathcal{A}_p \subseteq \mathcal{V}$ for all primes p. If a is of finite order n, then $\mathbb{Z}_n \in \mathcal{V}$ and for any prime p which divides n, also $\mathbb{Z}_p \in \mathcal{V}$ and thus $\mathcal{A}_p \subseteq \mathcal{V}$. $\square$

Note that the statement "$\mathcal{V}_1, \mathcal{V}_2, \ldots$ are minimal non-$\mathcal{V}$ varieties and every non-$\mathcal{V}$ variety contains some $\mathcal{V}_i$" immediately implies that the set $\{\mathcal{V}_1, \mathcal{V}_2, \ldots\}$ is the complete set of minimal non-$\mathcal{V}$ varieties.

Observe that in parts (iv), (vi) and (vii) of Theorem 5.5 we have varieties with minimal non-$\mathcal{V}$ subvarieties but without forbidden divisors, as we saw after Proposition 5.4.

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