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PRETTY CLEANNESS AND FILTER-REGULAR SEQUENCES

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Abstract. Let K be a field and $S = K[x_1, \ldots, x_n]$. Let I be a monomial ideal of S and $u_1, \ldots, u_r$ be monomials in S. We prove that if $u_1, \ldots, u_r$ form a filter-regular sequence on S/I, then S/I is pretty clean if and only if $S/(I, u_1, \ldots, u_r)$ is pretty clean. Also, we show that if $u_1, \ldots, u_r$ form a filter-regular sequence on S/I, then Stanley's conjecture is true for S/I if and only if it is true for $S/(I, u_1, \ldots, u_r)$. Finally, we prove that if $u_1, \ldots, u_r$ is a minimal set of generators for I which form either a d-sequence, proper sequence or strong s-sequence (with respect to the reverse lexicographic order), then S/I is pretty clean.

Keywords: almost clean module; clean module; d-sequence; filter-regular sequence; pretty clean module

MSC 2010: 13F20, 05E40

1. Introduction

Let R be a multigraded Noetherian ring and M a finitely generated multigraded R-module. (Here, "multigraded" stands for "$\mathbb{Z}^n$-graded".) A basic fact in commutative algebra says that there exists a finite filtration

$$\mathcal{F}\colon\ 0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

of multigraded submodules of M such that there are multigraded isomorphisms $M_i/M_{i-1} \cong R/\mathfrak{p}_i(-a_i)$ for some $a_i \in \mathbb{Z}^n$ and some multigraded prime ideals $\mathfrak{p}_i$ of R. Such a filtration of M is called a (multigraded) prime filtration. The set of prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ which define the cyclic quotients of $\mathcal{F}$ will be denoted by $\operatorname{Supp} \mathcal{F}$. It is known (and easy to see) that $\operatorname{Ass}_R M \subseteq \operatorname{Supp} \mathcal{F} \subseteq \operatorname{Supp}_R M$.

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Let $\operatorname{Min} M$ denote the set of minimal prime ideals of $\operatorname{Supp}_R M$. Dress [4] called a prime filtration $\mathcal{F}$ of M *clean* if $\operatorname{Supp} \mathcal{F} = \operatorname{Min} M$. Pretty clean filtrations were defined as a generalization of clean filtrations by Herzog and Popescu [7]. A prime filtration $\mathcal{F}$ is called *pretty clean* if for all $i < j$ for which $\mathfrak{p}_i \subseteq \mathfrak{p}_j$, it follows that $\mathfrak{p}_i = \mathfrak{p}_j$. If $\mathcal{F}$ is a pretty clean filtration of M, then $\operatorname{Supp} \mathcal{F} = \operatorname{Ass}_R M$; see [7], Corollary 3.4. The converse is not true in general as shown by some examples in [7] and [16]. The prime filtration $\mathcal{F}$ of M is called *almost clean* if $\operatorname{Supp} \mathcal{F} = \operatorname{Ass}_R M$.

The R-module M is called *clean* (or *pretty clean* or *almost clean*) if it admits a clean (or pretty clean or almost clean) filtration. Obviously, cleanness implies pretty cleanness and pretty cleanness implies almost cleanness.

Throughout, let K be a field and I a monomial ideal of the polynomial ring $S = K[x_1, \dots, x_n]$. In this paper, we always consider the ring S with its standard multigrading. So, an ideal J of S is multigraded if and only if J is a monomial ideal. When I is square-free, one has $\operatorname{Ass}_S S/I = \operatorname{Min} S/I$, and so the above three concepts coincide for S/I. If S/I is pretty clean, then [7], Theorem 6.5, asserts that Stanley's conjecture holds for S/I; see the paragraph preceding Theorem 3.6 for the statement of this conjecture.

Let $u_1, \dots, u_r$ be monomials in S. If $u_1, \dots, u_r$ is a regular sequence on S/I, then by [11], Theorem 2.1, S/I is pretty clean if and only if $S/(I, u_1, \dots, u_r)$ is pretty clean. In this paper, we pursuit this line of research not only for regular sequences, but also for other special types of sequences of monomials.

We show that the assertion of [11], Theorem 2.1, is also true for cleanness and almost cleanness. Also, we prove that if $u_1, \dots, u_r$ is a filter-regular sequence on S/I, then S/I is pretty clean if and only if $S/(I, u_1, \dots, u_r)$ is pretty clean. Next, we show that if $u_1, \dots, u_r$ form a filter-regular sequence on S/I, then Stanley's conjecture is true for S/I if and only if it is true for $S/(I, u_1, \dots, u_r)$.

Assume that $u_1, \dots, u_r$ is a minimal set of generators for I. We prove that if either $u_1, \dots, u_r$ is a d-sequence, proper sequence or strong s-sequence (with respect to the reverse lexicographic order), then S/I is pretty clean.

2. Regular sequences

We begin with the following preliminary results.

Lemma 2.1. *Let R be a commutative Noetherian ring, M an R-module and A an Artinian submodule of M. Then*

$$\operatorname{Ass}_R M = \operatorname{Ass}_R A \cup \operatorname{Ass}_R M/A.$$

P r o o f. It is well-known that

$$\operatorname{Ass}_R A \subseteq \operatorname{Ass}_R M \subseteq \operatorname{Ass}_R A \cup \operatorname{Ass}_R M/A.$$

On the other hand, [3], Lemma 2.2, yields that

$$\operatorname{Ass}_R M/A \subseteq \operatorname{Ass}_R M \cup \operatorname{Supp}_R A.$$

But A is Artinian, and so $\operatorname{Supp}_R A = \operatorname{Ass}_R A$. This implies our desired equality. □

Lemma 2.2. *Let R be a multigraded Noetherian ring, M a multigraded finitely generated R-module and A a multigraded Artinian submodule of M. If M/A is pretty clean (almost clean, respectively), then M is pretty clean (almost clean, respectively) too.*

P r o o f. Since A is an Artinian R-module, one has

$$\operatorname{Min} A = \operatorname{Ass}_R A = \operatorname{Supp}_R A \subseteq \operatorname{Max} R.$$

So obviously, if M/A is pretty clean, then M is pretty clean too. Also, by Lemma 2.1, almost cleanness of M/A implies almost cleanness of M. □

We denote the maximal monomial ideal $(x_1, \ldots, x_n)$ of the ring $S = K[x_1, \ldots, x_n]$ by $\mathfrak{m}$. For an S-module M, $H^i_{\mathfrak{m}}(M)$ denotes i-th local cohomology module of M with respect to $\mathfrak{m}$. If M is a multigraded finitely generated S-module, then $H^i_{\mathfrak{m}}(M)$ is a multigraded Artinian S-module for all i.

Example 2.3. Lemma 2.2 is not true for the cleanness. To this end, let $S = K[x, y]$ and $I = (x^2, xy)$. Set $M := S/I$ and $A := H^0_{\mathfrak{m}}(M)$. Clearly $A = (x)/I$, and so $M/A \cong S/(x)$. It is easy to see that M/A is clean while M is not clean.

Proposition 2.4. *Let M be a multigraded finitely generated S-module and A a multigraded Artinian submodule of M. Then M is pretty clean if and only if M/A is pretty clean.*

P r o o f. In view of Lemma 2.2, it remains to show that if M is pretty clean, then M/A is pretty clean. Let

$$\mathcal{F}\colon\ 0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

be a pretty clean filtration of M. For any S-module N, let $l_S(N)$ denote the length of N. First, by induction on $t := l_S(H^0_{\mathfrak{m}}(M))$, we show that $M/H^0_{\mathfrak{m}}(M)$ is pretty clean. For $t = 0$, there is nothing to prove. Now, assume that $t > 0$ and the claim holds for $t - 1$. Then $H^0_{\mathfrak{m}}(M) \neq 0$, and so $\mathfrak{m} \in \operatorname{Ass}_S M = \operatorname{Supp} \mathcal{F}$. Since the filtration $\mathcal{F}$ is pretty clean and $\operatorname{Ann}_S M_1 \subseteq \mathfrak{m}$, it follows that $M_1 \cong S/\mathfrak{m}$, and so $(M_1 :_M \mathfrak{m}^\infty) = H^0_{\mathfrak{m}}(M)$. Then, one has

$$H^0_{\mathfrak{m}}\Big(\frac{M}{M_1}\Big) = \frac{M_1 :_M \mathfrak{m}^\infty}{M_1} = \frac{H^0_{\mathfrak{m}}(M)}{M_1},$$

and so

$$l_S\Big(H^0_{\mathfrak{m}}\Big(\frac{M}{M_1}\Big)\Big) = l_S(H^0_{\mathfrak{m}}(M)) - l_S(M_1) = t - 1.$$

Obviously, M/M_1 is pretty clean, and so by the induction hypothesis, $(M/M_1)/H^0_{\mathfrak{m}}(M/M_1)$ is pretty clean. But,

$$\frac{M/M_1}{H^0_{\mathfrak{m}}(M/M_1)} = \frac{M/M_1}{H^0_{\mathfrak{m}}(M)/M_1} \cong \frac{M}{H^0_{\mathfrak{m}}(M)},$$

and hence $M/H^0_{\mathfrak{m}}(M)$ is pretty clean.

Since A is a multigraded Artinian submodule of M, one has $A \subseteq H^0_{\mathfrak{m}}(M)$. From the first part of the proof, we conclude that $(M/A)/(H^0_{\mathfrak{m}}(M)/A)$ is pretty clean. But $H^0_{\mathfrak{m}}(M)/A$ is a multigraded Artinian submodule of M/A, and so Lemma 2.2 implies that M/A is pretty clean. □

In what follows, we recall some needed notation and facts about monomial ideals. For each subset H of S, let $\operatorname{Mon} H$ denote the set of all monomials in H. For any monomial ideal I of S, there is a unique minimal generating set $\mathrm{G}(I)$ of I. Clearly, $\mathrm{G}(I)$ consists of finitely many monomials and there is no divisibility among different elements of $\mathrm{G}(I)$. Also for any nonempty subset T of $\operatorname{Mon} S$, set $\mathrm{G}(T) := \mathrm{G}((T))$. Clearly, $\mathrm{G}((T))$ is a finite subset of T. A monomial ideal of S is irreducible if and only if it is of the form $(x_{i_1}^{a_1}, \dots, x_{i_t}^{a_t})$, where $a_i \in \mathbb{N}$ for all i; see [6], Corollary 1.3.2. Moreover, $(x_{i_1}^{a_1}, \dots, x_{i_t}^{a_t})$ is $(x_{i_1}, \dots, x_{i_t})$-primary and each monomial ideal can be written as a finite intersection of irreducible monomial ideals. Let I be a monomial ideal of S and $\mathcal{P}$: $I = \bigcap\limits_{i=1}^{r} Q_i$ a primary decomposition of I such that each Q_i is an irreducible monomial ideal of S. We use notation $T_i(\mathcal{P})$ for $\mathrm{G}\Big(\operatorname{Mon}\Big(\bigcap\limits_{j=1}^{i-1} Q_j \setminus Q_i\Big)\Big)$. Notice that

$$T_1(\mathcal{P}) = \mathrm{G}(\operatorname{Mon}(S \setminus Q_1)) = \{1\}.$$

For proving our first theorem, we shall need the following lemma.

Lemma 2.5 ([14], Corollary 2.7)**.** *Let I be a monomial ideal of S. The following conditions are equivalent:*

a) *S/I is clean (or pretty clean or almost clean).*

b) *There exists a primary decomposition $\mathcal{P}\colon\ I=\bigcap_{j=1}^{r} Q_j$ of I, where each Q_j is an irreducible $\mathfrak{p}_j$-primary monomial ideal, such that*

   i) $\operatorname{ht}\mathfrak{p}_j \leqslant \operatorname{ht}\mathfrak{p}_{j+1}$ *for all j and* $\{\mathfrak{p}_1,\dots,\mathfrak{p}_r\}=\operatorname{Min} S/I$*,*
   (or $\operatorname{ht}\mathfrak{p}_j \leqslant \operatorname{ht}\mathfrak{p}_{j+1}$ *for all j or* $\{\mathfrak{p}_1,\dots,\mathfrak{p}_r\}=\operatorname{Ass}_S S/I$*) and*

   ii) *$T_j(\mathcal{P})$ is a singleton for all $1\leqslant j\leqslant r$.*

Next, we generalize [11], Theorem 2.1. It also extends [12], Corollary 4.10.

Theorem 2.6. *Let I be a monomial ideal of S and $u_1,\dots,u_c\in\operatorname{Mon} S$ a regular sequence on S/I. Then S/I is clean (or pretty clean or almost clean) if and only if $S/(I,u_1,\dots,u_c)$ is clean (or pretty clean or almost clean).*

P r o o f. By induction on c, it is enough to prove the case $c=1$. Let $u\in\operatorname{Mon} S$ be a non zero-divisor on S/I. Without loss of generality, we may and do assume that for some integer $0\leqslant t<n$, the only variables that divide u are $x_{t+1},\dots,x_n$. Then $u=\prod_{i=t+1}^{n} x_i^{a_i}$ for some natural integers $a_{t+1},\dots,a_n$ and $I=JS$ for some monomial ideal J of $S':=K[x_1,\dots,x_t]$.

First, we show that if S/I is clean (or pretty clean or almost clean), then $S/(I,u)$ is clean (or pretty clean or almost clean). Let $\mathcal{P}\colon\ I=\bigcap_{i=1}^{r} Q_i$ be a primary decomposition of I which satisfies the condition b) in Lemma 2.5. Let $1\leqslant e\leqslant r$. Since

$$\operatorname{Ass}_S S/I=\{\mathfrak{p}_1,\dots,\mathfrak{p}_r\}$$

and $\operatorname{Ass}_S S/Q_e=\{\mathfrak{p}_e\}$, it turns out that u is also a non zero-divisor on S/Q_e. Hence $Q_e=q_eS$ for some irreducible monomial ideal q_e of S'. Obviously,

$$\mathcal{P}'\colon\ (I,u)=\Bigl(\bigcap_{i=t+1}^{n}(Q_1,x_i^{a_i})\Bigr)\cap\Bigl(\bigcap_{i=t+1}^{n}(Q_2,x_i^{a_i})\Bigr)\cap\ldots\cap\Bigl(\bigcap_{i=t+1}^{n}(Q_r,x_i^{a_i})\Bigr)$$

is a primary decomposition of (I,u) and each $(Q_i,x_j^{a_j})$ is an irreducible $(\mathfrak{p}_i,x_j)$-primary monomial ideal. We are going to show that the condition b) in Lemma 2.5 holds for $\mathcal{P}'$. Clearly, $T_1(\mathcal{P}')$ is a singleton. For each $t+2\leqslant i\leqslant n$, we have

$$\begin{aligned}\mathrm{G}\Bigl(\operatorname{Mon}\Bigl(\bigcap_{j=t+1}^{i-1}(Q_1,x_j^{a_j})\setminus(Q_1,x_i^{a_i})\Bigr)\Bigr)\\ =\mathrm{G}\Bigl(\operatorname{Mon}\Bigl(\Bigl(Q_1,\prod_{j=t+1}^{i-1}x_j^{a_j}\Bigr)\setminus(Q_1,x_i^{a_i})\Bigr)\Bigr)=\Bigl\{\prod_{j=t+1}^{i-1}x_j^{a_j}\Bigr\}.\end{aligned}$$

Let $2 \leqslant i \leqslant r$, $t+1 \leqslant h \leqslant n$ and assume that $T_i(\mathcal{P}) = \{v\}$. Since

$$\Big(\Big(\bigcap_{j=1}^{i-1} \bigcap_{k=t+1}^{n} (Q_j, x_k^{a_k})\Big) \cap \Big(\bigcap_{l=t+1}^{h-1} (Q_i, x_l^{a_l})\Big)\Big) \setminus (Q_i, x_h^{a_h})$$

$$= \Big(\Big(\bigcap_{j=1}^{i-1} \Big(Q_j, \prod_{k=t+1}^{n} x_k^{a_k}\Big)\Big) \cap \Big(Q_i, \prod_{l=t+1}^{h-1} x_l^{a_l}\Big)\Big) \setminus (Q_i, x_h^{a_h}),$$

one has

$$\mathrm{G}\Big(\mathrm{Mon}\Big(\Big(\Big(\bigcap_{j=1}^{i-1} \bigcap_{k=t+1}^{n} (Q_j, x_k^{a_k})\Big) \cap \Big(\bigcap_{l=t+1}^{h-1} (Q_i, x_l^{a_l})\Big)\Big) \setminus (Q_i, x_h^{a_h})\Big)\Big) = \Big\{v \prod_{l=t+1}^{h-1} x_l^{a_l}\Big\}.$$

So, $T_i(\mathcal{P}')$ is a singleton for all i. On the other hand, we can easily deduce that

$$(*) \qquad \mathrm{Ass}_S \frac{S}{(I,u)} = \Big\{(\mathfrak{p}, x_k);\ \mathfrak{p} \in \mathrm{Ass}_S \frac{S}{I} \text{ and } t+1 \leqslant k \leqslant n\Big\},$$

$$(\dagger) \qquad \mathrm{Min} \frac{S}{(I,u)} = \Big\{(\mathfrak{p}, x_k);\ \mathfrak{p} \in \mathrm{Min} \frac{S}{I} \text{ and } t+1 \leqslant k \leqslant n\Big\},$$

and

$$(\ddagger) \qquad \mathrm{ht}(\mathfrak{p}, x_k) = \mathrm{ht}\,\mathfrak{p} + 1$$

for all $\mathfrak{p} \in \mathrm{Ass}_S S/I$ and all $t+1 \leqslant k \leqslant n$. Hence $\mathcal{P}'$ satisfies the condition b) in Lemma 2.5.

Conversely, let $S/(I,u)$ be clean (or pretty clean or almost clean). So, (I,u) has a primary decomposition $\mathcal{P}$ which satisfies the condition b) in Lemma 2.5. From $(*)$, we can conclude that $\mathcal{P}$ has the form

$$\mathcal{P}\colon\ (I,u) = (Q_1, x_{j_1}^{h_{j_1}}) \cap (Q_2, x_{j_2}^{h_{j_2}}) \cap \ldots \cap (Q_s, x_{j_s}^{h_{j_s}}),$$

where for each $1 \leqslant i \leqslant s$, $Q_i = q_i S$ for some irreducible monomial ideal q_i of S', $\sqrt{Q_i} \in \mathrm{Ass}_S S/I$ and $j_i \in \{t+1, \ldots, n\}$. It follows that $I = \bigcap_{i=1}^{s} Q_i$ is a primary decomposition of I. By deleting unneeded components, we get a primary decomposition

$$\mathcal{P}'\colon\ I = Q_{i_1} \cap Q_{i_2} \cap \ldots \cap Q_{i_l}$$

such that $i_1 < i_2 < \ldots < i_l$ and for each $1 \leqslant j \leqslant l$, $\bigcap_{k<j} Q_{i_k} \nsubseteq Q_{i_j}$ and $\bigcap_{k<j} Q_{i_k} = \bigcap_{m<i_j} Q_m$. We intend to show that $\mathcal{P}'$ satisfies the condition b) in Lemma 2.5. Since

$$\mathrm{Ass}_S S/I = \{\sqrt{Q_{i_1}}, \sqrt{Q_{i_2}}, \ldots, \sqrt{Q_{i_l}}\},$$

in view of (∗), (†) and (‡), we only need to indicate that each $T_i(\mathcal{P}')$ is a singleton. Let $1 \leqslant j \leqslant l$. Since $\bigcap\limits_{k<j} Q_{i_k} \nsubseteq Q_{i_j}$, it follows that there exists at least a monomial v in $\mathrm{G}\Big(\bigcap\limits_{k<j} Q_{i_k}\Big) \setminus Q_{i_j}$. We claim that v is unique. If there exists a monomial $w \neq v$ in $\mathrm{G}\Big(\bigcap\limits_{k<j} Q_{i_k}\Big) \setminus Q_{i_j}$, then since $\bigcap\limits_{k<j} Q_{i_k} = \bigcap\limits_{m<i_j} Q_m$, it turns out that v and w are belonging to $\mathrm{G}\Big(\bigcap\limits_{m<i_j} Q_m\Big) \setminus Q_{i_j}$. Denote i_j by d. Since $v, w \in S'$, we can conclude that v and w belong to

$$\mathrm{G}((Q_1, x_{j_1}^{h_{j_1}}) \cap (Q_2, x_{j_2}^{h_{j_2}}) \cap \ldots \cap (Q_{d-1}, x_{j_{d-1}}^{h_{j_{d-1}}})) \setminus (Q_d, x_{j_d}^{h_{j_d}}).$$

This contradicts the assumption that $T_d(\mathcal{P})$ is a singleton. Therefore, each $T_i(\mathcal{P}')$ is a singleton, as desired. □

As an immediate consequence, we obtain the following result; see [5], Proposition 2.2.

Corollary 2.7. *Let $u_1, \ldots, u_t \in \operatorname{Mon} S$ be a regular sequence on S. Then $S/(u_1, \ldots, u_t)$ is clean.*

3. Filter-regular sequences

Definition 3.1. Let M be a multigraded finitely generated S-module. A non-unit monomial u in S is called a *filter-regular element* on M if

$$u \notin \bigcup_{\mathfrak{p} \in \operatorname{Ass}_S M - \{\mathfrak{m}\}} \mathfrak{p}.$$

A sequence $u_1, \ldots, u_r$ of non-unit monomials in S is called a *filter-regular sequence* on M if for each $1 \leqslant i \leqslant r$, u_i is a filter-regular element on $M/(u_1, \ldots, u_{i-1})M$.

Lemma 3.2. *Let M be a multigraded finitely generated S-module. An element $1 \neq u \in \operatorname{Mon} S$ is a filter-regular element of M if and only if it is not a non zero-divisor of $M/H^0_{\mathfrak{m}}(M)$.*

P r o o f. Since $H^0_{\mathfrak{m}}(M)$ is Artinian and $H^0_{\mathfrak{m}}(M/(H^0_{\mathfrak{m}}(M))) = 0$, Lemma 2.1 yields that

$$\operatorname{Ass}_S \Big(\frac{M}{H^0_{\mathfrak{m}}(M)}\Big) = \operatorname{Ass}_S M - \{\mathfrak{m}\}.$$

Hence, by definition the claim is immediate. □

Theorem 3.3. *Let I be a monomial ideal of S and $u_1, \ldots, u_r \in \operatorname{Mon} S$ a filter-regular sequence on S/I. Then S/I is pretty clean if and only if $S/(I, u_1, \ldots, u_r)$ is pretty clean.*

Proof. By induction on r, it is enough to prove that for a monomial filter-regular element u of S/I, S/I is pretty clean if and only if $S/(I, u)$ is pretty clean. For convenience, we set $M := S/I$. By Proposition 2.4, M is pretty clean if and only if $M/H^0_{\mathfrak{m}}(M)$ is pretty clean. By Lemma 3.2, u is a non zero-divisor on $M/H^0_{\mathfrak{m}}(M)$. Hence, in view of the isomorphism

$$\frac{M/H^0_{\mathfrak{m}}(M)}{u(M/H^0_{\mathfrak{m}}(M))} \cong \frac{M}{uM + H^0_{\mathfrak{m}}(M)},$$

Theorem 2.6 yields that $M/H^0_{\mathfrak{m}}(M)$ is pretty clean if and only if $M/(uM + H^0_{\mathfrak{m}}(M))$ is pretty clean. On the other hand, as $(uM + H^0_{\mathfrak{m}}(M))/(uM)$ is a multigraded Artinian submodule of M/uM, by Proposition 2.4 and the isomorphism

$$\frac{M}{uM + H^0_{\mathfrak{m}}(M)} \cong \frac{M/uM}{(uM + H^0_{\mathfrak{m}}(M))/uM},$$

it turns out that $M/(uM + H^0_{\mathfrak{m}}(M))$ is pretty clean if and only if M/uM is pretty clean. Therefore, M is pretty clean if and only if M/uM is pretty clean. □

Corollary 3.4. *Let monomials $u_1, \ldots, u_r$ be a filter-regular sequence on S. Then $S/(u_1, \ldots, u_r)$ is pretty clean.*

Lemma 3.5. *Let M be a multigraded finitely generated S-module and let $u_1, \ldots, u_r \in \operatorname{Mon} S$ be a filter-regular sequence on M. If $\mathfrak{m} \in \operatorname{Ass}_S M$, then $\mathfrak{m} \in \operatorname{Ass}_S(M/(u_1, \ldots, u_r)M)$.*

Proof. By induction on r, it is enough to prove that if u is a monomial filter-regular element of M and $\mathfrak{m} \in \operatorname{Ass}_S M$, then $\mathfrak{m} \in \operatorname{Ass}_S M/(uM)$. Since $\mathfrak{m} \in \operatorname{Ass}_S M$, there exists $0 \neq x \in M$ such that $\mathfrak{m} = 0 :_S x$. Then, there exists a nonnegative integer t such that $x \in u^t M \setminus u^{t+1} M$. Hence $x = u^t y$ for some $y \in M \setminus uM$. Clearly, $0 :_S y \subset S$. Let $\mathfrak{p} \subset \mathfrak{m}$ be a prime ideal of S containing $0 :_S y$. Since u is a filter-regular element on M and $\mathfrak{p} \neq \mathfrak{m}$, it follows that $u/1 \in S_{\mathfrak{p}}$ is $M_{\mathfrak{p}}$-regular. Hence

$$(0 :_S x)_{\mathfrak{p}} = 0 :_{S_{\mathfrak{p}}} \frac{u^t}{1}\frac{y}{1} = 0 :_{S_{\mathfrak{p}}} \frac{y}{1} = (0 :_S y)_{\mathfrak{p}} \subseteq \mathfrak{p}S_{\mathfrak{p}},$$

and so

$$(0 :_S x) \subseteq (0 :_S x)_{\mathfrak{p}} \cap S \subseteq \mathfrak{p}S_{\mathfrak{p}} \cap S = \mathfrak{p}.$$

This is a contradiction, and so $\mathfrak{m}$ is the unique prime ideal of S containing $(0:_S y)$. So,

$$\mathfrak{m} = \sqrt{(0:_S y)} \subseteq \sqrt{(0:_S y + uM)} \subset S.$$

Therefore, $\sqrt{(0:_S y + uM)} = \mathfrak{m}$, and so $\mathfrak{m} \in \operatorname{Ass}_S M/(uM)$. □

A decomposition of S/I as direct sum of K-vector spaces of the form $\mathcal{D}\colon S/I = \bigoplus_{i=1}^{r} u_i K[Z_i]$, where u_i is a monomial in S and $Z_i \subseteq \{x_1, \ldots, x_n\}$, is called a *Stanley decomposition* of S/I. The number $\operatorname{sdepth} \mathcal{D} := \min\{|Z_i|\colon i = 1, \ldots, r\}$ is called the *Stanley depth* of $\mathcal{D}$. The *Stanley depth* of S/I is defined to be

$$\operatorname{sdepth} S/I := \max\{\operatorname{sdepth} \mathcal{D}\colon \mathcal{D} \text{ is a Stanley decomposition of } S/I\}.$$

Stanley conjectured [18] that $\operatorname{depth} S/I \leqslant \operatorname{sdepth} S/I$. This conjecture is known as Stanley's conjecture. Recently, this conjecture was extensively examined by several authors; see, e.g., [1], [2], [7], [5], [10], [11], [15], and [16]. On the other hand, the present third author [15] conjectured that there always exists a Stanley decomposition $\mathcal{D}$ of S/I such that the degree of each u_i is at most $\operatorname{reg} S/I$. We refer to this conjecture as h-regularity conjecture. It is known that for square-free monomial ideals, these two conjectures are equivalent.

Theorem 3.6. *Let I be a monomial ideal of S and $u_1, \ldots, u_r \in \operatorname{Mon} S$ a filter-regular sequence on S/I. Then Stanley's conjecture holds for S/I if and only if it holds for $S/(I, u_1, \ldots, u_r)$.*

Proof. By induction on r, it is enough to prove that if u is a monomial filter-regular element on S/I, then Stanley's conjecture holds for S/I if and only if it holds for $S/(I, u)$. First, assume that $\mathfrak{m} \in \operatorname{Ass}_S S/I$. Then $\operatorname{depth} S/I = 0$ and by Lemma 3.5, $\mathfrak{m} \in \operatorname{Ass}_S S/(I, u)$. So, $\operatorname{depth} S/(I, u) = 0$. Hence the claim is immediate in this case. Now, assume that $\mathfrak{m} \notin \operatorname{Ass}_S S/I$. Then u is a non zero-divisor on S/I, and so by [11], Theorem 1.1, Stanley's conjecture holds for S/I if and only if it holds for $S/(I, u)$. □

4. d-SEQUENCES

Definition 4.1. Let R be a commutative Noetherian ring, M a finitely generated R-module and $f_1, \ldots, f_t \in R$.

i) $f_1, \ldots, f_t$ is called a *d-sequence* on M if $f_1, \ldots, f_t$ is a minimal generating set of the ideal $(f_1, \ldots, f_t)$ and $(f_1, \ldots, f_i)M :_M f_{i+1}f_k = (f_1, \ldots, f_i)M :_M f_k$ for all $0 \leqslant i < t$ and all $k \geqslant i+1$. A d-sequence on R is simply called a d-sequence.

ii) $f_1, \ldots, f_t$ is called a *proper sequence* if $f_{i+1}H_j(f_1, \ldots, f_i; R) = 0$ for all $0 \leqslant i < t$ and all $j > 0$. Here $H_j(f_1, \ldots, f_i; R)$ denotes the j-th Koszul homology of R with respect to $f_1, \ldots, f_i$.

iii) Let $M = (g_1, \ldots, g_t)$ and $(a_{ij})_{s\times t}$ be a relation matrix of M. Then the symmetric algebra of M is defined by $\operatorname{Sym} M := R[y_1, \ldots, y_t]/J$, where $J = \Bigl(\sum\limits_{j=1}^{t} a_{1j}y_j, \ldots, \sum\limits_{j=1}^{t} a_{sj}y_j\Bigr)$. Let $<$ be a monomial order on the monomials in $y_1, \ldots, y_t$ with the property $y_1 < \ldots < y_t$. Set $I_i := (g_1, \ldots, g_{i-1}) :_R g_i$. Then $(I_1y_1, \ldots, I_ty_t) \subseteq \operatorname{in}_< J$. The sequence $g_1, \ldots, g_t$ is called an *s-sequence* (with respect to $<$) if $(I_1y_1, \ldots, I_ty_t) = \operatorname{in}_< J$. If in addition $I_1 \subseteq \ldots \subseteq I_t$, then $g_1, \ldots, g_t$ is called a *strong s-sequence.*

Definition 4.2. Let I be a (not necessarily square-free) monomial ideal of S with $\mathrm{G}(I) = \{u_1, \ldots, u_m\}$. A monomial u_t is called a leaf of $\mathrm{G}(I)$ if u_t is the only element in $\mathrm{G}(I)$ or there exists a $j \neq t$ such that $\gcd(u_t, u_i) \mid \gcd(u_t, u_j)$ for all $i \neq t$. In this case, u_j is called a branch of u_t. We say that I is a monomial ideal of forest type if every nonempty subset of $\mathrm{G}(I)$ has a leaf.

[17], Theorem 1.5, yields that if I is a monomial ideal of forest type, then S/I is pretty clean.

Lemma 4.3. *Let $u_1, \ldots, u_t$ be a sequence of monomials with the following properties:*

i) *there is no $i \neq j$ such that $u_i \mid u_j$; and*

ii) $\gcd(u_i, u_j) \mid u_k$ *for all* $1 \leqslant i < j < k \leqslant t$.

Then $I = (u_1, \ldots, u_t)$ is of forest type, and so S/I is pretty clean.

Proof. For every nonempty subset $A = \{u_{n_1}, \ldots, u_{n_s}\}$ of $\{u_1, \ldots, u_t\}$, we may and do assume that $n_1 < n_2 < \ldots < n_s$. Then obviously the first element of A is a leaf and the last element of A is a branch for that leaf. So, I is of forest type. Then [17], Theorem 1.5, implies that S/I is pretty clean. □

Proposition 4.4. *Let I be a monomial ideal of S with $\mathrm{G}(I) = \{u_1, \ldots, u_t\}$. If $u_1, \ldots, u_t$ is a d-sequence, proper sequence or strong s-sequence (with respect to the reverse lexicographic order), then S/I is pretty clean.*

Proof. By [8], Corollaries 3.3 and 3.4, any d-sequence is a strong s-sequence with respect to the reverse lexicographic order and $u_1, \ldots, u_t$ is a proper sequence if and only if it is a strong s-sequence with respect to the reverse lexicographic order. So, by the hypothesis and [19], Theorem 3.1, there is no $i \neq j$ such that $u_i \mid u_j$ and $\gcd(u_i, u_j) \mid u_k$ for all $1 \leqslant i < j < k \leqslant t$. Hence, by Lemma 4.3, S/I is pretty clean. □

Let I be a monomial ideal of S and u a monomial which is a d-sequence on S/I. The following example shows that it may happen that S/I is pretty clean, but $S/(I,u)$ is not.

Example 4.5. Let $I = (x_1x_2, x_2x_3, x_3x_4)$ be a monomial ideal of $S = K[x_1, x_2, x_3, x_4]$. It is easy to see that S/I is pretty clean and x_4x_1 is a d-sequence on S/I. But, by [16], Example 1.11, we know that $S/(I, x_4x_1) = S/(x_1x_2, x_2x_3, x_3x_4, x_4x_1)$ is not pretty clean.

We conclude the paper with the following result.

Corollary 4.6. *Let I be a monomial ideal of S. Assume that either:*

i) *I is generated by a filter-regular sequence; or*

ii) *I is generated by a d-sequence.*

Then both Stanley's and the h-regularity conjectures hold for S/I. Also, in each of these cases S/I is sequentially Cohen-Macaulay and $\operatorname{depth} S/I = \min\{\dim S/\mathfrak{p}\,;\ \mathfrak{p} \in \operatorname{Ass}_S S/I\}$.

P r o o f. In both cases i) and ii), it follows that S/I is pretty clean; see Corollary 3.4 and Proposition 4.4.

As S/I is pretty clean, [7], Theorem 6.5, asserts that Stanley's conjecture holds for S/I. In fact, by [9], Proposition 1.3, we have $\operatorname{depth} S/I = \operatorname{sdepth} S/I$. On the other hand, by [15], Theorem 4.7, the h-regularity conjecture holds for S/I.

Next, as S/I is pretty clean, [7], Corollary 4.3, implies that S/I is sequentially Cohen-Macaulay. In [13] this fact is reproved by a different argument and, in addition, it is shown that depth of S/I is equal to the minimum of the dimension of $S/\mathfrak{p}$, where $\mathfrak{p} \in \operatorname{Ass}_S S/I$. □

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