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STABILITY AND CONTAGION MEASURES FOR SPATIAL EXTREME VALUE ANALYZES

Cecília Fonseca, Helena Ferreira, Luísa Pereira and Ana Paula Martins

As part of global climate change an accelerated hydrologic cycle (including an increase in heavy precipitation) is anticipated (Trenberth [20, 21]). So, it is of great importance to be able to quantify high-impact hydrologic relationships, for example, the impact that an extreme precipitation (or temperature) in a location has on a surrounding region. Building on the Multivariate Extreme Value Theory we propose a contagion index and a stability index. The contagion index makes it possible to quantify the effect that an exceedance above a high threshold can have on a region. The stability index reflects the expected number of crossings of a high threshold in a region associated to a specific location $\mathbf{i}$, given the occurrence of at least one crossing at that location. We will find some relations with well-known extremal dependence measures found in the literature, which will provide immediate estimators. For these estimators an application to the annual maxima precipitation in Portuguese regions is presented.

Keywords: spatial extremes, max-stable processes, extremal dependence

Classification: 60G70

1. INTRODUCTION

The need to model and predict environmental extreme events such as hurricanes, floods, droughts, heat waves and other high impact events, which can lead to a devasting impacts, ranging from disturbances in ecosystems to economic impacts on society as well as loss of life, motivated the modelling of spatial extremes.

A common method of modeling spatial extremes is through max-stable processes. Max-stable processes are the natural analogues of the generalized extreme value distribution for modeling of extreme events in space and time. Under suitable conditions, these processes are asymptotically justified models for maxima of independent replications of random fields, and they are also suitable for the modeling of joint individual extreme measurements over high thresholds (Davison and Huser [3]).

Max-stable processes can be, for example, good approximations for annual maxima of daily spatial rainfall (Smith [17], Coles [2], Schlather [12], among others) and therefore have been widely applied to real data.

Briefly, a max-stable process $\mathbf{X} = \{X_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{R}^d}$ is the limit process of maxima of i.i.d.

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random fields $\left\{Y_{\mathbf{i}}^{(j)}\right\}_{\mathbf{i}\in\mathbb{R}^d}$, $j \geq 1$. Namely, for suitable $\{a_n(\mathbf{i}) > 0\}_{n\geq 1}$ and $\{b_n(\mathbf{i})\}_{n\geq 1}$ sequences of real constants,

$$X_{\mathbf{i}} = \lim_{n\to\infty} \frac{\bigvee_{j=1}^{n} Y_{\mathbf{i}}^{(j)} - b_n(\mathbf{i})}{a_n(\mathbf{i})}, \quad \mathbf{i} \in \mathbb{R}^d,$$

provided the limit exists.

We shall consider $d = 2$, that is $\mathbf{X} = \{X_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{R}^2}$. The distribution of $(X_{\mathbf{i}_1}, \ldots, X_{\mathbf{i}_k})$ is a Multivariate Extreme Value (MEV) distribution $G_{\mathbf{A}}$, $\mathbf{A} = \{\mathbf{i}_1, \ldots, \mathbf{i}_k\}$, and since one can transform one max-stable distribution into another one by a monotone transformation we can assume, without loss of generality, that the margins of $\mathbf{X}$ have a unit Fréchet distribution, $F(x) = \exp(-x^{-1}),\ x > 0$ (Resnick [11]). The distribution $G_{\mathbf{A}}$ can then be defined by

$$G_{\mathbf{A}}(x_1, \ldots, x_k) = \exp(-V_{\mathbf{A}}(x_1, \ldots, x_k)),\ x_i \in \mathbb{R}^+,\ i = 1, \ldots, k, \tag{1}$$

where $V_{\mathbf{A}}$ denotes the exponent function of the MEV distribution $G_{\mathbf{A}}$.

The exponent function summarizes the extremal dependence structure of $G_{\mathbf{A}}$ and the scalar $V_{\mathbf{A}}(1, \ldots, 1)$ defines the extremal coefficient $\epsilon_{\mathbf{A}}$ detailed in Schlather and Tawn [13], which summarizes the extremal dependence between the variables indexed in the region $\mathbf{A}$. This coefficient takes values between 1 and k, with a value of 1 corresponding to complete dependence and a value of k corresponding to complete independence. Its value can be thought of as the number of effectively independent locations among the k under consideration.

If we consider $\mathbf{A} = \{\mathbf{i}, \mathbf{j}\}$ we find the extremal coefficient of Tiago de Oliveira [19] which is related with the bivariate upper tail dependence coefficient $\lambda_{\{\mathbf{i},\mathbf{j}\}} = \lim_{u\uparrow 1} P\left(F(X_{\mathbf{i}}) > u | F(X_{\mathbf{j}}) > u\right)$, introduced in Sibuya [16], as $\epsilon_{\{\mathbf{i},\mathbf{j}\}} = 2 - \lambda_{\{\mathbf{i},\mathbf{j}\}}$.

Although these measures are very useful to analyze the dependence among extremal events, there remain important questions to be answered, for example, the influence of an extreme event on the regional smoothness of $\mathbf{X}$ and the contagion effect of an extreme event at a specific location over the variables of $\mathbf{X}$ indexed in a region of $\mathbb{R}^2$.

In Section 2, we propose measures of dependency to reflect the contagion and smoothness of $\mathbf{X}$ indexed in a region of $\mathbb{R}^2$, by the occurrence of an extreme event. The first measure that we propose, called contagion index, enables to quantify the impact that an exceedance of a high threshold can have on a region and we present its relation with bivariate extremal coefficients.

Clearly an extreme event could affect the smoothness of a random field over a region so, we also propose a stability index on a region $\mathbf{A}$ associated to a specific location $\mathbf{i}$, defined as the expected number of crossings of a high threshold u in $\mathbf{A}$ associated with $\mathbf{i}$, given that there is at least one crossing in $\mathbf{A}$ associated to $\mathbf{i}$. We also present some properties of this coefficient.

Based on relations of our indices with well-known dependence measures, for which estimators and respective properties have already been studied in the literature, in Section 3 we present estimators for the stability and contagion indices. The performance of the proposed estimators is analyzed in Section 4 with a max-stable M4 random field. Finally, Section 5 illustrates our measures through an application to the annual maxima precipitation in Portuguese regions. Section 6 is devoted to conclusions.

2. CONTAGION AND STABILITY INDICES

We first argue why there is a need for measures of dependency to reflect the contagion and stability of a region by the occurrence of an extreme event and why standard concepts introduced in Sibuya [16] are less suitable for the question at hand.

Indeed, why hydrologists will be interested in measures for contagion or stability of a region?

The fact is that the bivariate upper tail dependence coefficient focus on the occurrence of an extreme event at individual locations of the random field, without much attention for contagion ramifications.

We develop measures which do not hinge on a particular dependence structure and which link risks and probabilities directly.

The occurrence of an extreme event at a given location $\mathbf{i}$ may spread throughout a region of locations. In the following we define a measure for assessing the effect of an exceedance above a high threshold u at a specific location $\mathbf{i}$ on a region $\mathbf{A}$ of locations.

Definition 2.1. Let $X = \{X_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{R}^2}$ be a max-stable random field with unit Fréchet margins, F, and $\mathbf{A}$ a region of $\mathbb{R}^2$. The contagion index from the location $\mathbf{i}$ to the region $\mathbf{A}$ is defined as

$$CI(\mathbf{A},\mathbf{i}) = \lim_{u\uparrow 1} E\left(\sum_{\mathbf{j}\in\mathbf{A}} \mathbb{1}_{\{F(X_{\mathbf{j}})>u\}} \,\middle|\, F(X_{\mathbf{i}}) > u\right), \tag{2}$$

provided the limit exists.

The $CI(\mathbf{A},\mathbf{i})$ is the conditional expected number of exceedances above a high threshold u in $\mathbf{A}$, given $X_{\mathbf{i}}$ exceeds u, that is, the $CI(\mathbf{A},\mathbf{i})$ measures the impact that the event $\{X_{\mathbf{i}} > u\}$ has on the region $\mathbf{A}$.

We remark that the conditioning location $\mathbf{i}$ does not necessarily have to be in the region $\mathbf{A}$

The following proposition states that $CI(\mathbf{A},\mathbf{i})$ is directly linked with the bivariate extremal dependence coefficients $\epsilon_{\{\mathbf{i},\mathbf{j}\}}$, $\mathbf{j}\in\mathbf{A}$, and the tail dependence coefficients $\lambda_{\{\mathbf{i},\mathbf{j}\}}$, $\mathbf{j}\in\mathbf{A}$.

Proposition 2.2. For any max-stable random field with unit Fréchet margins F, $\mathbf{i}\in\mathbb{R}^2$ and $\mathbf{A}\subset\mathbb{R}^2$, we have

$$CI(\mathbf{A},\mathbf{i}) = \sum_{\mathbf{j}\in\mathbf{A}} \lambda_{\{\mathbf{i},\mathbf{j}\}} = 2\,|\mathbf{A}| - \sum_{\mathbf{j}\in\mathbf{A}} \epsilon_{\{\mathbf{i},\mathbf{j}\}}.$$

Proof. Observe that

$$\begin{aligned} CI(\mathbf{A},\mathbf{i}) &= \sum_{\mathbf{j}\in\mathbf{A}} \lim_{u\uparrow 1} P\left(F(X_{\mathbf{j}}) > u\,|F(X_{\mathbf{i}}) > u\right) \\ &= \sum_{\mathbf{j}\in\mathbf{A}} \lambda_{\{\mathbf{i},\mathbf{j}\}} = \sum_{\mathbf{j}\in\mathbf{A}} \left(2 - \epsilon_{\{\mathbf{i},\mathbf{j}\}}\right) = 2\,|\mathbf{A}| - \sum_{\mathbf{j}\in\mathbf{A}} \epsilon_{\{\mathbf{i},\mathbf{j}\}}. \end{aligned}$$

□

An $CI(\mathbf{A},\mathbf{i})$ close to $|\mathbf{A}|$ means that $\mathbf{i}$ has a high influence on $\mathbf{A}$, while an $CI(\mathbf{A},\mathbf{i})$ close to zero implies a negligible influence of $\mathbf{i}$ on $\mathbf{A}$. In other words, the higher the index, the higher the contagion effect of the event $\{X_{\mathbf{i}} > u\}$ on the region $\mathbf{A}$.

Remark 2.3. To gain some intuition for this measure, as a device for measuring dependence, consider two polar cases:

- Case 1. If $X_{\mathbf{i}}$ is independent of $X_{\mathbf{j}}$, for each $\mathbf{j} \in \mathbf{A}$, then $CI(\mathbf{A},\mathbf{i}) = 0$.
- Case 2. If, for each $\mathbf{j} \in \mathbf{A}$, $X_{\mathbf{j}}$ and $X_{\mathbf{i}}$ are totally dependent, then $CI(\mathbf{A},\mathbf{i}) = |\mathbf{A}|$.

Remark 2.4. We can extend the $CI(\mathbf{A},\mathbf{i})$ to the contagion index from a region $\mathbf{A}$ to a region $\mathbf{B}$, as follows

$$CI(\mathbf{A},\mathbf{B}) = \lim_{u\uparrow 1} E\left(\sum_{\mathbf{j}\in\mathbf{A}} \mathbb{1}_{\{F(X_{\mathbf{j}})>u\}} \,\middle|\, \bigcup_{\mathbf{i}\in\mathbf{B}} \{F(X_{\mathbf{i}}) > u\}\right).$$

This measure is related with the multivariate upper tail dependence coefficient (Schmidt [14]; Li [10]; Ferreira [5]), defined as

$$\lambda_{\mathbf{A},\mathbf{B}} = \lim_{u\uparrow 1} P\left(\bigcap_{\mathbf{j}\in\mathbf{A}} \{F(X_{\mathbf{j}}) > u\} \,\middle|\, \bigcap_{\mathbf{i}\in\mathbf{B}} \{F(X_{\mathbf{i}}) > u\}\right),$$

in the following way

$$CI(\mathbf{A},\mathbf{B}) = \sum_{\mathbf{j}\in\mathbf{A}} \frac{\sum_{\emptyset\neq\mathbf{J}\subseteq\mathbf{B}} (-1)^{|\mathbf{J}|+1}\lambda_{\mathbf{J},\{\mathbf{j}\}}}{\epsilon_{\mathbf{B}}}.$$

When we take $\mathbf{A} = \mathbf{B}$, we obtain the fragility index (FI) of the region $\mathbf{A}$. The FI was introduced in Geluk et al. [8] to measure the stability of a stochastic system. The system is called stable if $FI = 1$, otherwise it is called fragile.

In order to analyze the regional smoothness of a random field associated to a specific location we propose the following measure.

Definition 2.5. Let $\mathbf{X} = \{X_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{R}^2}$ be a max-stable random field with unit Fréchet margins, F, and $\mathbf{A}$ a region of $\mathbb{R}^2$. The stability index of the region $\mathbf{A}$ associated to a specific location $\mathbf{i} \in \mathbb{R}^2$, $SI(\mathbf{A},\mathbf{i})$, is defined as

$$SI(\mathbf{A},\mathbf{i}) = \lim_{u\uparrow 1} E\left(\sum_{\mathbf{j}\in\mathbf{A}} \mathbb{1}_{\{F(X_{\mathbf{i}})\leq u<F(X_{\mathbf{j}})\}} \,\middle|\, \sum_{\mathbf{j}\in\mathbf{A}} \mathbb{1}_{\{F(X_{\mathbf{i}})\leq u<F(X_{\mathbf{j}})\}} > 0\right),$$

provided the limit exists.

The $SI(\mathbf{A},\mathbf{i})$ is the conditional expected number of crossings (above a high threshold u) in $\mathbf{A}$ from a specific location $\mathbf{i}$, given that there is at least one crossing in $\mathbf{A}$ from $\mathbf{i}$.

If a max-stable random field $\mathbf{X}$ does not vary smoothly over a region $\mathbf{A}$, we will expect a large number of crossings of a high threshold in $\mathbf{A}$ associated to a specific location $\mathbf{i}$. A higher number of crossings signifies increased instability.

The next results highlight the connections between $SI(\mathbf{A},\mathbf{i})$ and the extremal coefficients.

Proposition 2.6. For any max-stable random field with unit Fréchet margins F, $\mathbf{i}\in\mathbb{R}^2$ and $\mathbf{A}\subset\mathbb{R}^2$, we have

$$SI(\mathbf{A},\mathbf{i})=\frac{|\mathbf{A}|-CI(\mathbf{A},\mathbf{i})}{\epsilon_{\{\mathbf{i}\}\cup\mathbf{A}}-1}.$$

Proof. Since

$$E\left(\sum_{\mathbf{j}\in\mathbf{A}}\mathbb{1}_{\{F(X_{\mathbf{i}})\le u<F(X_{\mathbf{j}})\}}\,\middle|\,\sum_{\mathbf{j}\in\mathbf{A}}\mathbb{1}_{\{F(X_{\mathbf{i}})\le u<F(X_{\mathbf{j}})\}}>0\right)$$
$$=\frac{\sum_{\mathbf{j}\in\mathbf{A}}P(F(X_{\mathbf{i}})\le u<F(X_{\mathbf{j}}))}{P\left(F(X_{\mathbf{i}})\le u,\bigcup_{\mathbf{j}\in A}\{F(X_{\mathbf{j}})>u\}\right)},$$

it follows that

$$SI(\mathbf{A},\mathbf{i})=\lim_{u\uparrow 1}\frac{|\mathbf{A}|\,u-\sum_{\mathbf{j}\in\mathbf{A}}u^{\epsilon_{\{\mathbf{i},\mathbf{j}\}}}}{u-u^{\epsilon_{\{\mathbf{i}\}\cup\mathbf{A}}}}=\frac{\sum_{\mathbf{j}\in\mathbf{A}}\epsilon_{\{\mathbf{i},\mathbf{j}\}}-|\mathbf{A}|}{\epsilon_{\{\mathbf{i}\}\cup\mathbf{A}}-1}.$$

□

Remark 2.7. If the random variables $X_{\mathbf{i}},X_{\mathbf{i}_1},\ldots,X_{\mathbf{i}_k}$, are totally dependent the stability index is not defined since, for all $\mathbf{j}\in\{\mathbf{i}_1,\ldots,\mathbf{i}_k\}$,

$$P\left(F(X_{\mathbf{i}})\le u<F(X_{\mathbf{j}})\right)=P\left(F(X_{\mathbf{j}})\le u\right)-P^{\varepsilon(\mathbf{i},\mathbf{j})}\left(F(X_{\mathbf{i}})\le u\right)=0.$$

If the random variables are independent we have $SI(\mathbf{A},\mathbf{i})=1$.

Proposition 2.8. Under the conditions of Proposition 2.6, we have

$$\frac{\sum_{\mathbf{j}\in\mathbf{A}}\epsilon_{\{\mathbf{i},\mathbf{j}\}}-|\mathbf{A}|}{|\mathbf{A}|}\le SI(\mathbf{A},\mathbf{i})\le\frac{\sum_{\mathbf{j}\in\mathbf{A}}\epsilon_{\{\mathbf{i},\mathbf{j}\}}-|\mathbf{A}|}{\bigvee_{\mathbf{j}\in\mathbf{A}}\epsilon_{\{\mathbf{i},\mathbf{j}\}}-1}.$$

Proof. Just observe that $\epsilon_{\{\mathbf{i}\}\cup\mathbf{A}}\le|\mathbf{A}|+1$ and $\epsilon_{\{\mathbf{i}\}\cup\mathbf{A}}\ge\bigvee_{\mathbf{j}\in\mathbf{A}}\epsilon_{\{\mathbf{i},\mathbf{j}\}}$. □

An important point to keep in mind is that conditional probabilities do not necessarily imply causation. However this set of measures do provide important insights into the inter-linkages and the likelihood of contagion of an extreme event in a region of locations.

In combination, both the CI and SI provide a valuable tool for analyzing risk factor from complementary perspectives.

We next focus on the estimation of the stability and contagion indices.

3. ESTIMATION

As previously stated, the contagion and stability indices relate with the extremal coefficients of Tiago de Oliveira [19] and Schlather and Tawn [13], which can be expressed by the exponent function given in (1).

There are several references in literature on the estimation of the exponent function. For a survey we refer to Krajina [9] and Beirlant [1].

It is known that parametric estimation methods are efficient if the distribution model under consideration is true, but they suffer from biased estimates otherwise. Non parametric estimation procedures avoid this type of model error. However, they are usually based on an arbitrarily chosen parameter k corresponding to the number of top order statistics to be used on the estimation of a high quantile of F, which relates to the usual variance-bias problem: if k is too small, then the estimator tends to have a large variance, where if k is too large, then the bias tends to dominate. Some methods of choosing an optimal k are discussed in Einmahl *et al.* [4].

In order to overcome the problem of the optimal choice of k, Ferreira and Ferreira [6] developed another approach. Based on the following relation

$$\epsilon_{\mathbf{A}} = V_{\mathbf{A}}(1,1,\ldots,1) = \frac{E(M(\mathbf{A}))}{1-E(M(\mathbf{A}))}, \quad \text{where} \quad M(\mathbf{A}) = \bigvee_{\mathbf{i}\in\mathbf{A}} F_{\mathbf{i}}(X_{\mathbf{i}}),$$

the estimator of $\epsilon_{\mathbf{A}}$ proposed in Ferreira and Ferreira [6] is defined as

$$\widehat{\epsilon}_{\mathbf{A}} = \frac{\overline{M(\mathbf{A})}}{1-\overline{M(\mathbf{A})}},$$

where $\overline{M(\mathbf{A})}$ is the sample mean,

$$\overline{M(\mathbf{A})} = \frac{1}{n}\sum_{j=1}^{n}\bigvee_{\mathbf{i}\in\mathbf{A}} \widehat{F}_{\mathbf{i}}(X_{\mathbf{i}}^{(j)})$$

and $\widehat{F}_{\mathbf{i}}$, $\mathbf{i}\in\mathbf{A}$, is the (modified) empirical distribution function of $F_{\mathbf{i}}$,

$$\widehat{F}_{\mathbf{i}}(u) = \frac{1}{n+1}\sum_{j=1}^{n} \mathbb{1}_{\left\{X_{\mathbf{i}}^{(j)}\le u\right\}}.$$

With this estimator of the extremal coefficient and the relations established in Propositions 2.2 and 2.6 we propose, respectively, the following estimators for the contagion index $CI(\mathbf{A},\mathbf{i})$ and the stability index $SI(\mathbf{A},\mathbf{i})$,

$$\widehat{CI}(\mathbf{A},\mathbf{i}) = 2\,|\mathbf{A}| - \sum_{\mathbf{j}\in\mathbf{A}} \widehat{\epsilon}_{\{\mathbf{i},\mathbf{j}\}}$$

and

$$\widehat{SI}(\mathbf{A},\mathbf{i}) = \frac{\sum_{\mathbf{j}\in\mathbf{A}}\widehat{\epsilon}_{\{\mathbf{i},\mathbf{j}\}} - |\mathbf{A}|}{\widehat{\epsilon}_{\{\mathbf{i}\}\cup\mathbf{A}} - 1},$$

which are consistent given the consistency of the estimators $\widehat{\epsilon}_{\{\mathbf{i},\mathbf{j}\}}$ and $\widehat{\epsilon}_{\{\mathbf{i}\}\cup\mathbf{A}}$ already stated in Ferreira and Ferreira [6].

Estimator efficiency is assessed through an M4 random field which will be introduced in the following section.

4. AN M4 RANDOM FIELD

It is well known that the class of max-stable processes called maxima of moving maxima processes or simply M4 processes, introduced by Smith and Weissman [18], is particularly well adapted to model the extreme behaviour of several time series (Zhang and Smith [22]). Motivated by this application of M4 processes, we extended the model M4 for the random fields theory and we expect that they provide future real applications. An M4 random field $\mathbf{X} = \{X_\mathbf{i}\}_{\mathbf{i}\in\mathbb{N}^2}$ is defined as

$$X_\mathbf{i} = \max_{l\geq 1} \max_{-\infty<m<+\infty} a_{lm\mathbf{i}} Z_{l,1-m}, \quad \mathbf{i} \in \mathbb{N}^2, \tag{3}$$

where $\{Z_{l,n}\}_{l\geq 1, n\in\mathbb{N}}$ is a family of independent unit Fréchet random variables and, for each $\mathbf{i} \in \mathbb{N}^2$, $\{a_{lm\mathbf{i}}\}_{l\geq 1, m\in\mathbb{N}}$ are non-negative constants such that $\sum_{l=1}^{+\infty}\sum_{m=-\infty}^{+\infty} a_{lm\mathbf{i}} = 1$. By considering that the distribution of $(X_{\mathbf{i}_1}, \ldots, X_{\mathbf{i}_k})$ is characterized by the copula

$$C(u_{\mathbf{i}_1}, \ldots, u_{\mathbf{i}_k}) = \prod_{l=1}^{+\infty}\prod_{m=-\infty}^{+\infty} \bigwedge_{\mathbf{i}\in\{\mathbf{i}_1,\ldots,\mathbf{i}_k\}} u_\mathbf{i}^{a_{lm\mathbf{i}}}, \quad u_{\mathbf{i}_j} \in [0,1],\ j = 1, \ldots, k, \tag{4}$$

it was shown in Fonseca et al. [7] that the random field $\mathbf{X} = \{X_\mathbf{i}\}_{\mathbf{i}\in\mathbb{N}^2}$ is max-stable and the exponent function of the distribution of $(X_{\mathbf{i}_1}, \ldots, X_{\mathbf{i}_k})$ is given by

$$V_\mathbf{A}(x_1, \ldots, x_k) = \sum_{l=1}^{+\infty}\sum_{m=-\infty}^{+\infty}\bigvee_{j=1}^{k}\left(x_j^{-1} a_{lm\mathbf{i}_j}\right),\ x_j \in \mathbb{R},\ j = 1, \ldots, k,\ \mathbf{A} = \{\mathbf{i}_1, \ldots, \mathbf{i}_k\}.$$

So

$$CI(\mathbf{A}, \mathbf{i}) = 2\,|\mathbf{A}| - \sum_{\mathbf{j}\in\mathbf{A}}\sum_{l=1}^{+\infty}\sum_{m=-\infty}^{+\infty}(a_{lm\mathbf{i}} \vee a_{lm\mathbf{j}}) \tag{5}$$

and

$$SI(\mathbf{A}, \mathbf{i}) = \frac{\sum_{\mathbf{j}\in\mathbf{A}}\sum_{l=1}^{+\infty}\sum_{m=-\infty}^{+\infty}(a_{lm\mathbf{i}} \vee a_{lm\mathbf{j}}) - |\mathbf{A}|}{\sum_{l=1}^{+\infty}\sum_{m=-\infty}^{+\infty}\left(\bigvee_{\mathbf{j}\in\mathbf{A}} a_{lm\mathbf{j}} \vee a_{lm\mathbf{i}}\right) - 1}. \tag{6}$$

To assess the performance of the estimators of the stability and contagion indices given in Section 3, we shall consider, in what follows, examples with a finite number of signature patterns ($1 \leq l \leq L$) and a finite range of sequential dependencies ($M_1 \leq m \leq M_2$).

Example 4.1. Let us consider that for each location $\mathbf{i} \in \mathbb{N}^2$ with even abscissa we have $a_{11\mathbf{i}} = \frac{4}{5}$, $a_{12\mathbf{i}} = \frac{1}{5}$ and otherwise $a_{11\mathbf{i}} = \frac{1}{4} = 1 - a_{12\mathbf{i}}$. The values of $(a_{11\mathbf{i}}, a_{12\mathbf{i}})$ determine the moving pattern or signature pattern of the random field, which in this case corresponds to one pattern ($L = 1$).

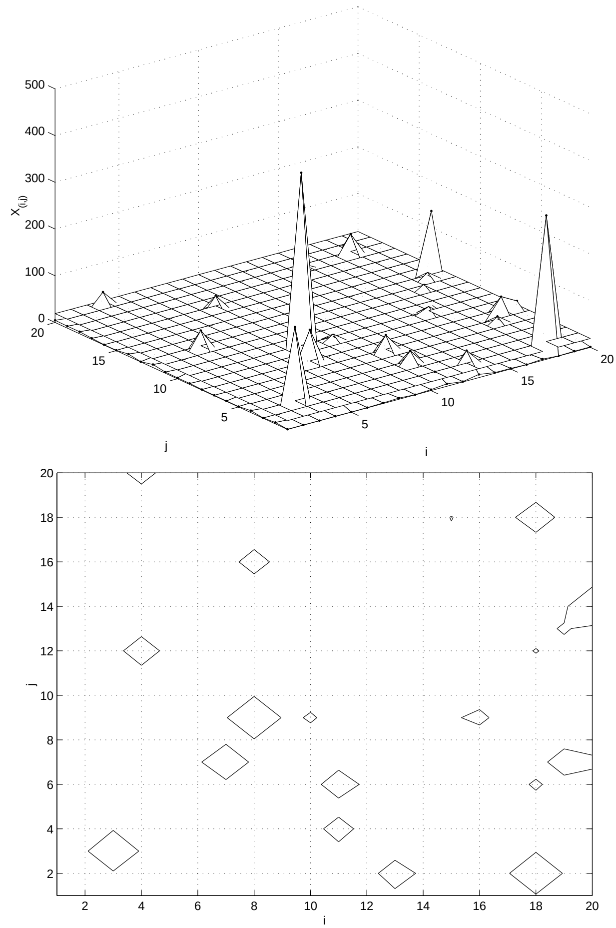

Fig. 1. Simulation of the M4 as defined in Example 4.1 (left) and the contour at $x_{(i,j)} = 19.0219$, the 95 % quantile (right).

Let $\mathbf{A}^{(k)} = \{s_j^k(\mathbf{i}) : \mathbf{i} = (3,3) \wedge j \in \{1,2,\ldots,8\}\}$, where $s_j^k(\mathbf{i}) = (s_j \circ \ldots \circ s_j)(\mathbf{i})$, k times, $k \geq 1$, $s_j^0(\mathbf{i}) = \mathbf{i}$, and $s_j(\mathbf{i})$, $j = 1,\ldots,8$, denote the neighbors of $\mathbf{i}$ defined as follows:

$$s_1(\mathbf{i}) = (i_1+1, i_2), \quad s_2(\mathbf{i}) = \mathbf{i}+\mathbf{1}, \quad s_3(\mathbf{i}) = (i_1, i_2+1), \quad s_4(\mathbf{i}) = (i_1-1, i_2+1),$$
$$s_5(\mathbf{i}) = (i_1-1, i_2), \quad s_6(\mathbf{i}) = \mathbf{i}-\mathbf{1}, \quad s_7(\mathbf{i}) = (i_1, i_2-1), \quad s_8(\mathbf{i}) = (i_1+1, i_2-1).$$

The matrix of the bivariate extremal coefficients, $\epsilon_{\{s_j^k(\mathbf{i}),\mathbf{i}\}}$, $j = 1,\ldots,8$, $k \geq 1$, provide insight into the likelihood of contagion from $\mathbf{i} = (3,3)$ to its neighbors $s_j^k(\mathbf{i})$, $j = 1,\ldots,8$, although without specifying the size of the impact which is given by $CI(\mathbf{A}^{(k)}, \mathbf{i})$. We obtain

$$\begin{bmatrix} \epsilon_{\{s_4^k(\mathbf{i}),\mathbf{i}\}} & \epsilon_{\{s_3^k(\mathbf{i}),\mathbf{i}\}} & \epsilon_{\{s_2^k(\mathbf{i}),\mathbf{i}\}} \\ \epsilon_{\{s_5^k(\mathbf{i}),\mathbf{i}\}} & \epsilon_{\{\mathbf{i},\mathbf{i}\}} & \epsilon_{\{s_1^k(\mathbf{i}),\mathbf{i}\}} \\ \epsilon_{\{s_6^k(\mathbf{i}),\mathbf{i}\}} & \epsilon_{\{s_7^k(\mathbf{i}),\mathbf{i}\}} & \epsilon_{\{s_8^k(\mathbf{i}),\mathbf{i}\}} \end{bmatrix} = \begin{bmatrix} \frac{31}{20} & 1 & \frac{31}{20} \\ \frac{31}{20} & 1 & \frac{31}{20} \\ \frac{31}{20} & 1 & \frac{31}{20} \end{bmatrix}$$

$$= \begin{cases} \begin{bmatrix} 31/20 & 1 & 31/20 \\ 31/20 & 1 & 31/20 \\ 31/20 & 1 & 31/20 \end{bmatrix}, & \text{if } k \text{ is odd} \\ \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, & \text{if } k \text{ is even} \end{cases}$$

and

$$CI(\mathbf{A}^{(k)}, \mathbf{i}) = \begin{cases} 4,7, & \text{if } k \text{ is odd} \\ 8, & \text{if } k \text{ is even.} \end{cases}$$

The stability index of the region $\mathbf{A}^{(k)}$ associated with $\mathbf{i} = (3,3)$ is given by

$$SI(\mathbf{A}^{(k)}, \mathbf{i}) = 6,$$

when k is odd and it is not defined for k even, because the variables are totally dependent.

The results of the application of the estimators $\widehat{CI}(\mathbf{A}, \mathbf{i})$ and $\widehat{SI}(\mathbf{A}, \mathbf{i})$ are presented in Tables 1 and 2.

number of random fields	$\widehat{CI}$	MSE
100	8	1.64e-29
500	8	6.04e-29
1000	8	1.18e-28

Tab. 1. Results with 100 replications of M4 random field defined in Example 4.1, where $CI(\mathbf{A}^{(k)}, \mathbf{i}) = 8$, when k is even. $\widehat{CI}$ denotes the mean of estimated values of the contagion index and MSE the estimated mean squared error.

	Contagion I.		Stability I.	
number of random fields	$\widehat{CI}$	MSE	$\widehat{SI}$	MSE
100	4.7848	0.1232	6	3.60e-29
500	4.7449	0.0270	6	1.97e-29
1000	4.7081	0.0126	6	1.97e-29

Tab. 2. Results with 100 replications of M4 random field defined in Example 4.1, where $CI(\mathbf{A}^{(k)}, \mathbf{i}) = 4.7$ and $SI(\mathbf{A}^{(k)}, \mathbf{i}) = 6$ when k is odd. $\widehat{CI}$ denotes the mean of estimated values of the contagion index and MSE the estimated mean squared error.

Example 4.2. Now, we shall consider one example with four signature patterns ($L = 4$). Lets assume that for each location $\mathbf{i} = (i_1, j_1) \in \mathbb{N}^2$ with

(1) $i_1 > j_1 \quad \wedge \quad i_1$ even $\quad \wedge \quad j_1$ odd ,

$$a_{11\mathbf{i}} = a_{12\mathbf{i}} = a_{21\mathbf{i}} = a_{22\mathbf{i}} = \tfrac{1}{8},$$
$$a_{31\mathbf{i}} = a_{32\mathbf{i}} = a_{41\mathbf{i}} = a_{42\mathbf{i}} = \tfrac{1}{8},$$

(2) $i_1 \leq j_1 \quad \wedge \quad i_1$ even $\quad \wedge \quad j_1$ odd ,

$$a_{11\mathbf{i}} = a_{12\mathbf{i}} = \tfrac{2}{17}, \quad a_{21\mathbf{i}} = \tfrac{5}{17}, \quad a_{22\mathbf{i}} = \tfrac{4}{17},$$
$$a_{31\mathbf{i}} = a_{32\mathbf{i}} = a_{41\mathbf{i}} = a_{42\mathbf{i}} = \tfrac{1}{17},$$

(3) $i_1 > j_1 \quad \wedge \quad i_1$ odd $\quad \wedge \quad j_1$ even ,

$$a_{11\mathbf{i}} = \tfrac{1}{20}, \quad a_{12\mathbf{i}} = \tfrac{2}{20}, \quad a_{21\mathbf{i}} = \tfrac{3}{20}, \quad a_{22\mathbf{i}} = \tfrac{4}{20},$$
$$a_{31\mathbf{i}} = \tfrac{5}{20}, \quad a_{32\mathbf{i}} = \tfrac{3}{20}, \quad a_{41\mathbf{i}} = a_{42\mathbf{i}} = \tfrac{1}{20},$$

(4) $i_1 \leq j_1 \quad \wedge \quad i_1$ odd $\quad \wedge \quad j_1$ even ,

$$a_{11\mathbf{i}} = \tfrac{1}{36}, \quad a_{12\mathbf{i}} = \tfrac{2}{36}, \quad a_{21\mathbf{i}} = \tfrac{3}{36}, \quad a_{22\mathbf{i}} = \tfrac{4}{36},$$
$$a_{31\mathbf{i}} = \tfrac{5}{36}, \quad a_{32\mathbf{i}} = \tfrac{6}{36}, \quad a_{41\mathbf{i}} = \tfrac{7}{36}, \quad a_{42\mathbf{i}} = \tfrac{8}{36},$$

(5) $i_1 > j_1 \quad \wedge \quad i_1$ even $\quad \wedge \quad j_1$ even ,

$$a_{11\mathbf{i}} = \tfrac{1}{40}, \quad a_{12\mathbf{i}} = \tfrac{2}{40}, \quad a_{21\mathbf{i}} = \tfrac{3}{40}, \quad a_{22\mathbf{i}} = \tfrac{4}{40},$$
$$a_{31\mathbf{i}} = \tfrac{5}{40}, \quad a_{32\mathbf{i}} = \tfrac{6}{40}, \quad a_{41\mathbf{i}} = \tfrac{7}{40}, \quad a_{42\mathbf{i}} = \tfrac{12}{40},$$

(6) $i_1 \leq j_1 \quad \wedge \quad i_1$ even $\quad \wedge \quad j_1$ even ,

$$a_{11\mathbf{i}} = \tfrac{1}{45}, \quad a_{12\mathbf{i}} = \tfrac{2}{45}, \quad a_{21\mathbf{i}} = \tfrac{3}{45}, \quad a_{22\mathbf{i}} = \tfrac{4}{45},$$
$$a_{31\mathbf{i}} = \tfrac{6}{45}, \quad a_{32\mathbf{i}} = \tfrac{8}{45}, \quad a_{41\mathbf{i}} = \tfrac{9}{45}, \quad a_{42\mathbf{i}} = \tfrac{12}{45},$$

(7) $i_1 > j_1 \quad \wedge \quad i_1$ odd $\quad \wedge \quad j_1$ odd ,

$$a_{11\mathbf{i}} = \tfrac{1}{50}, \quad a_{12\mathbf{i}} = \tfrac{7}{50}, \quad a_{21\mathbf{i}} = \tfrac{3}{50}, \quad a_{22\mathbf{i}} = \tfrac{4}{50},$$
$$a_{31\mathbf{i}} = \tfrac{6}{50}, \quad a_{32\mathbf{i}} = \tfrac{8}{50}, \quad a_{41\mathbf{i}} = \tfrac{9}{50}, \quad a_{42\mathbf{i}} = \tfrac{12}{50},$$

(8) $i_1 \leq j_1 \quad \wedge \quad i_1$ odd $\quad \wedge \quad j_1$ odd ,

$$a_{11\mathbf{i}} = \tfrac{1}{60}, \quad a_{12\mathbf{i}} = \tfrac{7}{60}, \quad a_{21\mathbf{i}} = \tfrac{3}{60}, \quad a_{22\mathbf{i}} = \tfrac{14}{60},$$
$$a_{31\mathbf{i}} = \tfrac{6}{60}, \quad a_{32\mathbf{i}} = \tfrac{8}{60}, \quad a_{41\mathbf{i}} = \tfrac{9}{60}, \quad a_{42\mathbf{i}} = \tfrac{12}{60}.$$

The values of $(a_{l1\mathbf{i}}, a_{l2\mathbf{i}})$ for $l = 1, \ldots, 4$ define the four signature patterns of the random field.

For $\mathbf{A} = \{(4,3),(3,4),(2,3),(2,2),(3,2),(4,2)\}$ and $\mathbf{i} = (3,3)$ the bivariate extremal coefficients are

$$\epsilon_{\{(4,3),\mathbf{i}\}} = \tfrac{29}{24}, \quad \epsilon_{\{(3,4),\mathbf{i}\}} = \tfrac{71}{60}, \quad \epsilon_{\{(2,3),\mathbf{i}\}} = \tfrac{275}{204},$$

$$\epsilon_{\{(2,2),\mathbf{i}\}} = \tfrac{73}{60}, \quad \epsilon_{\{(3,2),\mathbf{i}\}} = \tfrac{13}{10}, \quad \epsilon_{\{(4,2),\mathbf{i}\}} = \tfrac{6}{5}$$

and from (5) we have $CI(\mathbf{A}, \mathbf{i}) \approx 4.5353$ and by (6) we obtain $SI(\mathbf{A}, \mathbf{i}) \approx 2.0712$.

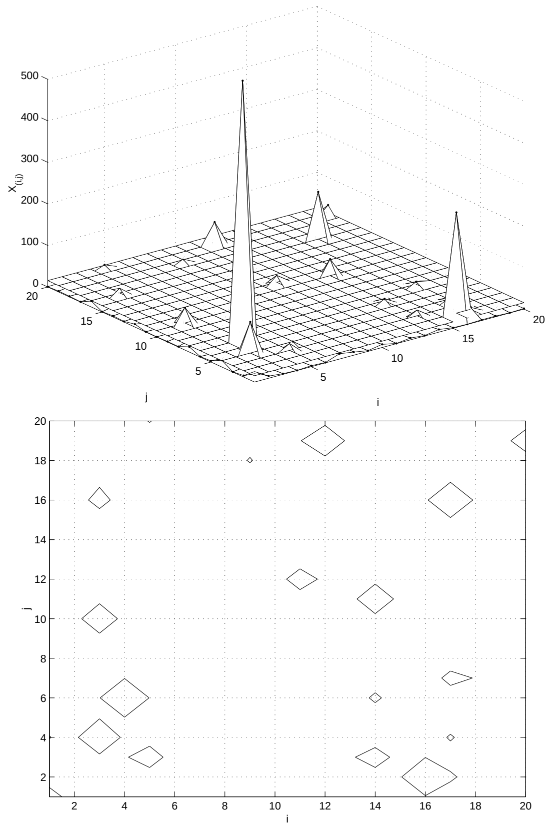

Fig. 2. Simulation of the M4 as defined in Example 4.2 (left) and the contour at $x_{(i,j)} = 15.3417$, the 95% quantile (right).

The Table 3 shows the results of the application of the estimators $\widehat{CI}(\mathbf{A}, \mathbf{i})$ and $\widehat{SI}(\mathbf{A}, \mathbf{i})$ to the Example 4.2.

As we can see, in both examples, the estimated values are very close to the true values of the coefficients. These results show that the simple non-parametric estimators $\widehat{CI}$ and $\widehat{SI}$ are a promising tool for assessing regional contagion effects and regional smoothness for these random fields.

5. AN APPLICATION TO PRECIPITATION DATA

Even though Portugal is a small country, it has a wide variety of landforms, climatic conditions and soils. The major difference is between the mountains regions of the north and the great rolling plains of the south. The Central Cordillera formed by the mountains of Sintra, Montejunto and Estrela, divides Portugal into northern and southern regions and creates a physical barrier for precipitation.

	Contagion I.		Stability I.	
number of random fields	$\widehat{CI}$	MSE	$\widehat{SI}$	MSE
100	4.5463	0.0202	2.0451	0.0178
500	4.5309	0.0047	2.0606	0.0033
1000	4.5339	0.0021	2.0721	0.0015

Tab. 3. Results with 100 replications of M4 random field defined in Example 4.2, where $CI(\mathbf{A}, \mathbf{i}) \approx 4.5353$ e $SI(\mathbf{A}, \mathbf{i}) \approx 2.0712$. $\widehat{CI}$ and $\widehat{SI}$ denotes the mean of estimated values of the contagion and stability index, respectively, and MSE the estimated mean squared error.

The spatial distribution of mean annual rainfall in Portugal reveals a sharp contrast between north and south. The amounts of precipitation is significantly higher in the north than in the south.

In the following we study the influence of extreme precipitation occuring at Lagoa Comprida, located in North-West part of 'Serra da Estrela', the highest mountain in Continental Portugal and part of the Central Cordillera, on the north and south regions {Gouveia, Oliveira do Hospital, Seia} and {Penamacor, Barragem Cabeço Monteiro}, respectively. We used annual maxima values of daily maxima precipitation recorded over 32 years, in six Portuguese stations (Figure 3), provided by the Portuguese National System of Water Resources (http://snirh.pt).

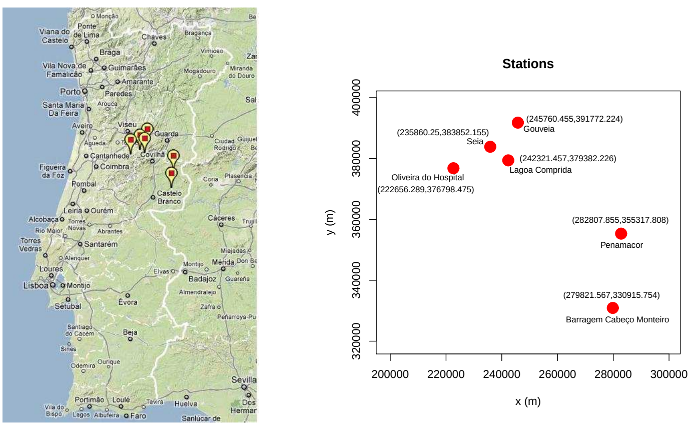

Fig. 3. The locations of the stations where precipitation data were collected, obtained from Portuguese National System of Water Resources (left) and their representation in Lambert coordinates (right).

Since the data are maxima over a long period of time, we assumed that they are independent over the years in each location. We also assumed that the random field is max-stable with unknown marginal distributions so data were previously transformed at each site so that they have a standard Fréchet distribution.

The estimated values of the contagion and stability indices from Lagoa Comprida to the regions {Gouveia, Oliveira do Hospital, Seia} and {Penamacor, Barragem Cabeço Monteiro} are presented in Table 4.

A	$\widehat{CI}(\mathbf{A}$, Lagoa Comprida)	$\widehat{SI}(\mathbf{A}$, Lagoa Comprida)
{Gouveia, Oliveira do Hospital, Seia}	0.9669	1.6017
{Penamacor, Barragem Cabeço Monteiro}	0.0089	1.2224

Tab. 4. Estimates of the contagion and stability indices from Lagoa Comprida to the regions {Gouveia, Oliveira do Hospital, Seia} and {Penamacor, Barragem Cabeço Monteiro}.

The results suggest that Lagoa Comprida has a higher influence on the region {Gouveia, Oliveira do Hospital, Seia} in terms of precipitation amounts and this region is smoother when compared to region {Penamacor, Barragem Cabeço Monteiro}. This result is consistent with the previously stated that the Central Cordillera creates a physical barrier for precipitation in Portugal.

6. CONCLUSION

In this work we introduced two new measures the contagion index and the stability index. The contagion index enables to quantify the impact that an exceedance of a high threshold can have on a region and the stability index allows us to analyze the regional smoothness of a random field associated to a specific location. With these indices we are able to analyze, for instance, the influence that an extreme precipitation (temperature) at a specific location could have on a neighboring region and its effect on the regional smoothness.

Besides the theoretical study of these measures, estimators were proposed and a simulation study was carried out to evaluate their behavior. Applications to precipitation data from Portuguese regions were also presented. The simulation results show the good performance of the proposed estimators, when considering an M4 random field.

All the simulations presented in this paper were done in R statistical computing program (`http://cran.rproject.org/`).

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REFERENCES

[1] J. Beirlant, Y. Goegebeur, J. Segers, and J. Teugels: Statistics of Extremes: Theory and Applications. John Wiley, 2004.

[2] S. G. Coles: Regional modelling of extreme storms via max-stable processes. J. R. Stat. Soc. Ser. B *55* (1993), 797–816.

[3] A. C. Davison and R. Huser: Space-time modelling of extreme events. J. R. Stat. Soc. Ser. B *76* (2013), 439–461.

[4] J. Einmahl, J. Li, and R. Liu: Extreme value theory approach to simultaneous monitoring and thresholding of multiple risk indicators. CentER Discussion Paper (Int. Rep. 2006-104) Econometrics, 2006.

[5] H. Ferreira: Dependence between two multivariate extremes. Stat. Probab. Lett. *81* (2011), 586–591.

[6] H. Ferreira and M. Ferreira: On extremal dependence: some contributions. Test *21* (2012), 566–583.

[7] C. Fonseca, L. Pereira, H. Ferreira, and A. P. Martins: Generalized madogram and pairwise dependence of maxima over two disjoint regions of a random field. arXiv: `http://arxiv.org/pdf/1104.2637v2.pdf`, 2012.

[8] J. L. Geluk, L. De Haan, and C. G. De Vries: Weak and strong financial fragility. Tinbergen Institute Discussion Paper, TI 2007-023/2.

[9] A. Krajina: An M-Estimator of Multivariate Dependence Concepts. Tilburg University Press, Tilburg 2010.

[10] H. Li: Orthant tail dependence of multivariate extreme value distributions. J. Multivariate Anal. *46* (2009), 262–282.

[11] S. I. Resnick: Extreme Values, Regular Variation and Point Processes. Springer-Verlag, Berlin 1987.

[12] M. Schlather: Models for stationary max-stable random fields. Extremes *5* (2002), 33–44.

[13] M. Schlather and J. Tawn: A dependence measure for multivariate and spatial extreme values: Properties and inference. Biometrika *90* (2003), 139–156.

[14] R. Schmidt: Tail dependence for elliptically countered distributions. Math. Methods Oper. Res. *55* (2002), 301–327.

[15] R. Schmidt and U. Stadmüller: Non parametric estimation of tail dependence. Scand. J. Stat. *33* (2006), 307–335.

[16] M. Sibuya: Bivariate extreme statistics. Ann. Inst. Stat. Math. *11* (1960), 195–210.

[17] R. L. Smith: Max-stable processes and spatial extremes. Unpublished manuscript. `http://www.stat.unc.edu/postscript/rs/spatex.pdf`, 1990.

[18] R. L. Smith and I. Weissman: Characterization and estimation of the multivariate extremal index. Technical Report, Department of Statistics, University of North Carolina. `http://www.stat.unc.edu/postscript/rs/extremal.pdf`, 1996.

[19] J. Tiago de Oliveira: Structure theory of bivariate extremes, extensions. Est. Mat., Est. and Econ. *7* (1992/93), 165–195.

[20] K. E. Trenberth: Atmospheric moisture residence times and cycling implications for rainfall rates and climate change. Climate Change *39* (1998), 667–694.

[21] K. E. Trenberth: Conceptual framework for changes of extremes of the hydrological cycle with climate change. Climate Change *42* (1999) 327–339.

[22] Z. Zhang and R. L. Smith: The behavior of multivariate maxima of moving maxima processes. J. Appl. Probab. *41* (2004), 1113–1123.

Cecília Fonseca, Instituto Politécnico da Guarda. Portugal.
e-mail: cfonseca@ipg.p

Helena Ferreira, Departamento de Matemática, Universidade da Beira Interior. Portugal.
e-mail: helenaf@ubi.pt

Luísa Pereira, Departamento de Matemática, Universidade da Beira Interior. Portugal.
e-mail: lpereira@ubi.pt

Ana Paula Martins, Departamento de Matemática, Universidade da Beira Interior. Portugal.
e-mail: amartins@ubi.pt