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DERIVED CONES TO REACHABLE SETS OF A NONLINEAR DIFFERENTIAL INCLUSION

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Abstract. We consider a nonlinear differential inclusion defined by a set-valued map with nonconvex values and we prove that the reachable set of a certain variational inclusion is a derived cone in the sense of Hestenes to the reachable set of the initial differential inclusion. In order to obtain the continuity property in the definition of a derived cone we use a continuous version of Filippov's theorem for solutions of our differential inclusion. As an application, in finite dimensional spaces, we obtain a sufficient condition for local controllability along a reference trajectory.

Keywords: derived cone; m-dissipative operator; local controllability

MSC 2010: 34A60, 93C15

1. Introduction

The concept of the derived cone to an arbitrary subset of a normed space has been introduced by M. Hestenes in [8] and successfully used to obtain necessary optimality conditions in Control Theory. Afterwards, this concept has been largely ignored in favor of other concepts of tangent cones that may intrinsically be associated with a point of a given set: the cone of interior directions, the contingent, the quasitangent and, above all, Clarke's tangent cone (e.g., [1]). Mirică ([10], [11]) obtained "an intersection property" of derived cones that allowed a conceptually simple proof and significant extensions of the maximum principle in optimal control; moreover, other properties of derived cones may be used to obtain controllability and other results in the qualitative theory of control systems. In our previous papers [4]–[7] we identified derived cones to the reachable sets of certain classes of discrete and differential inclusions in terms of a variational inclusion associated with the initial discrete or differential inclusion.

In the present note we consider differential inclusions of the form

$$x' \in Ax + F(t, x), \quad x(0) \in X_0, \tag{1.1}$$

where X is a separable Banach space, A is an m-dissipative operator on X, $F\colon [0,T] \times X \to \mathcal{P}(X)$ is a set valued map and $X_0 \subset X$ is a closed set.

Our aim is to prove that the reachable set of a certain variational inclusion is a derived cone in the sense of Hestenes to the reachable set of the problem (1.1). In order to obtain the continuity property in the definition of a derived cone we shall use a continuous version of Filippov's theorem for solutions of differential inclusions (1.1) obtained in [3]. As an application, when X is finite dimensional, we obtain a sufficient condition for local controllability along a reference trajectory.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a normed space.

Definition 2.1 ([8]). A subset $M \subset X$ is said to be a *derived set* to $E \subset X$ at $x \in E$ if for any finite subset $\{v_1, \ldots, v_k\} \subset M$ there exist $s_0 > 0$ and a continuous mapping $a(\cdot)\colon [0, s_0]^k \to E$ such that $a(0) = x$ and $a(\cdot)$ is (conically) differentiable at $s = 0$ with the derivative $\mathrm{col}[v_1, \ldots, v_k]$ in the sense that

$$\lim_{\mathbb{R}^k_+ \ni \theta \to 0} \frac{\left\|a(\theta) - a(0) - \sum_{i=1}^k \theta_i v_i\right\|}{\|\theta\|} = 0.$$

We shall write in this case that the derivative of $a(\cdot)$ at $s = 0$ is given by

$$\mathrm{D}a(0)\theta = \sum_{i=1}^k \theta_j v_j \quad \forall \theta = (\theta_1, \ldots, \theta_k) \in \mathbb{R}^k_+ := [0, \infty)^k.$$

A subset $C \subset X$ is said to be a *derived cone* of E at x if it is a derived set and also a convex cone.

For the basic properties of derived sets and cones we refer to M. Hestenes [8]; we recall that if M is a derived set, then $M \cup \{0\}$, as well as the convex cone generated by M, is also a derived set, hence a derived cone.

On the other hand, the up-to-date experience in Nonsmooth Analysis shows that for some problems, the use of one of the intrinsic tangent cones may be preferable. From the multitude of intrinsic tangent cones in the literature (e.g. [1]), the contingent, the quasitangent (intermediate) and Clarke's tangent cones, defined, respectively, by

$$K_xE = \Big\{v \in X\,;\ \exists\, s_m \to 0+,\ \exists\, x_m \to x,\ x_m \in E\colon\ \frac{x_m - x}{s_m} \to v\Big\},$$
$$Q_xE = \Big\{v \in X\,;\ \forall s_m \to 0+,\ \exists\, x_m \to x,\ x_m \in E\colon\ \frac{x_m - x}{s_m} \to v\Big\},$$
$$C_xE = \Big\{v \in X\,;\ \forall\,(x_m, s_m) \to (x, 0+),\ x_m \in E,\ \exists\, y_m \in E\colon\ \frac{y_m - x_m}{s_m} \to v\Big\}$$

seem to be among the most often used in the study of different problems involving nonsmooth sets and mappings.

A useful property of the derived cone, obtained by Hestenes ([8], Theorem 4.7.4) is stated in the next lemma.

Lemma 2.2 ([8])**.** *Let $X = \mathbb{R}^n$. Then $x \in \operatorname{int}(E)$ if and only if $C = \mathbb{R}^n$ is a derived cone at $x \in E$ to E.*

The property in Lemma 2.2 is not satisfied by every tangent cone. For example, the contingent and quasitangent cones satisfy only: if $x \in \operatorname{int}(E)$ then $K_xE = \mathbb{R}^n$ ($Q_xE = \mathbb{R}^n$). The converse statement is not true. Clarke's tangent cone satisfies the property in Lemma 2.2, but Clarke's tangent cone often reduces to $\{0\}$. The advantage in using derived cones must be regarded with respect to this property and, therefore, in obtaining a result as in Theorem 3.4 below.

Corresponding to each type of tangent cone, say τ_xE, one may introduce (e.g. [1]) the *set-valued directional derivative* of a multifunction $G(\cdot)\colon E \subset X \to \mathcal{P}(X)$ (in particular, of a single-valued mapping) at a point $(x, y) \in \operatorname{Graph}(G)$ as follows:

$$\tau_yG(x; v) = \{w \in X\,;\ (v, w) \in \tau_{(x,y)} \operatorname{Graph}(G)\}, \quad \forall v \in \tau_xE.$$

We recall that a set-valued map $A(\cdot)\colon X \to \mathcal{P}(X)$ is said to be a *convex* or *closed convex process* if $\operatorname{Graph}(A(\cdot)) \subset X \times X$ is, respectively, a convex or closed convex cone.

Let us denote by I the interval $[0, T]$ and let X be a real separable Banach space with the norm $\|\cdot\|$ and with the corresponding metric $d(\cdot,\cdot)$. Denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I, by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. Recall that the Pompeiu-Hausdorff distance between nonempty closed subsets $A, B \subset X$ is defined by $d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}$, $d^*(A, B) = \sup\{d(a, B);\ a \in A\}$, where $d(x, B) = \inf_{y \in B} d(x, y)$. As usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x(\cdot)\colon I \to X$ endowed with the norm $\|x(\cdot)\|_C = \sup_{t \in I} \|x(t)\|$ and by $L^1(I,X)$ the Banach space of all (Bochner) integrable functions $x(\cdot)\colon I\to X$ endowed with the norm $\|x(\cdot)\|_1=\int_I\|x(t)\|\,\mathrm{d}t$.

For $x,y\in X$ we denote by $(x,y)=\lim\limits_{h\to 0}(\|x+hy\|^2-\|x\|^2)/(2h)$ the left sided directional derivative of $\frac{1}{2}\|\cdot\|^2$ at x in the direction y.

Consider an operator $A\colon X\to\mathcal{P}(X)$. By $D(A)$ we denote its domain. We recall that A is called *dissipative* if $(x_1-x_2,y_1-y_2)\leqslant 0$ for any $x_1,x_2\in D(A)$ and $y_1\in Ax_1$, $y_2\in Ax_2$. A is called *m-dissipative* if it is dissipative and $R(I-\lambda A)=X$ for any (equivalently, for some) $\lambda>0$.

Let $A\colon D(A)\subset X\to\mathcal{P}(X)$ be m-dissipative and $f(\cdot)\in L^1(I,X)$ and consider the differential equation

$$x'\in Ax+f(t). \tag{2.1}$$

A mapping $x(\cdot)\colon I\to X$ is called a *strong solution* of (2.1) if $x(t)\in D(A)$ a.e. on $(0,T)$, $x(\cdot)$ is locally absolute continuous on $(0,T]$ and there exists $g\in L^1_{\mathrm{loc}}((0,T],X)$ such that $g(t)\in Ax(t)$ a.e. on $(0,T)$ and $x'(t)=g(t)+f(t)$ a.e. on $(0,T)$.

It is well-known that if X is reflexive then for any $x_0\in D(A)$ equation (2.1) has a unique strong solution on I which satisfies $x(0)=x_0$ (e.g. [2]). In general, equation (2.1) need not have strong solutions and a way to overcome this difficulty is the concept of C^0-solutions (e.g. [9]).

Definition 2.3. A function $x(\cdot)\in C(I,X)$ is called a C^0-*solution* of (2.1) if it satisfies: for every $c\in(0,T)$ and $\varepsilon>0$ there exist

(i) $0<t_1<\ldots<c\leqslant t_n<T$, $t_k-t_{k-1}\leqslant\varepsilon$, $t_0=0$ for $k=1,2,\ldots,n$;

(ii) $f_1,f_2,\ldots,f_n\in X$ with $\sum\limits_{k=1}^{n}\int_{t_{k-1}}^{t_k}\|f(t)-f_k\|\,\mathrm{d}t\leqslant\varepsilon$;

(iii) $v_0,v_1,\ldots,v_n\in X$ with $(v_k-v_{k-1})/(t_k-t_{k-1})\in Av_k+f_k$ for $k=1,2,\ldots,n$ and $\|x(t)-v_k\|\leqslant\varepsilon$ for $t\in[t_{k-1},t_k)$, $k=1,2,\ldots,n$.

According to [9], if A is m-dissipative, $f\in L^1(I,X)$ and $x_0\in\overline{D(A)}$, there exists a unique C^0-solution of (2.1) with $x(0)=x_0$. Denote by $x(\cdot,x_0,f)\colon I\to\overline{D(A)}$ the unique C^0-solution of (2.1) which satisfies $x(0,x_0,f)=x_0$.

Theorem 2.4 ([10])**.** *Let X be a real Banach space, let $A\colon D(A)\subseteq X\to X$ be an m-dissipative operator, let $\xi,\eta\in\overline{D(A)}$ and $f,g\in L^1(I,X)$. Then for any $t\in I$*

$$\|x(t,\xi,f)-x(t,\eta,g)\|\leqslant\|\xi-\eta\|+\int_0^t\|f(s)-g(s)\|\,\mathrm{d}s. \tag{2.2}$$

Next we are concerned with the Cauchy problem

$$x'\in Ax+F(t,x),\quad x(0)=x_0, \tag{2.3}$$

where A is an m-dissipative operator on the separable Banach space X, $F\colon\ I\times X\to\mathcal{P}(X)$ and $x_0\in X$.

A continuous mapping $x\colon\ I\to\overline{D(A)}$ is said to be a C^0-*solution* of problem (2.3) if $x(0)=x_0$ and there exists $f(\cdot)\in L^1(I,X)$ with $f(t)\in F(t,x(t))$ a.e. on I and $x(\cdot)$ is a C^0-solution on I of equation (2.1) in the sense of Definition 2.1.

We shall call $(x(\cdot),f(\cdot))$ a *trajectory-selection pair* of (2.3) if $f(t)\in F(t,x(t))$ a.e. on I and $x(\cdot)$ is a C^0-solution of (2.3).

Hypothesis 2.5. (i) *$F(\cdot,\cdot)\colon\ I\times X\to\mathcal{P}(X)$ has nonempty closed values and is $\mathcal{L}(I)\otimes\mathcal{B}(X)$ measurable.*

(ii) *There exists $L(\cdot)\in L^1(I,\mathbb{R}_+)$ such that, for any $t\in I$, $F(t,\cdot)$ is $L(t)$-Lipschitz in the sense that*

$$d_H(F(t,x_1),F(t,x_2))\leqslant L(t)\|x_1-x_2\|\quad\forall\,x_1,x_2\in X.$$

The main tool in characterizing derived cones to reachable sets of semilinear differential inclusions is a certain version of Filippov's theorem for differential inclusion (2.3).

Hypothesis 2.6. *Let S be a separable metric space, $X_0\subset\overline{D(A)}$ a closed set, let $a_0(\cdot)\colon\ S\to X_0$ and $c(\cdot)\colon\ S\to(0,\infty)$ be given continuous mappings.*

The continuous mappings $g(\cdot)\colon\ S\to L^1(I,X)$, $y(\cdot)\colon\ S\to C(I,X)$ are given such that for any $s\in S$, $y(s)(\cdot)$ is a C^0-solution of $x'\in Ax+g(s)(t)$, $x(0)\in X_0$ and there exists a continuous function $q(\cdot)\colon\ S\to L^1(I,\mathbb{R}_+)$ such that $d(g(s)(t),F(t,y(s)(t)))\leqslant q(s)(t)$ a.e. $t\in I$, for every $s\in S$.

Theorem 2.7 ([3])**.** *Assume that Hypotheses 2.5 and 2.6 are satisfied. Then there exist $M>0$ and a continuous function $x(\cdot)\colon\ S\to L^1(I,X)$ such that for any $s\in S$, $x(s)(\cdot)$ is a C^0-solution of the problem*

$$x'\in Ax+F(t,x),\quad x(0)=a_0(s),$$

satisfying for any $(t,s)\in I\times S$

$$\|x(s)(t)-y(s)(t)\|\leqslant M\left[c(s)+\|a_0(s)-y(s)(0)\|+\int_0^t q(s)(u)\,\mathrm{d}u\right]. \tag{2.4}$$

3. The main result

Our object of study is the reachable set of (2.3) defined by

$$R_F(T,X_0):=\{x(T);\ x(\cdot)\text{ is a solution of (2.3)}\}.$$

We consider a certain variational second-order differential inclusion and prove that the reachable set of this variational inclusion from derived cones $C_0\subset X$ to X_0 at time T is a derived cone to the reachable set $R_F(T,X_0)$.

Throughout this section we assume

Hypothesis 3.1. (i) *Hypothesis* 2.5 *is satisfied and* $X_0\subset\overline{D(A)}$ *is a closed set.*

(ii) $(z(\cdot),f(\cdot))\in C(I,X)\times L^1(I,X)$ *is a trajectory-selection pair of* (2.3) *and a family* $P(t,\cdot)\colon X\to\mathcal{P}(X)$, $t\in I$ *of convex processes satisfying the condition*

$$P(t,u)\subset Q_{f(t)}F(t,\cdot)(z(t);u)\quad\forall\,u\in\operatorname{dom}(P(t,\cdot)),\ \textit{a.e. } t\in I\tag{3.1}$$

is given and defines the variational inclusion

$$v'(t)\in Av(t)+P(t,v(t)).\tag{3.2}$$

Remark 3.2. As a family of convex processes $P(t,\cdot)$, $t\in I$, satisfying condition (3.1) one may take, for example, Clarke's convex-valued directional derivatives $C_{f(t)}F(t,\cdot)(z(t);\cdot)$.

We recall (e.g. [1]) that, since $F(t,\cdot)$ is assumed to be Lipschitz a.e. on I, the quasitangent directional derivative is given by

$$Q_{f(t)}F(t,\cdot)((z(t);u))=\Big\{w\in X;\ \lim_{\theta\to0^+}\frac{1}{\theta}d(f(t)+\theta w,F(t,z(t)+\theta u))=0\Big\}.\tag{3.3}$$

Theorem 3.3. *Assume that Hypothesis* 3.1 *is satisfied, let* $C_0\subset X$ *be a derived cone to* X_0 *at* $z(0)$*. Then the reachable set* $R_P(T,C_0)$ *of* (3.2) *is a derived cone to* $R_F(T,X_0)$ *at* $z(T)$*.*

Proof. In view of Definition 2.1, let $\{v_1,\dots,v_m\}\subset R_P(T,C_0)$, hence such that there exist trajectory-selection pairs $(u_1(\cdot),g_1(\cdot)),\dots,(u_m(\cdot),g_m(\cdot))$ of the variational inclusion (3.2) such that

$$u_j(T)=v_j,\quad u_j(0)\in C_0,\quad j=1,2,\dots,m.\tag{3.4}$$

Since $C_0\subset X$ is a derived cone to X_0 at $z(0)$ there exists a continuous mapping $a_0\colon S=[0,\theta_0]^m\to X_0$, such that

$$a_0(0)=z(0),\quad \mathrm{D}a_0(0)s=\sum_{j=1}^m s_ju_j(0)\quad\forall\,s\in\mathbb{R}_+^m.\tag{3.5}$$

Further on, for any $s = (s_1, \ldots, s_m) \in S$ and $t \in I$ we denote

$$
\begin{aligned}
(3.6) \qquad y(s)(t) &= z(t) + \sum_{j=1}^{m} s_j u_j(t), \\
g(s)(t) &= f(t) + \sum_{j=1}^{m} s_j g_j(t), \\
q(s)(t) &= d(g(s)(t), F(t, y(s)(t)))
\end{aligned}
$$

and prove that $y(\cdot)$, $q(\cdot)$ satisfy the hypothesis of Theorem 2.7.

Using the lipschitzianity of $F(t, \cdot, \cdot)$ we have that for any $s \in S$, the measurable function $q(s)(\cdot)$ in (3.6) it is also integrable.

$$
\begin{aligned}
q(s)(t) &= d(g(s)(t), F(t, y(s)(t))) \\
&\leqslant \sum_{j=1}^{m} s_j \|g_j(t)\| + d_H(F(t, z(t)), F(t, y(s)(t))) \\
&\leqslant \sum_{j=1}^{m} s_j \|g_j(t)\| + L(t) \sum_{j=1}^{m} s_j \|u_j(t)\|.
\end{aligned}
$$

Moreover, the mapping $s \to q(s)(\cdot) \in L^1(I, \mathbb{R}_+)$ is continuous (in fact Lipschitzian) since for any $s, s' \in S$ one may write succesively

$$
\begin{aligned}
\|q(s)(\cdot) - q(s')(\cdot)\|_1 &= \int_0^T \|q(s)(t) - q(s')(t)\| \, \mathrm{d}t \\
&\leqslant \int_0^T [\|g(s)(t) - g(s')(t)\| + d_H(F(t, y(s)(t)), F(t, y(s')(t)))] \, \mathrm{d}t \\
&\leqslant \|s - s'\| \biggl(\sum_{j=1}^{m} \int_0^T [\|g_j(t)\| + L(t)\|u_j(t)\|] \, \mathrm{d}t \biggr).
\end{aligned}
$$

Let us define $S_1 := S \setminus \{(0, \ldots, 0)\}$ and $c(\cdot)\colon S_1 \to (0, \infty)$, $c(s) := \|s\|^2$. Theorem 2.7 yields the existence of a continuous function $x(\cdot)\colon S_1 \to C(I, X)$ such that for any $s \in S_1$, $x(s)(\cdot)$ is a C^0-solution of (2.3) with the property (2.4).

For $s = 0$ we define $x(0)(t) = y(0)(t) = z(t)$ for all $t \in I$. Obviously, $x(\cdot)\colon S \to C(I, X)$ is also continuous. Finally, we define the function $a(\cdot)\colon S \to R_F(T, X_0)$ by $a(s) = x(s)(T)$ for all $s \in S$. Obviously, $a(\cdot)$ is continuous on S and satisfies $a(0) = z(T)$.

To complete the proof we need to show that $a(\cdot)$ is differentiable at $s_0 = 0 \in S$ and its derivative is given by $\mathrm{D}a(0)(s) = \sum\limits_{j=1}^{m} s_j v_j$ for all $s \in \mathbb{R}_+^m$, which is equivalent to the fact that

$$\lim_{s\to 0}\frac{1}{\|s\|}\left(\|a(s)-a(0)-\sum_{j=1}^{m}s_jv_j\|\right)=0. \tag{3.7}$$

From (2.4) we obtain

$$\begin{aligned}\frac{1}{\|s\|}\|a(s)-a(0)-\sum_{j=1}^{m}s_jv_j\| &\leqslant \frac{1}{\|s\|}\|x(s)(T)-y(s)(T)\|\\ &\leqslant M\|s\|+\frac{M}{\|s\|}\left\|a_0(s)-z(0)-\sum_{j=1}^{m}s_ju_j(0)\right\|+M\int_0^T\frac{q(s)(u)}{\|s\|}\,\mathrm{d}u\end{aligned}$$

and therefore in view of (3.5), relation (3.7) is implied by the following property of the mapping $q(\cdot)$ in (3.6):

$$\lim_{s\to 0}\frac{q(s)(t)}{\|s\|}=0\quad \text{a.e. } t\in I. \tag{3.8}$$

In order to prove the last property, we note that, since $P(t,\cdot)$ is a convex process for any $s\in S$, one has

$$\sum_{j=1}^{m}\frac{s_j}{\|s\|}g_j(t)\in P\left(t,\ \sum_{j=1}^{m}\frac{s_j}{\|s\|}u_j(t)\right)\subset Q_{f(t)}F(t,\cdot)\left(z(t);\ \sum_{j=1}^{m}\frac{s_j}{\|s\|}u_j(t)\right)\quad \text{a.e. } t\in I.$$

Hence by (3.3) we obtain

$$\lim_{h\to 0+}\frac{1}{h}d\left(f(t)+h\sum_{j=1}^{m}\frac{s_j}{\|s\|}g_j(t),F\left(t,z(t)+h\sum_{j=1}^{m}\frac{s_j}{\|s\|}u_j(t)\right)\right)=0. \tag{3.9}$$

In order to prove that (3.9) implies (3.8) we consider the compact metric space $S_+^{m-1}=\{\sigma\in\mathbb{R}_+^m;\ \|\sigma\|=1\}$ and the real function $\varphi_t(\cdot,\cdot)\colon (0,\theta_0]\times S_+^{m-1}\to\mathbb{R}_+$ defined by

$$\varphi_t(h,\sigma)=\frac{1}{h}d\left(f(t)+h\sum_{j=1}^{m}\sigma_jg_j(t),F\left(t,z(t)+h\sum_{j=1}^{m}\sigma_ju_j(t)\right)\right), \tag{3.10}$$

where $\sigma=(\sigma_1,\dots,\sigma_m)$, which according to (3.9) has the property

$$\lim_{\theta\to 0^+}\varphi_t(\theta,\sigma)=0\quad \forall\sigma\in S_+^{m-1}\ \text{a.e. } t\in I. \tag{3.11}$$

Using the fact that $\varphi_t(\theta,\cdot)$ is Lipschitzian and the fact that S_+^{m-1} is a compact metric space, from (3.11) it follows easily (e.g. Proposition 4.4 in [7]) that $\lim\limits_{\theta\to 0^+}\max\limits_{\sigma\in S_+^{m-1}}\varphi_t(\theta,\sigma)=0$, which implies that $\lim\limits_{s\to 0}\varphi_t(\|s\|,s/\|s\|)=0$ a.e. $t\in I$ and the proof is complete. □

An application of Theorem 3.3 concerns local controllability of the differential inclusion in (2.3) along a reference trajectory, $z(\cdot)$ at time T, in the sense that

$$z(T) \in \operatorname{int}(R_F(T, X_0)).$$

Theorem 3.4. *Let $X = \mathbb{R}^n$, $z(\cdot), F(\cdot,\cdot)$ and $P(\cdot,\cdot)$ satisfy Hypothesis 3.1, let $C_0 \subset X$ be a derived cone to X_0 at $z(0)$. If the variational differential inclusion in (3.2) is controllable at T, in the sense that $R_P(T, C_0) = \mathbb{R}^n$, then the differential inclusion (2.3) is locally controllable along $z(\cdot)$ at time T.*

Proof. The proof is a straightforward application of Lemma 2.2 and of Theorem 3.3. □

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