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SEMI-SLANT RIEMANNIAN MAPS INTO ALMOST HERMITIAN MANIFOLDS

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Abstract. We introduce semi-slant Riemannian maps from Riemannian manifolds to almost Hermitian manifolds as a generalization of semi-slant immersions, invariant Riemannian maps, anti-invariant Riemannian maps and slant Riemannian maps. We obtain characterizations, investigate the harmonicity of such maps and find necessary and sufficient conditions for semi-slant Riemannian maps to be totally geodesic. Then we relate the notion of semi-slant Riemannian maps to the notion of pseudo-horizontally weakly conformal maps, which are useful for proving various complex-analytic properties of stable harmonic maps from complex projective space and give many examples of such maps.

Keywords: Riemannian map; semi-slant Riemannian map; harmonic map; totally geodesic map

MSC 2010: 53C15, 53C43

1. Introduction

It is known that complex techniques in physics have been very effective tools for understanding spacetime geometry [18]. Indeed, complex manifolds have two interesting classes of Kähler manifolds. One is Calabi-Yau manifolds, which have their applications in superstring theory [7]. The other one is Teichmüller spaces applicable to relativity [26]. For complex methods in general relativity, see [12].

Let $\overline{M}$ be a Kähler manifold with complex structure J and M a Riemannian manifold isometrically immersed in $\overline{M}$. We note that many types of submanifolds can be defined depending on the behaviour of the tangent bundle of the submanifold under the action of the complex structure of the ambient manifold. A submanifold M is called *holomorphic (complex)* if $J(T_pM) \subset T_pM$ for every $p \in M$, where T_pM denotes the tangent space to M at the point p. M is called *totally real* if $J(T_pM) \subset T_pM^{\perp}$ for every $p \in M$, where $T_pM^{\perp}$ denotes the normal space to M

at the point p. The submanifold M is called a *CR-submanifold* [5] if there exists a differentiable distribution $\mathcal{D}\colon\ p \to \mathcal{D}_p \subset T_pM$ such that $\mathcal{D}$ is invariant with respect to J and the complementary distribution $\mathcal{D}^\perp$ is anti-invariant with respect to J. The submanifold M is called *slant* ([8] and [9]) if for every nonzero vector X tangent to M the angle $\theta(X)$ between JX and T_pM is constant, i.e., it does not depend on the choice of $p \in M$ and $X \in T_pM$. A slant or a CR-submanifold is called *proper* if it is neither holomorphic nor totally real. It is easy to see that there is no inclusion relation between proper CR-submanifolds and proper slant submanifolds. Therefore, as a generalization of slant submanifolds and CR-submanifolds, semi-slant submanifolds were defined in [22].

Riemannian submersions between Riemannian manifolds were studied by O'Neill [21] and Gray [16]. Later, such submersions were considered between manifolds with differentiable structures. As an analogue of holomorphic submanifolds, Watson defined almost Hermitian submersions between almost Hermitian manifolds and he showed that the base manifold and each fiber have the same kind of structure as the total space in most cases [27]. We note that almost Hermitian submersions have been extended to almost contact manifolds [10], locally conformal Kähler manifolds [20] and quaternion Kähler manifolds [17] (see [13] for details concerning Riemannian submersions between Riemannian manifolds equipped with additional complex, contact, locally conformal or quaternion Kähler structures).

In 1992, Fischer introduced Riemannian maps between Riemannian manifolds in [14] as a generalization of isometric immersions and Riemannian submersions. Let $F\colon\ (M_1, g_1) \longrightarrow (M_2, g_2)$ be a smooth map between Riemannian manifolds such that $0 < \operatorname{rank} F < \min\{m, n\}$, where $\dim M_1 = m$ and $\dim M_2 = n$. Then we denote the kernel space of F_* by $\ker F_*$ and consider the orthogonal complementary space $(\ker F_*)^\perp$ to $\ker F_*$. Then the tangent bundle of M_1 has the following decomposition:

$$TM_1 = \ker F_* \oplus (\ker F_*)^\perp.$$

We denote the range of F_{*p_1} by $\operatorname{range} F_{*p_1}$ for $p_1 \in M_1$ and consider the orthogonal complementary space $(\operatorname{range} F_{*p_1})^\perp$ to $\operatorname{range} F_{*p_1}$ in the tangent space $T_{p_2}M_2$, $p_2 = F(p_1)$. Since $\operatorname{rank} F < \min\{m, n\}$, we always have $(\operatorname{range} F_{*p_1})^\perp$. Thus the tangent space $T_{p_2}M_2$ has the following decomposition

$$T_{p_2}M_2 = (\operatorname{range} F_{*p_1}) \oplus (\operatorname{range} F_{*p_1})^\perp.$$

Now, a smooth map $F\colon\ (\overset{m}{M_1}, g_1) \longrightarrow (\overset{n}{M_2}, g_2)$ is called a *Riemannian map* at $p_1 \in M_1$ if the horizontal restriction $\overset{h}{F_{*p_1}}\colon\ (\ker F_{*p_1})^\perp \longrightarrow (\operatorname{range} F_{*p_1})$ is a linear isometry between the inner product spaces $((\ker F_{*p_1})^\perp, g_1(p_1)|_{(\ker F_{*p_1})^\perp})$ and

$(\operatorname{range} F_{*p_1}, g_2(p_2)|_{(\operatorname{range} F_{*p_1})})$. Therefore, Fischer stated in [14] that a Riemannian map is a map which is as isometric as it can be. In another words, a smooth map $F\colon (M_1, g_1) \longrightarrow (M_2, g_2)$ between Riemannian manifolds (M_1, g_1) and (M_2, g_2) is called a *Riemannian map* if it satisfies the equation

$$g_2(F_*X, F_*Y) = g_1(X, Y) \tag{1.1}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$. It follows that isometric immersions and Riemannian submersions are particular Riemannian maps with $\ker F_* = \{0\}$ and $(\operatorname{range} F_*)^\perp = \{0\}$. It is known that a Riemannian map is a subimmersion [14]. It is also important to note that Riemannian maps satisfy the eikonal equation which is a bridge between geometric optics and physical optics. For Riemannian maps and their applications in spacetime geometry, see [15].

As a generalization of holomorphic submanifolds and totally real submanifolds, invariant Riemannian maps and anti-invariant Riemannian maps from Riemannian manifolds to almost Hermitian manifolds were introduced in [25]. Semi-invariant Riemannian maps from Riemannian manifolds to almost Hermitian manifolds were defined and the geometry of such maps was studied in [24] as a generalization of invariant and anti-invariant Riemannian maps. On the other hand, slant Riemannian maps were introduced in [23] and it was shown that such maps include slant submanifolds (therefore holomorphic immersions and totally real immersions), invariant Riemannian maps and anti-invariant Riemannian maps.

In this paper, we introduce semi-slant Riemannian maps from Riemannian manifolds to almost Hermitian manifolds. We show that such maps include semi-slant immersions (therefore holomorphic immersions, totally real immersions, slant immersions), invariant Riemannian maps, anti-invariant Riemannian maps, semi-invariant Riemannian maps and slant Riemannian maps. We obtain characterizations of semi-slant Riemannian maps and investigate the harmonicity of such maps. We also investigate necessary and sufficient conditions for semi-slant Riemannian maps to be totally geodesic. Moreover, we show that every semi-slant Riemannian map is a pseudo-horizontally weakly conformal map. In this direction, we find necessary and sufficient conditions for such a map to be a pseudo-homothetic map. We recall that the notion of pseudo-horizontally weakly conformal maps was introduced in [6] to study the stability of harmonic maps into irreducible Hermitian symmetric spaces of compact type, later such maps have been studied in [2], [3] and [19].

2. Preliminaries

Let $F\colon\ (M, g_M) \mapsto (N, g_N)$ be a C^∞-map. The second fundamental form of F is given by

$$(\nabla F_*)(X, Y) := \nabla^F_X F_* Y - F_*(\nabla_X Y) \quad \text{for } X, Y \in \Gamma(TM), \tag{2.1}$$

where ∇^F is the pullback connection and we denote conveniently by ∇ the Levi-Civita connections of the metrics g_M and g_N [4]. The tension field of F is defined by $\tau(F) := \operatorname{trace}(\nabla F_*)$.

Now, we consider the harmonicity of the map F. Given a C^∞-map F from a Riemannian manifold (M, g_M) to a Riemannian manifold (N, g_N), we define a function $e(F)\colon\ M \mapsto [0, \infty)$ given by

$$e(F)(x) := \frac{1}{2}|(F_*)_x|^2, \quad x \in M,$$

where $|(F_*)_x|$ denotes the Hilbert-Schmidt norm of $(F_*)_x$ [4]. Then we call $e(F)$ the *energy density* of F. Let K be a compact domain of M, i.e., K is the compact closure $\overline{U}$ of a nonempty connected open subset U of M. The *energy integral* of F over K is the integral of its energy density:

$$E(F; K) := \int_K e(F) v_{g_M} = \frac{1}{2}\int_K |F_*|^2 v_{g_M},$$

where v_{g_M} is the volume form on (M, g_M). Let $C^\infty(M, N)$ denote the space of all C^∞-maps from M to N. A C^∞-map $F\colon\ M \mapsto N$ is said to be *harmonic* if it is a critical point of the energy functional $E(\cdot; K)\colon\ C^\infty(M, N) \mapsto \mathbb{R}$ for any compact domain $K \subset M$. By the result of J. Eells and J. Sampson [11], we know that the map F is harmonic if and only if the tension field $\tau(F) = \operatorname{trace}(\nabla F_*) = 0$. On the other hand, a map F is called *totally geodesic* if $(\nabla F_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM)$ [4].

Denote the range of F_* by $\operatorname{range} F_*$ as a subset of the pullback bundle $F^{-1}TN$. With its orthogonal complement $(\operatorname{range} F_*)^\perp$ we have the following decomposition:

$$F^{-1}TN = \operatorname{range} F_* \oplus (\operatorname{range} F_*)^\perp.$$

Moreover, we get

$$TM = \ker F_* \oplus (\ker F_*)^\perp.$$

We now recall the following result from [25], which will be very useful when we investigate the geometry of semi-slant Riemannian maps.

Lemma 2.1. *Let F be a Riemannian map from a Riemannian manifold (M, g_M) to a Riemannian manifold (N, g_N). Then*

$$(\nabla F_*)(X, Y) \in \Gamma((\operatorname{range} F_*)^\perp) \quad \textit{for } X, Y \in \Gamma((\ker F_*)^\perp).$$

We also recall the following result, which shows that a Riemannian map satisfies the eikonal equation, which is a bridge between geometric optics and physical optics.

Lemma 2.2. *Let F be a Riemannian map from a Riemannian manifold (M, g_M) to a Riemannian manifold (N, g_N). Then the map F satisfies a generalized eikonal equation, see* [14]

$$2e(F) = \|F_*\|^2 = \operatorname{rank} F.$$

As we know, $\|F_*\|^2$ is a continuous function on M and $\operatorname{rank} F$ is integer-valued, so $\operatorname{rank} F$ is locally constant. Hence, if M is connected, then $\operatorname{rank} F$ is a constant function [1].

3. Semi-slant Riemannian maps

In this section we are going to define semi-slant Riemannian maps and obtain their characterizations. We also investigate the geometry of such maps. But we first need to recall the following notions. Let $F\colon (N, g_N) \mapsto (M, g_M, J)$ be a C^∞-map. We call the map F an *invariant Riemannian map* [25] if F is a Riemannian map and $J((\operatorname{range} F_*)_{F(p)}) = (\operatorname{range} F_*)_{F(p)}$ for $p \in N$, where $(\operatorname{range} F_*)_{F(p)} := (F_*)_p((\ker(F_*)_p)^\perp)$. The map F is said to be an *anti-invariant Riemannian map* [25] if F is a Riemannian map and $J((\operatorname{range} F_*)_{F(p)}) \subset ((\operatorname{range} F_*)_{F(p)})^\perp$ for $p \in N$. The map F is said to be a *slant Riemannian map* [23] if F is a Riemannian map and the angle $\theta = \theta(X)$ between JF_*X and the space $F_*((\ker(F_*)_p)^\perp)$ is constant for nonzero $X \in (\ker(F_*)_p)^\perp$ and $p \in N$. We call the angle θ a *slant angle*. We are now ready to present the following definition, which is a generalization of the above Riemannian maps.

Definition 3.1. Let (N, g_N) be a Riemannian manifold and (M, g_M, J) an almost Hermitian manifold. A Riemannian map $F\colon (N, g_N) \mapsto (M, g_M, J)$ is called a *semi-slant Riemannian map* if there is a distribution $\mathcal{D}_1 \subset (\ker F_*)^\perp$ such that

$$(\ker F_*)^\perp = \mathcal{D}_1 \oplus \mathcal{D}_2, \quad J(F_*\mathcal{D}_1) = F_*\mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between JF_*X and the space $F_*(\mathcal{D}_2)_p$ is constant for nonzero $X \in (\mathcal{D}_2)_p$ and $p \in N$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $(\ker F_*)^\perp$.

We call the angle θ a *semi-slant angle*.

Let $F\colon (N, g_N) \mapsto (M, g_M, J)$ be a semi-slant Riemannian map. Then for $X \in \Gamma((\ker F_*)^\perp)$, we write

$$X = PX + QX, \tag{3.1}$$

where $PX \in \Gamma(\mathcal{D}_1)$ and $QX \in \Gamma(\mathcal{D}_2)$.

For $U \in \Gamma(\operatorname{range} F_*)$, we get

$$JU = \varphi U + \omega U, \tag{3.2}$$

where $\varphi U \in \Gamma(\operatorname{range} F_*)$ and $\omega U \in \Gamma((\operatorname{range} F_*)^\perp)$.

For $V \in \Gamma((\operatorname{range} F_*)^\perp)$, we have

$$JV = BV + CV, \tag{3.3}$$

where $BV \in \Gamma(\operatorname{range} F_*)$ and $CV \in \Gamma((\operatorname{range} F_*)^\perp)$.

For $Y \in \Gamma(TN)$, we obtain

$$Y = \mathcal{V}Y + \mathcal{H}Y, \tag{3.4}$$

where $\mathcal{V}Y \in \Gamma(\ker F_*)$ and $\mathcal{H}Y \in \Gamma((\ker F_*)^\perp)$.

Define the tensors $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \tag{3.5}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F \tag{3.6}$$

for $E, F \in \Gamma(TN)$.

For $W \in \Gamma(F^{-1}TM)$, we write

$$W = \overline{P}W + \overline{Q}W, \tag{3.7}$$

where $\overline{P}W \in \Gamma(\operatorname{range} F_*)$ and $\overline{Q}W \in \Gamma((\operatorname{range} F_*)^\perp)$.

For $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma((\operatorname{range} F_*)^\perp)$, define

$$\widehat{\nabla}_X^F F_*Y := \overline{P}\nabla_X^F F_*Y, \tag{3.8}$$

$$\mathcal{S}_V F_*Y := -\overline{P}\nabla_X^F V, \tag{3.9}$$

$$\nabla_X^{F\perp} V := \overline{Q}\nabla_X^F V. \tag{3.10}$$

Then

$$\nabla^F_X V = -\mathcal{S}_V F_* Y + \nabla^{F\perp}_X V \tag{3.11}$$

and $\nabla^{F\perp}$ is a connection on $(\operatorname{range} F_*)^\perp$ such that $\nabla^{F\perp} g_M = 0$.

For $X, Y \in \Gamma((\ker F_*)^\perp)$, define

$$(\nabla^F_X \varphi) F_* Y := \widehat{\nabla}^F_X \varphi F_* Y - \varphi \widehat{\nabla}^F_X F_* Y, \tag{3.12}$$

$$(\nabla^F_X \omega) F_* Y := \nabla^{F\perp}_X \omega F_* Y - \omega \widehat{\nabla}^F_X F_* Y. \tag{3.13}$$

Then we have

$$(\nabla^F_X \varphi) F_* Y = \mathcal{S}_{\omega F_* Y} F_* X + B(\nabla F_*)(X, Y), \tag{3.14}$$

$$(\nabla^F_X \omega) F_* Y = C(\nabla F_*)(X, Y) - (\nabla F_*)(X, Y') \tag{3.15}$$

for some $Y' \in \Gamma((\ker F_*)^\perp)$ with $F_* Y' = \varphi F_* Y$.

We call the tensor φ *parallel* if $\nabla^F \varphi = 0$ and the tensor ω is *parallel* if $\nabla^F \omega = 0$.

Then we easily obtain the following lemma:

Lemma 3.2. *Let (M, g_M, J) be a Kähler manifold and (N, g_N) a Riemannian manifold. Let $F\colon (N, g_N) \mapsto (M, g_M, J)$ be a semi-slant Riemannian map. Then we get*

(a) $\widehat{\nabla}^F_X \varphi F_* Y = \varphi \widehat{\nabla}^F_X F_* Y + B\overline{Q}\nabla^F_X F_* Y$,
$\overline{Q}\nabla^F_X \varphi F_* Y = \omega \widehat{\nabla}^F_X F_* Y + C\overline{Q}\nabla^F_X F_* Y$
for $X, Y \in \Gamma(\mathcal{D}_1)$;

(b) $\widehat{\nabla}^F_X F_* Y' - \mathcal{S}_{\omega F_* Y} F_* X = \varphi \widehat{\nabla}^F_X F_* Y + B\overline{Q}\nabla^F_X F_* Y$,
$\overline{Q}\nabla^F_X F_* Y' + \nabla^{F\perp}_X \omega F_* Y = \omega \widehat{\nabla}^F_X F_* Y + C\overline{Q}\nabla^F_X F_* Y$
for $X, Y \in \Gamma(\mathcal{D}_2)$ where $Y' \in \Gamma((\ker F_*)^\perp)$ and $F_* Y' = \varphi F_* Y$.

Similarly to Theorem 3.1 of [23], we have the following theorem:

Theorem 3.3. *Let F be a semi-slant Riemannian map from a Riemannian manifold (N, g_N) to an almost Hermitian manifold (M, g_M, J) with the semi-slant angle θ. Then we obtain*

$$\varphi^2 F_* X = -\cos^2\theta \cdot F_* X \quad \textit{for } X \in \Gamma(\mathcal{D}_2). \tag{3.16}$$

Remark 3.4. It is easy to check that the converse of Theorem 3.3 is also true. Furthermore, we get

$$g_M(\varphi F_* X, \varphi F_* Y) = \cos^2\theta\, g_M(F_* X, F_* Y), \tag{3.17}$$

$$g_M(\omega F_* X, \omega F_* Y) = \sin^2\theta\, g_M(F_* X, F_* Y) \tag{3.18}$$

for $X, Y \in \Gamma(\mathcal{D}_2)$, hence with $\theta \in [0, \pi/2)$, there is locally an orthonormal frame $\{F_*e_1, \sec\theta\varphi F_*e_1, \ldots, F_*e_k, \sec\theta\varphi F_*e_k\}$ of $F_*\mathcal{D}_2$ for some $\{e_1, \ldots, e_k\} \subset \Gamma(\mathcal{D}_2)$.

Let F be a C^∞-map from a Riemannian manifold (N, g_N) into a Riemannian manifold (M, g_M). Then the adjoint map ${}^*(F_*)_p$ of the differential $(F_*)_p$, $p \in N$, is given by

$$g_M((F_*)_pX, Z) = g_N(X, {}^*(F_*)_pZ) \quad \text{for } X \in T_pN \text{ and } Z \in T_{F(p)}M. \tag{3.19}$$

Moreover, if the map F is a Riemannian map, then we easily have

$$(F_*)_p{}^*(F_*)_pZ = Z \quad \text{for } Z \in (\operatorname{range} F_*)_{F(p)}$$

and

$${}^*(F_*)_p(F_*)_pX = X \quad \text{for } X \in (\ker(F_*)_p)^\perp,$$

so the linear map

$${}^*(F_*)_p\colon\ (\operatorname{range} F_*)_{F(p)} \mapsto (\ker(F_*)_p)^\perp$$

is an isomorphism.

Define $\mathcal{Q} := {}^*(F_*)\varphi(F_*)$. Using Theorem 3.3, we get the following result:

Corollary 3.5. *Let F be a semi-slant Riemannian map from a Riemannian manifold (N, g_N) to an almost Hermitian manifold (M, g_M, J) with the semi-slant angle θ. Then we obtain*

$$\mathcal{Q}^2X = -\cos^2\theta \cdot X \quad \textit{for } X \in \Gamma(\mathcal{D}_2). \tag{3.20}$$

Similarly to Lemma 3.2 of [23], we have the following lemma:

Lemma 3.6. *Let F be a semi-slant Riemannian map from a Riemannian manifold (N, g_N) to a Kähler manifold (M, g_M, J) with the semi-slant angle θ. If the tensor ω is parallel, then we get*

$$(\nabla F_*)(\mathcal{Q}X, \mathcal{Q}Y) = -\cos^2\theta \cdot (\nabla F_*)(X, Y) \quad \textit{for } X, Y \in \Gamma(\mathcal{D}_2). \tag{3.21}$$

P r o o f. Assume that the tensor ω is parallel.
Then by (3.15), we obtain

$$C(\nabla F_*)(X, Y) = (\nabla F_*)(X, \mathcal{Q}Y) \quad \text{for } X, Y \in \Gamma(\mathcal{D}_2).$$

Interchanging the role of X and Y implies

$$C(\nabla F_*)(Y,X) = (\nabla F_*)(Y,\mathcal{Q}X).$$

Since the tensor ∇F_* is symmetric, we have

$$(\nabla F_*)(X,\mathcal{Q}Y) = (\nabla F_*)(Y,\mathcal{Q}X),$$

so

$$(\nabla F_*)(\mathcal{Q}X,\mathcal{Q}Y) = (\nabla F_*)(X,\mathcal{Q}^2Y) = -\cos^2\theta\cdot(\nabla F_*)(X,Y).$$

□

Theorem 3.7. *Let F be a semi-slant Riemannian map from a Riemannian manifold (N,g_N) to a Kähler manifold (M,g_M,J) with the semi-slant angle $\theta\in[0,\pi/2)$. If the tensor ω is parallel, then F is harmonic if and only if all the fibers $F^{-1}(y)$ are minimal submanifolds of N for $y\in M$.*

Proof. We know

$$TN = (\ker F_*)\oplus(\ker F_*)^{\perp} = (\ker F_*)\oplus\mathcal{D}_1\oplus\mathcal{D}_2.$$

Moreover, all the fibers $F^{-1}(y)$ are minimal submanifolds of N for $y\in M$ if and only if $\operatorname{trace}(\nabla F_*)|_{(\ker F_*)}=0$.

Since $JF_*\mathcal{D}_1 = F_*\mathcal{D}_1$, there is locally an orthonormal frame $\{F_*v_1, JF_*v_1,\ldots, F_*v_l, JF_*v_l\}$ of $F_*\mathcal{D}_1$, so $\{v_1,\mathcal{Q}v_1,\ldots,v_l,\mathcal{Q}v_l\}$ is locally an orthonormal frame of $\mathcal{D}_1$. We can also choose locally an orthonormal frame $\{e_1,\sec\theta\mathcal{Q}e_1,\ldots,e_k,\sec\theta\mathcal{Q}e_k\}$ of $\mathcal{D}_2$.

It is easy to show that $\mathcal{Q}^2v_i=-v_i$ for $1\leqslant i\leqslant l$.

Since ω is parallel, by using (3.15) and the proof of Lemma 3.6, we get

$$\begin{aligned}\operatorname{trace}(\nabla F_*)|_{\mathcal{D}_1} &= \sum_{i=1}^{l}\{(\nabla F_*)(v_i,v_i)+(\nabla F_*)(\mathcal{Q}v_i,\mathcal{Q}v_i)\}\\ &= \sum_{i=1}^{l}\{(\nabla F_*)(v_i,v_i)+(\nabla F_*)(v_i,\mathcal{Q}^2v_i)\}\\ &= \sum_{i=1}^{l}\{(\nabla F_*)(v_i,v_i)-(\nabla F_*)(v_i,v_i)\}=0.\end{aligned}$$

Furthermore, by using Corollary 3.5,

$$\begin{aligned}\operatorname{trace}(\nabla F_*)|_{\mathcal{D}_2} &= \sum_{j=1}^{k}\{(\nabla F_*)(e_j,e_j)+(\nabla F_*)(\sec\theta\mathcal{Q}e_j,\sec\theta\mathcal{Q}e_j)\}\\ &= \sum_{j=1}^{k}\{(\nabla F_*)(e_j,e_j)+\sec^2\theta(\nabla F_*)(\mathcal{Q}e_j,\mathcal{Q}e_j)\}\\ &= \sum_{j=1}^{k}\{(\nabla F_*)(e_j,e_j)+\sec^2\theta(\nabla F_*)(e_j,\mathcal{Q}^2e_j)\}\\ &= \sum_{j=1}^{k}\{(\nabla F_*)(e_j,e_j)-(\nabla F_*)(e_j,e_j)\}=0.\end{aligned}$$

Therefore, the result follows. $\square$

Remark 3.8. Comparing Theorem 3.7 with Theorem 3.2 of [23], we see that the conditions for such maps to be harmonic are the same for slant Riemannian maps and semi-slant Riemannian maps.

We study the conditions for such a map F to be totally geodesic.

Theorem 3.9. *Let F be a semi-slant Riemannian map from a Riemannian manifold (N,g_N) to a Kähler manifold (M,g_M,J) with the semi-slant angle $\theta\in(0,\pi/2)$. Then the map F is totally geodesic if and only if*

(a) *all the fibers $F^{-1}(y)$ are totally geodesic for $y\in M$,*
(b) *the horizontal distribution $(\ker F_*)^\perp$ is a totally geodesic foliation,*
(c) $g_M(\widehat{\nabla}^F_X F_*Y',BV)+g_M(\overline{Q}\nabla^F_X\varphi F_*Y,CV)=0$ *for $X\in\Gamma((\ker F_*)^\perp)$, $V\in\Gamma((\operatorname{range}F_*)^\perp)$, and $Y\in\Gamma(\mathcal{D}_1)$ with $\varphi F_*Y=F_*Y'$ and $Y'\in\Gamma(\mathcal{D}_1)$,*
(d) $g_M(\mathcal{S}_{\omega F_*Y}F_*X,BV)=g_M(\nabla^{F\perp}_X\omega F_*Y,CV)-g_M(\nabla^{F\perp}_X\omega\varphi F_*Y,V)$ *for $X,Y\in\Gamma(\mathcal{D}_2)$ and $V\in\Gamma((\operatorname{range}F_*)^\perp)$.*

Proof. Given $U_1,U_2\in\Gamma(\ker F_*)$ and $X\in\Gamma((\ker F_*)^\perp)$, we have

$$g_M((\nabla F_*)(U_1,U_2),F_*X)=-g_M(F_*\nabla_{U_1}U_2,F_*X)=-g_N(\nabla_{U_1}U_2,X),$$

so $(\nabla F_*)(U_1,U_2)=0$ for $U_1,U_2\in\Gamma(\ker F_*)$ if and only if (a).

For $U\in\Gamma(\ker F_*)$ and $X,Y\in\Gamma((\ker F_*)^\perp)$,

$$\begin{aligned}g_M((\nabla F_*)(X,U),F_*Y) &= -g_M(F_*\nabla_X U,F_*Y)\\ &= -g_N(\nabla_X U,Y)=g_N(U,\nabla_X Y).\end{aligned}$$

Hence, $(\nabla F_*)(X, U) = 0$ for $U \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$ if and only if (b).

If $X \in \Gamma((\ker F_*)^\perp)$, $Y \in \Gamma(\mathcal{D}_1)$, and $V \in \Gamma((\operatorname{range} F_*)^\perp)$, then by using Lemma 2.1, we get

$$\begin{aligned} g_M((\nabla F_*)(X,Y),V) &= g_M(\nabla^F_X F_*Y, V) \\ &= g_M(\nabla^F_X \varphi F_*Y, JV) \\ &= g_M(\widehat{\nabla}^F_X F_*Y', BV) + g_M(\overline{Q}\nabla^F_X \varphi F_*Y, CV) \end{aligned}$$

for some $Y' \in \Gamma(\mathcal{D}_1)$ with $\varphi F_*Y = F_*Y'$, so $(\nabla F_*)(X,Y) = 0$ for $X \in \Gamma((\ker F_*)^\perp)$ and $Y \in \Gamma(\mathcal{D}_1)$ if and only if (c).

Given $X, Y \in \Gamma(\mathcal{D}_2)$ and $V \in \Gamma((\operatorname{range} F_*)^\perp)$, we obtain

$$\begin{aligned} g_M((\nabla F_*)(X,Y),V) &= g_M(\nabla^F_X F_*Y, V) \\ &= -g_M(\nabla^F_X J(\varphi F_*Y + \omega F_*Y), V) \\ &= \cos^2\theta g_M(\nabla^F_X F_*Y, V) - g_M(\nabla^F_X \omega\varphi F_*Y, V) \\ &\quad + g_M(\nabla^F_X \omega F_*Y, BV + CV), \end{aligned}$$

so with some elementary calculations, $(\nabla F_*)(X,Y) = 0$ for $X, Y \in \Gamma(\mathcal{D}_2)$ if and only if (d).

Therefore, we have the result. □

4. Semi-slant Riemannian maps and PHWC maps

Let F be a map from a Riemannian manifold (M_1, g_1) to a Kähler manifold (M_2, g_2, J), where g_1 and g_2 are Riemannian metrics on M_1 and M_2, respectively, and J is the complex structure on M_2. For any point $p \in M_1$, we denote the adjoint map of the tangent map $F_{*p}\colon T_pM_1 \longrightarrow T_{F(p)}M_2$ by ${}^*F_{*p}\colon T_{F(p)}M_2 \longrightarrow T_pM_1$. If $\operatorname{range} F_{*p}$ is J-invariant, then we can define an almost complex structure $\widehat{J}_p$ on the horizontal space $(\ker F_{*p})^\perp$ by

$$\widehat{J}_p = F_{*p}^{-1} \circ J_{F(p)} \circ F_{*p}.$$

If the spaces $\operatorname{range} F_{*p}$ are J-invariant for all p, then the almost complex structure on $(\ker F_*)^\perp$ is defined by

$$\widehat{J} = F_*^{-1} \circ J \circ F_*.$$

The map F is called *pseudo-horizontally weakly conformal (PHWC)* at p if and only if $\operatorname{range} F_{*p}$ is J-invariant and $g_1|_{(\ker F_{*p})^\perp}$ is $\widehat{J}_p$-Hermitian. The map F is called *pseudo-horizontally weakly conformal* if and only if it is pseudo-horizontally weakly conformal at any point of M_1. A pseudo-horizontally weakly conformal map F from a Riemannian manifold (M_1, g_1) to a Kähler manifold (M_2, g_2, J) is called *pseudo-horizontally homothetic* if and only if $\widehat{J}$ is parallel in horizontal directions, i.e., $\nabla^1_X \widehat{J} = 0$ for $X \in \Gamma((\ker F_*)^\perp)$ (for more details, see [2], [3]).

Proposition 4.1. *Let F be a semi-slant Riemannian map from a Riemannian manifold (M_1, g_1) to an almost Hermitian manifold (M_2, g_2, J). Then F is a PHWC map.*

Proof. For $X \in \Gamma((\ker F_*)^\perp)$, we define $\widetilde{J}F_*(X) = JF_*(PX) + \sec\theta\varphi F_*(QX)$. Then it is easy to see that $\widetilde{J}$ is a complex structure on $(\operatorname{range} F_*)$ and $\operatorname{range} F_*$ is invariant with respect to $\widetilde{J}$. We now define $\widehat{J}X = {}^*F_* \circ J \circ F_*(PX) + \sec\theta\, {}^*F_*\varphi F_*(QX)$. Then it follows that $\widehat{J}$ is a complex structure on $(\ker F_*)^\perp$, thus $((\ker F_*)^\perp, \widehat{J})$ is an almost complex distribution. Considering $\widehat{g} = g_1|_{(\ker F_*)^\perp}$, by direct computation we obtain

$$\widehat{g}(\widehat{J}X, \widehat{J}Y) = \widehat{g}(X, Y)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$. Thus $\widehat{g}$ is $\widehat{J}$-Hermitian and $((\ker F_*)^\perp, \widehat{g}, \widehat{J})$ is an almost Hermitian distribution. Therefore, F is a PHWC map. $\square$

We now give necessary and sufficient conditions for a semi-slant Riemannian map F from a Riemannian manifold (M_1, g_1) to a Kähler manifold (M_2, g_2, J) to be pseudo-horizontally homothetic. We denote ${}^*F_* JF_*$, $\overset{F}{\nabla}_X JF_*(PY) - JF_*(P\mathcal{H}\nabla^1_X Y)$ and $\overset{F}{\nabla}_X \varphi F_*(QY) - \varphi F_*(Q\mathcal{H}\nabla^1_X Y)$ by $\widetilde{\mathcal{Q}}$, $(\nabla_X \varphi_1)F_*(PY)$ and $(\nabla_X \varphi_2)F_*(QY)$, respectively, where ∇^1 denotes the Levi-Civita connection on M_1.

Theorem 4.2. *Let F be a semi-slant Riemannian map from a Riemannian manifold (M_1, g_1) to a Kähler manifold (M_2, g_2, J). Then F is a pseudo-horizontally homothetic map if and only if*

$$\begin{aligned}(\nabla F_*)(X, \widetilde{\mathcal{Q}}(PY)) &+ \sec\theta(\nabla F_*)(X, \mathcal{Q}(QX)) \\ &= (\nabla_X \varphi_1)F_*(PY) + \sec\theta(\nabla_X \varphi_2)F_*(QY)\end{aligned}$$

and

$$g_2(F_*(PY), J(\nabla F_*)(X, U)) = \sec\theta g_2(\varphi F_*(QY), (\nabla F_*)(X, U))$$

for $X \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$.

Proof. First of all, we have

$$(\nabla_X \widehat{J})Y = \nabla^1_X \widehat{J}Y - \widehat{J}\mathcal{H}\nabla^1_X Y$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$. Hence we obtain

$$\begin{aligned}(\nabla_X \widehat{J})Y &= \nabla^1_X {}^*F_* JF_*(PY) + \sec\theta {}^*F_*\varphi F_*(QY)\\ &\quad - {}^*F_* JF_*(P\mathcal{H}\nabla^1_X Y) - \sec\theta {}^*F_*\varphi F_*(Q\mathcal{H}\nabla^1_X Y).\end{aligned}$$

Then by direct computation we have

$$(\nabla_X \widehat{J})Y = \nabla^1_X \widetilde{\mathcal{Q}}(PY) + \sec\theta \nabla^1_X \mathcal{Q}(QY) - \widetilde{\mathcal{Q}}(P\mathcal{H}\nabla^1_X Y) - \sec\theta \mathcal{Q}(Q\mathcal{H}\nabla^1_X Y)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$. Thus, using (2.1) we get

$$\begin{aligned}F_*(\nabla_X \widehat{J})Y &= -(\nabla F_*)(X, \widetilde{\mathcal{Q}}(PY)) - \sec\theta((\nabla F_*)(X, \mathcal{Q}(QY))\\ &\quad + \nabla^F_X JF_*(PY) - JF_*(P\mathcal{H}\nabla^1_X Y)\\ &\quad + \sec\theta \nabla^F_X \varphi F_*(QY) - \sec\theta\varphi F_*(Q\mathcal{H}\nabla^1_X Y)).\end{aligned}$$

On the other hand, since $\mathcal{Q}Y, \widetilde{\mathcal{Q}}Y \in \Gamma((\ker F_*)^\perp)$, we have

$$g_1((\nabla^1_X \widehat{J})Y, U) = -\sec\theta g_1(\mathcal{Q}(QY)Y, \nabla^1_X U) - g_1(\widetilde{\mathcal{Q}}(PX), \nabla^1_X U)$$

for $U \in \Gamma(\ker F_*)$. Then using the adjoint map ${}^*F_*$ and (2.1) we obtain

$$g_1((\nabla^1_X \widehat{J})Y, U) = \sec\theta g_2(\varphi F_*(QY), (\nabla F_*)(X, U)) - g_2(F_*(PY), J(\nabla F_*)(, X, U)).$$

This completes the proof. □

5. Examples

Note that given an Euclidean space $\mathbb{R}^{2n}$ with coordinates $(y_1, y_2, \dots, y_{2n})$, we can canonically choose an almost complex structure J on $\mathbb{R}^{2n}$ as follows:

$$\begin{aligned}&J\Big(a_1\frac{\partial}{\partial y_1} + a_2\frac{\partial}{\partial y_2} + \dots + a_{2n-1}\frac{\partial}{\partial y_{2n-1}} + a_{2n}\frac{\partial}{\partial y_{2n}}\Big)\\ &\quad = -a_2\frac{\partial}{\partial y_1} + a_1\frac{\partial}{\partial y_2} + \dots - a_{2n}\frac{\partial}{\partial y_{2n-1}} + a_{2n-1}\frac{\partial}{\partial y_{2n}},\end{aligned}$$

where $a_1, \dots, a_{2n} \in \mathbb{R}$. Throughout this section, we will use this notation.

Example 5.1. Let F be an invariant Riemannian map from a Riemannian manifold (M, g_M) to an almost Hermitian manifold (N, g_N, J) [25]. Then the map F is a semi-slant Riemannian map with $\mathcal{D}_1 = (\ker F_*)^\perp$.

Example 5.2. Let F be an anti-invariant Riemannian map from a Riemannian manifold (M, g_M) to an almost Hermitian manifold (N, g_N, J) [25]. Then the map F is a semi-slant Riemannian map with $\mathcal{D}_2 = (\ker F_*)^\perp$ and the semi-slant angle $\theta = \pi/2$.

Example 5.3. Let F be a semi-invariant Riemannian map from a Riemannian manifold (M, g_M) to an almost Hermitian manifold (N, g_N, J) [24]. Then the map F is a semi-slant Riemannian map with the semi-slant angle $\theta = \pi/2$.

Example 5.4. Let F be a slant Riemannian map from a Riemannian manifold (M, g_M) to an almost Hermitian manifold (N, g_N, J) with the slant angle θ [23]. Then the map F is a semi-slant Riemannian map with $\mathcal{D}_2 = (\ker F_*)^\perp$ and the semi-slant angle θ.

Example 5.5. Let (M, g_M) be an m-dimensional Riemannian manifold and (N, g_N, J) a $2n$-dimensional almost Hermitian manifold. Let F be a Riemannian map from the Riemannian manifold (M, g_M) to the almost Hermitian manifold (N, g_N, J) with $\operatorname{rank} F = 2n - 1$. Then the map F is a semi-slant Riemannian map with $F_*\mathcal{D}_2 = J((F_*[(\ker F_*)^\perp])^\perp)$ and the semi-slant angle $\theta = \pi/2$.

Example 5.6. Define a map $F\colon\ \mathbb{R}^8 \mapsto \mathbb{R}^6$ by

$$F(x_1, x_2, \dots, x_8) = (y_1, y_2, \dots, y_6) = \Big(x_3, \frac{x_4 - x_5}{\sqrt{6}}, \frac{x_4 - x_5}{\sqrt{3}}, c, x_2, x_1\Big),$$

where c is constant. Then the map F is a semi-slant Riemannian map with

$$\ker F_* = \Big\langle \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} \Big\rangle,$$
$$\mathcal{D}_1 = \Big\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \Big\rangle, \quad \mathcal{D}_2 = \Big\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5} \Big\rangle,$$
$$F_*\mathcal{D}_1 = \Big\langle \frac{\partial}{\partial y_5}, \frac{\partial}{\partial y_6} \Big\rangle, \quad F_*\mathcal{D}_2 = \Big\langle \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} + \sqrt{2}\frac{\partial}{\partial y_3} \Big\rangle,$$

and the semi-slant angle θ with $\cos\theta = 1/\sqrt{3}$.

Example 5.7. Define a map $F\colon\ \mathbb{R}^9 \mapsto \mathbb{R}^6$ by

$$\begin{aligned} F(x_1, x_2, \dots, x_9) &= (y_1, y_2, \dots, y_6) \\ &= \Big(x_1, x_9, x_3, \frac{(x_4 + x_5)\cos\alpha}{\sqrt{2}}, \frac{(x_4 + x_5)\sin\alpha}{\sqrt{2}}, \beta\Big), \end{aligned}$$

where α and β are constant with $\alpha \in (0, \pi/2)$. Then the map F is a semi-slant Riemannian map with

$$\ker F_* = \Big\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} \Big\rangle,$$
$$\mathcal{D}_1 = \Big\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_9} \Big\rangle, \quad \mathcal{D}_2 = \Big\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} \Big\rangle,$$
$$F_*\mathcal{D}_1 = \Big\langle \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} \Big\rangle, \quad F_*\mathcal{D}_2 = \Big\langle \frac{\partial}{\partial y_3}, \sqrt{2}\cos\alpha\frac{\partial}{\partial y_4} + \sqrt{2}\sin\alpha\frac{\partial}{\partial y_5} \Big\rangle,$$

and the semi-slant angle $\theta = \alpha$.

Example 5.8. Define a map $F\colon \mathbb{R}^7 \mapsto \mathbb{R}^6$ by

$$F(x_1, x_2, \dots, x_7) = (y_1, y_2, \dots, y_6) = (x_2\sin\alpha, 0, x_3, x_5, x_2\cos\alpha, x_7),$$

where $\alpha \in (0, \frac{\pi}{2})$. Then the map F is a semi-slant Riemannian map with

$$\ker F_* = \Big\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_6} \Big\rangle,$$
$$\mathcal{D}_1 = \Big\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5} \Big\rangle, \quad \mathcal{D}_2 = \Big\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_7} \Big\rangle,$$
$$F_*\mathcal{D}_1 = \Big\langle \frac{\partial}{\partial y_3}, \frac{\partial}{\partial y_4} \Big\rangle, \quad F_*\mathcal{D}_2 = \Big\langle \sin\alpha\frac{\partial}{\partial y_1} + \cos\alpha\frac{\partial}{\partial y_5}, \frac{\partial}{\partial y_6} \Big\rangle,$$

and the semi-slant angle $\theta = \alpha$.

Example 5.9. Define a map $F\colon \mathbb{R}^6 \mapsto \mathbb{R}^8$ by

$$F(x_1, x_2, \dots, x_6) = (y_1, y_2, \dots, y_8) = \Big(x_1, \frac{x_2 + x_3}{2}, \frac{x_2 + x_3}{2}, 0, 0, 0, x_5, x_6\Big).$$

Then the map F is a semi-slant Riemannian map with

$$\ker F_* = \Big\langle \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \Big\rangle,$$
$$\mathcal{D}_1 = \Big\langle \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6} \Big\rangle, \quad \mathcal{D}_2 = \Big\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \Big\rangle,$$
$$F_*\mathcal{D}_1 = \Big\langle \frac{\partial}{\partial y_7}, \frac{\partial}{\partial y_8} \Big\rangle, \quad F_*\mathcal{D}_2 = \Big\langle \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3} \Big\rangle,$$

and the semi-slant angle $\theta = \pi/4$.

References

[1] *R. Abraham, J. E. Marsden, T. Ratiu*: Manifolds, Tensor Analysis, and Applications (2nd edition). Applied Mathematical Sciences 75, Springer, New York, 1988.

[2] *M. A. Aprodu, M. Aprodu*: Implicitly defined harmonic PHH submersions. Manuscr. Math. *100* (1999), 103–121.

[3] *M. A. Aprodu, M. Aprodu, V. Brînzănescu*: A class of harmonic submersions and minimal submanifolds. Int. J. Math. *11* (2000), 1177–1191.

[4] *P. Baird, J. C. Wood*: Harmonic Morphisms Between Riemannian Manifolds. London Mathematical Society Monographs. New Series 29, Clarendon Press, Oxford University Press, Oxford, 2003.

[5] *A. Bejancu*: Geometry of CR-Submanifolds. Mathematics and Its Applications (East European Series) 23, D. Reidel Publishing Co., Dordrecht, 1986.

[6] *D. Burns, F. Burstall, P. de Bartolomeis, J. Rawnsley*: Stability of harmonic maps of Kähler manifolds. J. Differ. Geom. *30* (1989), 579–594.

[7] *P. Candelas, G. T. Horowitz, A. Strominger, E. Witten*: Vacuum configurations for superstrings. Nuclear Phys. B (electronic only) *258* (1985), 46–74.

[8] *B.-Y. Chen*: Geometry of Slant Submanifolds. Katholieke Universiteit Leuven, Dept. of Mathematics, Leuven, 1990.

[9] *B.-Y. Chen*: Slant immersions. Bull. Aust. Math. Soc. *41* (1990), 135–147.

[10] *D. Chinea*: Almost contact metric submersions. Rend. Circ. Mat. Palermo (2) *34* (1985), 89–104.

[11] *J. J. Eells, J. H. Sampson*: Harmonic mappings of Riemannian manifolds. Am. J. Math. *86* (1964), 109–160.

[12] *G. Esposito*: From spinor geometry to complex general relativity. Int. J. Geom. Methods Mod. Phys. *2* (2005), 675–731.

[13] *M. Falcitelli, S. Ianus, A. M. Pastore*: Riemannian Submersions and Related Topics. World Scientific, River Edge, New York, 2004.

[14] *A. E. Fischer*: Riemannian maps between Riemannian manifolds. Mathematical aspects of classical field theory. Proc. of the AMS-IMS-SIAM Joint Summer Research Conf., Seattle, Washington, USA, 1991 (M. J. Gotay et al., eds.). Contemp. Math. 132, American Mathematical Society, Providence, 1992, pp. 331–366.

[15] *E. García-Río, D. N. Kupeli*: Semi-Riemannian Maps and Their Applications. Mathematics and Its Applications 475, Kluwer Academic Publishers, Dordrecht, 1999.

[16] *A. Gray*: Pseudo-Riemannian almost product manifolds and submersions. J. Math. Mech. *16* (1967), 715–737.

[17] *S. Ianuş, R. Mazzocco, G. E. Vîlcu*: Riemannian submersions from quaternionic manifolds. Acta Appl. Math. *104* (2008), 83–89.

[18] *D. E. Lerner, P. D. Sommers* eds.: Complex Manifold Techniques in Theoretical Physics. Research Notes in Mathematics 32, Pitman Advanced Publishing Program, San Francisco, 1979.

[19] *E. Loubeau, R. Slobodeanu*: Eigenvalues of harmonic almost submersions. Geom. Dedicata *145* (2010), 103–126.

[20] *J. C. Marrero, J. Rocha*: Locally conformal Kähler submersions. Geom. Dedicata *52* (1994), 271–289.

[21] *B. O'Neill*: The fundamental equations of a submersion. Mich. Math. J. *13* (1966), 459–469.

[22] *N. Papaghiuc*: Semi-slant submanifolds of a Kaehlerian manifold. An. Ştiinţ. Univ. Al. I. Cuza Iaşi, Ser. Nouă, Mat. *40* (1994), 55–61.

[23] *B. Şahin*: Slant Riemannian maps to Kähler manifolds. Int. J. Geom. Methods Mod. Phys. *10* (2013), Paper No. 1250080, 12 pages.

[24] *B. Şahin*: Semi-invariant Riemannian maps to Kähler manifolds. Int. J. Geom. Methods Mod. Phys. *8* (2011), 1439–1454.
[25] *B. Şahin*: Invariant and anti-invariant Riemannian maps to Kähler manifolds. Int. J. Geom. Methods Mod. Phys. *7* (2010), 337–355.
[26] *A. J. Tromba*: Teichmüller Theory in Riemannian Geometry: Based on Lecture Notes by Jochen Denzler. Lectures in Mathematics ETH Zürich, Birkhäuser, Basel, 1992.
[27] *B. Watson*: Almost Hermitian submersions. J. Differ. Geom. *11* (1976), 147–165.

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