Irena Hinterleitner; Josef Mikeš; Patrik Peška

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ON F_2^ε-PLANAR MAPPINGS OF (PSEUDO-) RIEMANNIAN MANIFOLDS

Irena Hinterleitner, Josef Mikeš, and Patrik Peška

Abstract. We study special F-planar mappings between two n-dimensional (pseudo-) Riemannian manifolds. In 2003 Topalov introduced PQ^ε-projectivity of Riemannian metrics, $\varepsilon \neq 1, 1+n$. Later these mappings were studied by Matveev and Rosemann. They found that for $\varepsilon = 0$ they are projective.

We show that PQ^ε-projective equivalence corresponds to a special case of F-planar mapping studied by Mikeš and Sinyukov (1983) and F_2-planar mappings (Mikeš, 1994), with $F = Q$. Moreover, the tensor P is derived from the tensor Q and the non-zero number ε. For this reason we suggest to rename PQ^ε as F_2^ε. We use earlier results derived for F- and F_2-planar mappings and find new results.

For these mappings we find the fundamental partial differential equations in closed linear Cauchy type form and we obtain new results for initial conditions.

1. Introduction

Diffeomorphisms and automorphisms of geometrically generalized manifolds constitute one of the current main directions in differential geometry. Many papers are devoted to geodesic, almost geodesic, quasigeodesic, holomorphically projective, F-planar mappings and many others. The investigation of special manifolds with affine connection, (pseudo-) Riemannian, e-Kählerian and e-Hermitian spaces, give one of the most important area, see [1] – [33]. For example, T. Levi-Civita [15] used geodesic mappings for modeling mechanical processes, and A.Z. Petrov [27] used quasigeodesic mappings for modeling in theoretical physics. More general mappings were studied by Hrdina, Slovák and Vašík, see [10], [11] and [12].

The PQ^ε-projective equivalence between n-dimensional Riemannian manifolds were introduced by Topalov [32], P and Q are tensors of type $(1,1)$ for which $PQ = \varepsilon\,\mathrm{Id}$, $\varepsilon \in \mathbb{R}$, $\varepsilon \neq 1, 1+n$. It follows immediately from their definition that PQ^ε-projective equivalence is the correspondence occurring in the earlier studied F-planar mappings (Mikeš, Sinyukov [24]) and $F = Q$. We prove that these mappings are F_2-planar mappings (Mikeš [18]), which generalize geodesic and holomorphically projective mappings, see [25, 29, 33].

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In paper [32] by Topalov and paper [16] by Matveev and Rosemann, some properties of this equivalence were studied and among other things it was shown that if $\varepsilon = 0$ this equivalence is projective. This is the reason, why we study PQ^ε-projective equivalence where $\varepsilon \neq 0$ only. With a detailed analysis, we found that the tensor P, with all of its properties, is derived from the tensor Q and the number ε, so that $P = \varepsilon\, F^{-1}$. According to these facts, we renamed PQ^ε-projective equivalence as F_2^ε-planar mapping (for which $F \equiv Q$).

In this paper we study F_2^ε-projective mappings between (pseudo-) Riemannian manifolds for $\varepsilon \neq 0$. For these mappings we find a fundamental system of closed linear equations in covariant derivatives and we obtain new results for initial conditions. We proved that a set of (pseudo-) Riemannian manifolds with $F^2 \neq \varepsilon\, \mathrm{Id}$, on which some (pseudo-) Riemannian manifold admits F_2^ε-projective mappings, depends on no more than $n(n-1)/2$ parameters.

2. On F-planar mappings

Let $A_n = (M, \nabla, F)$ be an n-dimensional manifold M with affine connection ∇, and affinor structure F, i.e. a tensor field of type $(1,1)$.

Definition 1 ([24], [25, p. 213])**.** A curve ℓ, which is given by the equations $\ell = \ell(t)$, $\lambda(t) = d\ell(t)/dt$ $(\neq 0)$, $t \in I$, where t is a parameter, is called *F-planar*, if its tangent vector $\lambda(t_0)$, for any initial value t_0 of the parameter t, remains under parallel translation along the curve ℓ, in the distribution generated by the vector functions λ and $F\lambda$ along ℓ.

In accordance with this definition, ℓ is F-planar if and only if the following condition holds ([24], [25, p. 213]): $\nabla_{\lambda(t)}\lambda(t) = \varrho_1(t)\lambda(t) + \varrho_2(t)F\lambda(t)$, where ϱ_1 and ϱ_2 are some functions of the parameter t.

We consider two spaces $A_n = (M, \nabla, F)$ and $\bar{A}_n = (\bar{M}, \bar{\nabla}, \bar{F})$ with torsion-free affine connections ∇ and $\bar{\nabla}$, respectively. Affine structures F and $\bar{F}$ are defined on A_n, resp. $\bar{A}_n$.

Definition 2 (Mikeš, Sinyukov [24], see [25, p. 213])**.** A diffeomorphism f between manifolds with affine connection A_n and $\bar{A}_n$ is called an *F-planar mapping* if any F-planar curve in A_n is mapped onto an $\bar{F}$-planar curve in $\bar{A}_n$.

Assume an F-planar mapping $f\colon A_n \to \bar{A}_n$. Since f is a diffeomorphism, we can suppose local coordinate charts on M and $\bar{M}$, respectively, such that locally, $f\colon A_n \to \bar{A}_n$ maps points onto points with the same coordinates, and $\bar{M} = M$. We always suppose that ∇, $\bar{\nabla}$ and the affinors F, $\bar{F}$ are defined on M $(\equiv \bar{M})$. The following theorem holds.

Theorem 1. *An F-planar mapping f from A_n onto $\bar{A}_n$ preserves F-structures (i.e. $\bar{F} = a\,F + b\,\mathrm{Id}$, a, b are some functions on M), and is characterized by the following condition*

$$P(X,Y) = \psi(X)\cdot Y + \psi(Y)\cdot X + \varphi(X)\cdot FY + \varphi(Y)\cdot FX \tag{1}$$

for any vector fields X, Y, where $P = f^*\bar{\nabla} - \nabla$ is the deformation tensor field of f, ψ and φ are some linear forms on M.

This theorem was proved by Mikeš and Sinyukov [24] for finite dimension $n > 3$, a more concise proof of this theorem for $n > 3$ and also a proof for $n = 3$ was given by I. Hinterleitner and Mikeš [3], [25, p. 214].

We remind the following types of F-planar mappings from manifolds A_n with affine connection ∇ onto (pseudo-) Riemannian manifolds $\bar{V}_n$ with metric $\bar{g}$:

Definition 3 ([18], [25, p. 225])**.** (1) An F-planar mapping of a manifold $A_n = (M, \nabla)$ with affine connection onto a (pseudo-) Riemannian manifold $\bar{V}_n = (M, \bar{g})$ is called an F_1*-planar mapping* if the metric tensor $\bar{g}$ satisfies the condition

$$\bar{g}(X, FX) = 0\,, \quad \text{for all} \quad X\,. \tag{2}$$

(2) An F_1-planar mapping $A_n \to \bar{V}_n$ is called an F_2*-planar mapping* if the one-form ψ is gradient-like, i.e. $\psi(X) = \nabla_X \Psi$, where Ψ is a function on A_n.

If a manifold A_n admits F_2-planar mapping onto $\bar{V}_n$, then the following equations are satisfied (Mikeš [18], see [25, p. 230]):

$$\nabla_k a^{ij} = \lambda^i \delta^j_k + \lambda^j \delta^i_k + \xi^i F^j_k + \xi^j F^i_k\,, \tag{3}$$

where

$$a^{ij} = e^{2\psi} \bar{g}^{ij}\,, \qquad \lambda^i = -a^{i\alpha}\psi_\alpha\,, \qquad \xi^i = -a^{i\alpha}\varphi_\alpha\,, \tag{4}$$

where ψ_j, φ_i, F^h_i are components of ψ, φ, F and $\bar{g}^{ij}$ are components of the inverse matrix to the metric $\bar{g}$. From (2) and (4) follows that $a^{i\alpha}F^j_\alpha + a^{j\alpha}F^i_\alpha = 0$.

It is clear to see that if A_n is a (pseudo-) Riemannian manifold $V_n = (M, g)$ with metric tensor g, after lowering indices in (3), we obtain

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik} + \xi_i F_{jk} + \xi_j F_{ik}\,, \tag{5}$$

where $a_{ij} = a^{\alpha\beta} g_{i\alpha} g_{j\beta}$, $\lambda_i = g_{i\alpha}\lambda^\alpha$, $\xi_i = g_{i\alpha}\xi^\alpha$, $F_{ik} = g_{i\alpha}F^\alpha_k$. Evidently $a_{i\alpha}F^\alpha_j + a_{j\alpha}F^\alpha_i = 0$.

3. PQ^ε-projective Riemannian manifolds

3.1. **Definition of PQ^ε-projective Riemannian manifolds.** Let g and $\bar{g}$ be two Riemannian metrics on an n-dimensional manifold M. Consider the $(1,1)$-tensors P, Q which are satisfying the following conditions:

$$\begin{gathered} PQ = \varepsilon\,\mathrm{Id}\,, \quad g(X, PX) = 0\,, \quad \bar{g}(X, PX) = 0\,, \\ g(X, QX) = 0\,, \quad \bar{g}(X, QX) = 0\,, \end{gathered} \tag{6}$$

for all X and where $\varepsilon \neq 1, n+1$ is a real number. These conditions are written in a different way in [16] (formula (1)).

Definition 4 ([32])**.** The metrics $g, \bar{g}$ are called PQ^ε*-projective* if for the 1-form Φ the Levi-Civita connections ∇ and $\bar{\nabla}$ of g and $\bar{g}$ satisfy

$$(\bar{\nabla} - \nabla)_X Y = \Phi(X)Y + \Phi(Y)X - \Phi(PX)QY - \Phi(PY)QX \tag{7}$$

for all X, Y.

Remark 1. Two metrics g and $\bar{g}$ are denoted by the synonym PQ^{ε}-projective if they are PQ^{ε}-*projective equivalent*. On the other hand this notation can be seen from the point of view of mappings. Assume two Riemannian manifolds (M, g) and $(\bar{M}, \bar{g})$. A diffeomorphism $f\colon M \to \bar{M}$ allows to identify the manifolds M and $\bar{M}$. For this reason we can speak about PQ^{ε}-projective mappings (or more precisely diffeomorphisms) between (M, g) and $(\bar{M}, \bar{g})$, when equations (6) and (7) hold. In these formulas $\bar{g}$ and $\bar{\nabla}$ mean in fact the pullbacks $f^*\bar{g}$ and $f^*\bar{\nabla}$.

Comparing formulas (1) and (7) we make sure that PQ^{ε}-projective equivalence is a special case of the F-planar mapping between Riemannian manifolds (M, g) and $(M, \bar{g})$. Evidently, this is if $\psi \equiv \Phi$, $F \equiv Q$ and $\varphi(\cdot) = -\Phi(P(\cdot))$.

Moreover, it follows elementary from (7) that ψ is a gradient-like form, see [32], thus a PQ^{ε}-projective equivalence is a special case of an F_2-planar mapping.

Therefore the PQ^{ε}-projective equivalence formula (3), after lowering the indices i and j by the metric g, has the following form [32]:

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik} - \lambda_\alpha P_i^\alpha g_{j\beta} Q_k^\beta - \lambda_\alpha P_j^\alpha g_{i\beta} Q_k^\beta \,. \tag{8}$$

From conditions (4) and (6) we obtain $a(X, PX) = 0$ and $a(X, QX) = 0$ for all X, and equivalently in local form

$$a_{i\alpha} P_j^\alpha + a_{j\alpha} P_i^\alpha = 0 \qquad \text{and} \qquad a_{i\alpha} Q_j^\alpha + a_{j\alpha} Q_i^\alpha = 0 \,. \tag{9}$$

3.2. **New results about PQ^{ε}-projective Riemannian manifolds for $\varepsilon \neq 0$.** Next, we will study PQ^{ε}-projective mappings for $\varepsilon \neq 0$. From the condition $PQ = \varepsilon\,\mathrm{Id}$, it follows

$$P = \varepsilon\, Q^{-1} \,. \tag{10}$$

This implies that P depends on Q and ε. Moreover two conditions in (6) depend on the other ones, i.e. in the definition of PQ^{ε}-projective mappings we can restrict on the conditions $g(X, QX) = 0$, $\bar{g}(X, QX) = 0$, $PQ = \varepsilon\,\mathrm{Id}$. This fact implies the following lemma:

Lemma 1. *If Q satisfies the conditions $g(X, QX) = 0$ and $\bar{g}(X, QX) = 0$ for $\varepsilon \neq 0$, then we obtain $g(X, PX) = 0$ and $\bar{g}(X, PX) = 0$.*

Proof. We can write the first conditions (6) for g in the local form as $g_{i\alpha} Q_j^\alpha + g_{j\alpha} Q_i^\alpha = 0$. These equations we contract with $\bar{Q}_k^i \bar{Q}_l^j$, where $\bar{Q} = Q^{-1}$, after some calculations we obtain

$$g_{li} \bar{Q}_k^i + g_{kj} \bar{Q}_l^j = 0 \,,$$

i.e. $g(X, Q^{-1}X) = 0$ for all X. From that follows $g(X, PX) = 0$ for all X. Analogically it holds also for the metric $\bar{g}$. □

4. F_2^{ε}-projective mapping with $\varepsilon \neq 0$

Due to the above properties, from formula (7) and Lemma 1, we can simplify the Definition 4.

Let g and $\bar g$ be two (pseudo-) Riemannian metrics on an n-dimensional manifold M. Consider the regular $(1,1)$-tensors F which are satisfying the following conditions

$$g(X,FX)=0 \qquad \text{and} \qquad \bar g(X,FX)=0\,. \tag{11}$$

for all X.

Definition 5. The metrics g and $\bar g$ are called *F_2^ε-projective* if for a certain gradient-like form ψ the Levi-Civita connections ∇ and $\bar\nabla$ of g and $\bar g$ satisfy

$$(f^*\bar\nabla-\nabla)_X Y=\psi(X)Y+\psi(Y)X-\varepsilon\,\psi(F^{-1}X)FY-\varepsilon\,\psi(F^{-1}Y)FX, \tag{12}$$

for all vector fields X,Y and for all $x\in M$, ε is a non-zero constant.

From the discussion in section 3 we obtain the following proposition:

Proposition 1. *A PQ^ε-projective metrics can be understood as an F_2^ε-planar mapping with*

$$P=\varepsilon F^{-1} \qquad \textit{and} \qquad Q=F\,. \tag{13}$$

We can rewrite formula (12) in the form

$$\bar\Gamma^h_{ij}=\Gamma^h_{ij}+\psi_{(i}\delta^h_{j)}-\psi_\alpha P^\alpha_{(i}Q^h_{j)}\,. \tag{14}$$

Contracting h and j we get

$$\bar\Gamma^\alpha_{i\alpha}=\Gamma^\alpha_{i\alpha}+(n+1-\varepsilon)\cdot\psi_i\,.$$

Because $\varepsilon\neq n+1$ there is a function Ψ which is defined 1-form $\psi=\nabla\Psi$, i.e. $\psi_i=\partial\Psi/\partial x^i$, where $\Psi=\dfrac{1}{n+1-\varepsilon}\ \ln\sqrt{\left|\dfrac{\det\bar g}{\det g}\right|}$.

We obtain the following theorem:

Theorem 2. *If a (pseudo-) Riemannian manifold (M,g,F) with regular structure F, for which $F^2\neq\kappa\,\mathrm{Id}$ and $g(X,FX)=0$ for all X, admits an F_2^ε-projective mapping onto a (pseudo-) Riemannian manifold $(\bar M,\bar g)$, then the linear system of differential equations*

$$\nabla_k a_{ij}=\lambda_i g_{jk}+\lambda_j g_{ik}-\lambda_\alpha P^\alpha_i g_{j\beta}F^\beta_k-\lambda_\alpha P^\alpha_j g_{i\beta}F^\beta_k \tag{15}$$

and

$$a_{i\alpha}F^\alpha_j+a_{j\alpha}F^\alpha_i=0 \tag{16}$$

hold, where $P=\varepsilon F^{-1}$, $\lambda_i=a_{\alpha\beta}T^{\alpha\beta}_i$ and $T^{\alpha\beta}_i$ is a certain tensor obtained from g_{ij} and F^h_i.

Proof. We will study the fundamental equations of an F_2^ε-planar mapping $V_n\to\bar V_n$. From Proposition 1 follows, that formula (8) with help (13) has the form (15). From (14) and Lemma 1 we may deduce the validity of condition (16).

Now we covariantly differentiate (16) and obtain

$$\nabla_k a_{i\alpha}F^\alpha_j+\nabla_k a_{j\alpha}F^\alpha_i=\overset{1}{T}_{ijk}\,, \tag{17}$$

where $\overset{1}{T}_{ijk}=-a_{i\alpha}\nabla_k F^\alpha_j-a_{j\alpha}\nabla_k F^\alpha_i$.

Using formula (15), we obtain

$$\begin{aligned}\lambda_i g_{\alpha k} F^\alpha_j + \lambda_\alpha F^\alpha_j g_{ik} - \lambda_\beta P^\beta_i g_{\alpha\gamma} F^\alpha_j F^\gamma_k - \varepsilon\lambda_j g_{i\alpha} F^\alpha_k + \lambda_j g_{\alpha k} F^\alpha_i + \lambda_\alpha F^\alpha_i g_{jk}\\ - \lambda_\beta P^\beta_j g_{\alpha\gamma} F^\alpha_i F^\gamma_k - \varepsilon\lambda_i g_{j\alpha} F^\alpha_k = \overset{1}{T}_{ijk}\,.\end{aligned} \tag{18}$$

After some calculation we get

$$\begin{aligned}(\varepsilon+1)(g_{\alpha k} F^\alpha_j \lambda_i + g_{\alpha k} F^\alpha_i \lambda_j) + \lambda_\alpha F^\alpha_j g_{ik} + \lambda_\alpha F^\alpha_i g_{jk}\\ - \lambda_\alpha P^\alpha_i g_{\beta\gamma} F^\beta_j F^\gamma_k - \lambda_\alpha P^\alpha_j g_{\beta\gamma} F^\beta_i F^\gamma_k = \overset{1}{T}_{ijk}\,.\end{aligned} \tag{19}$$

By cyclic permutation of the indces i, j, k we obtain

$$\begin{aligned}\lambda_\alpha F^\alpha_j g_{ik} + \lambda_\alpha F^\alpha_i g_{jk} + \lambda_\alpha F^\alpha_k g_{ij} - \lambda_\alpha P^\alpha_i g_{\beta\gamma} F^\beta_j F^\gamma_k - \lambda_\alpha P^\alpha_j g_{\beta\gamma} F^\beta_i F^\gamma_k\\ - \lambda_\alpha P^\alpha_k g_{\beta\gamma} F^\beta_i F^\gamma_j = \overset{1}{T}_{ijk} + \overset{1}{T}_{jki} + \overset{1}{T}_{kij}\,.\end{aligned} \tag{20}$$

Next, we will subtract equations (19) and (20):

$$(\varepsilon+1)(g_{\alpha k} F^\alpha_j \lambda_i + g_{\alpha k} F^\alpha_i \lambda_j) - \lambda_\alpha F^\alpha_k g_{ij} + \lambda_\alpha P^\alpha_k g_{\beta\gamma} F^\beta_i F^\gamma_j = \overset{2}{T}_{ijk}\,, \tag{21}$$

where $\overset{2}{T}_{ijk} = -\overset{1}{T}_{jki} - \overset{1}{T}_{kij}$.

We write the homogeneous linear equation to equation (21)

$$g_{\alpha k} F^\alpha_j A_i + g_{\alpha k} F^\alpha_i A_j - B_k g_{ij} + C_k g_{\beta\gamma} F^\beta_i F^\gamma_j = 0\,, \tag{22}$$

where $A_i = (\varepsilon+1)\lambda_i$, $B_k = \lambda_\alpha F^\alpha_k$, $C_k = \lambda_\alpha P^\alpha_k$.

Now we prove that (22) has only trivial solution. From that follows that $\lambda_i = \overset{3}{T}$, i.e. is a linear combination of the tensor components a_{ij} with coefficients generated by g and F on V_n.

If $A_i \neq 0$, from (22) follows rank $\|g_{\alpha k} F^\alpha_j\| \leq 3$, in the other case $g_{\alpha k} F^\alpha_j$ we can decompose into 3 bivectors.

And because the tensors g and F are regular, follows that rank $\|g_{\alpha k} F^\alpha_j\| = n$. We suppose that $n \geq 4$. From that follows $A_i = 0$. Then equation (22) has the following form

$$- B_k g_{ij} + C_k g_{\beta\gamma} F^\beta_i F^\gamma_j = 0\,. \tag{23}$$

If B_k or $C_k \neq 0$:

$$g_{\beta\gamma} F^\beta_i F^\gamma_j = \rho g_{ij}\,, \tag{24}$$

where ρ is a function.

We multiply formula (24) by P^i_k. From that follows $F^2 = \kappa\,\mathrm{Id}$, where κ is a function, which is in contradiction with our assumption. For this reason in the formula (22) we suppose that $A_i = B_i = C_i = 0$. Therefore $\lambda_\alpha F^\alpha_k = \overset{3}{T}_k$, where $\overset{3}{T}_k$ is a tensor which is a linear combination of a_{ij} with coefficients generated by g and F. Let be $G = F^{-1}$, then $\lambda_i = \overset{3}{T}_k G^k_i$. This means $\lambda_i = a_{\alpha\beta} T^{\alpha\beta}_i$. □

5. F_2^ε-PLANAR MAPPINGS WITH THE $\bar{g} = k \cdot g$ CONDITION

From the properties of equations (15) and (16) follows a new result for F_2^ε-planar mappings, for which $F^2 \neq \kappa\,\mathrm{Id}$. These conditions we suppose for the whole studied (pseudo-) Riemannian manifolds (M, g, F). The system of equations (15) has the form of partial linear differential equations of Cauchy type in covariant derivative with respect to the unknown functions $a_{ij}(x)$. From the theory of this system (see [25, pp. 46–49]) follows that the system of equation (15) for initial condition at the point $x_0 \in M$

$$a_{ij}(x_0) = \overset{0}{a}_{ij} \tag{25}$$

has only one unique solution.

Due to this, the general solution of (15) depends on the real parameters which can be, for example, the conditions (25). Because a_{ij} is symmetric, conditions can not be more then $n(n+1)/2$. Moreover, condition (16) implies further reduction of the parameters.

The structure F at the point x_0 can be written in Jordan's form as $F_i^i = \lambda_i$, $F_i^{i+1} = \mu_i = 0, 1$ and the other components are vanishing. Because $\det F \neq 0$, all $\lambda_i \neq 0$. We do not exclude that λ_i are complex numbers (in this case the transformation equations are complex at the point x_0).

Substituting $i = j$ to equation (16), we obtain $a_{ii}\lambda_i + a_{ii+1}\mu_{i+1} = 0$ (formally $\mu_{n+1} \equiv 0$), i.e. the diagonal components a_{ii} depend on the other components.

This implies that the maximum number of the independent components of $\overset{0}{a}_{ij}$, which is not greater than $n(n-1)/2 - n$, i.e. $n(n-1)/2$ parameters.

Therefore this theorem is valid.

Theorem 3. *A set of (pseudo-) Riemannian manifolds (M, g, F), $\det F \neq 0$ and $F^2 \neq \kappa\,\mathrm{Id}$, on which some (pseudo-) Riemannian manifold admits an F_2^ε-projective mapping, depends on not more than $n(n-1)/2$ parameters.*

We have the following theorem.

Theorem 4. *Let $V_n = (M, g, F)$ and $\bar{V}_n = (M, \bar{g}, F)$ be (pseudo-) Riemannian manifolds with $F^2 \neq \kappa\,\mathrm{Id}$ and V_n, $\bar{V}_n$ have in F_2^ε-planar correspondence.*

If the condition $\bar{g} = k \cdot g$ is valid for $x_0 \in M$, then g and $\bar{g}$ are homothetic in M, i.e.

$$\bar{g}(x) = k \cdot g(x)\,, \tag{26}$$

for all $x \in M$, with $k = \mathrm{const}$.

Proof. In the assumption of Theorem 4, Theorem 2 is valid. Then equation (15) holds. For the initial condition (26) there is no more than one unique solution. On the other hand, a trivial solution of equations (15) is $\bar{g} = k \cdot g$, and it satisfies the initial condition (26). The given mapping is homothetic. □

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References

[1] Chudá, H., Shiha, M., *Conformal holomorphically projective mappings satisfying a certain initial condition*, Miskolc Math. Notes **14** (2) (2013), 569–574.

[2] Hinterleitner, I., *On holomorphically projective mappings of e-Kähler manifolds*, Arch. Mat. (Brno) **48** (2012), 333–338.

[3] Hinterleitner, I., Mikeš, J., *On F-planar mappings of spaces with affine connections*, Note Mat. **27** (2007), 111–118.

[4] Hinterleitner, I., Mikeš, J., *Fundamental equations of geodesic mappings and their generalizations*, J. Math. Sci. **174** (5) (2011), 537–554.

[5] Hinterleitner, I., Mikeš, J., *Projective equivalence and spaces with equi-affine connection*, J. Math. Sci. **177** (2011), 546–550, transl. from Fundam. Prikl. Mat. **16** (2010), 47–54.

[6] Hinterleitner, I., Mikeš, J., *Geodesic Mappings and Einstein Spaces*, Geometric Methods in Physics, Birkhäuser Basel, 2013, `arXiv:1201.2827v1[math.DG],2012`.

[7] Hinterleitner, I., Mikeš, J., *On holomorphically projective mappings from manifolds with equiaffine connection onto Kähler manifolds*, Arch. Math. (Brno) **49** (5) (2013), 295–302.

[8] Hinterleitner, I., Mikeš, J., Stránská, J., *Infinitesimal F-planar transformations*, Russ. Math. **52** (2008), 13–18, transl. from Izv. Vyssh. Uchebn. Zaved., Mat. (2008), 16–22.

[9] Hrdina, J., *Almost complex projective structures and their morphisms*, Arch. Mat. (Brno) **45** (2009), 255–264.

[10] Hrdina, J., Slovák, J., *Generalized planar curves and quaternionic geometry*, Ann. Global Anal. Geom. **29** (4) (2006), 349–360.

[11] Hrdina, J., Slovák, J., *Morphisms of almost product projective geometries*, Proc. 10th Int. Conf. on Diff. Geom. and its Appl., DGA 2007, Olomouc. Hackensack, NJ: World Sci., 2008, pp. 253–261.

[12] Hrdina, J., Vašík, P., *Generalized geodesics on almost Cliffordian geometries*, Balkan J. Geom. Appl. **17** (1) (2012), 41–48.

[13] Jukl, M., Juklová, L., Mikeš, J., *Some results on traceless decomposition of tensors*, J. Math. Sci. (New York) **174** (2011), 627–640.

[14] Lami, R.J.K. al, Škodová, M., Mikeš, J., *On holomorphically projective mappings from equiaffine generally recurrent spaces onto Kählerian spaces*, Arch. Math. (Brno) **42** (5) (2006), 291–299.

[15] Levi-Civita, T., *Sulle transformationi delle equazioni dinamiche*, Ann. Mat. Milano **24** Ser. 2 (1886), 255–300.

[16] Matveev, V., Rosemann, S., *Two remarks on PQ^{ε}-projectivity of Riemanninan metrics*, Glasgow Math. J. **55** (1) (2013), 131–138.

[17] Mikeš, J., *On holomorphically projective mappings of Kählerian spaces*, Ukr. Geom. Sb., Kharkov **23** (1980), 90–98.

[18] Mikeš, J., *Special F-planar mappings of affinely connected spaces onto Riemannian spaces*, Mosc. Univ. Math. Bull. **49** (1994), 15–21, translation from Vestn. Mosk. Univ., Ser. 1 (1994), 18–24.

[19] Mikeš, J., *Holomorphically projective mappings and their generalizations*, J. Math. Sci. (New York) **89** (1998), 1334–1353.

[20] Mikeš, J., Chudá, H., Hinterleitner, I., *Conformal holomorphically projective mappings of almost Hermitian manifolds with a certain initial condition*, Int. J. Geom. Methods in Modern Phys. **11** (5) (2014), Article Number 1450044.

[21] Mikeš, J., Pokorná, O., *On holomorphically projective mappings onto almost Hermitian spaces*, 8th Int. Conf. Opava, 2001, pp. 43–48.

[22] Mikeš, J., Pokorná, O., *On holomorphically projective mappings onto Kählerian spaces*, Rend. Circ. Mat. Palermo (2) Suppl. **69** (2002), 181–186.

[23] Mikeš, J., Shiha, M., Vanžurová, A., *Invariant objects by holomorphically projective mappings of Kähler space*, 8th Int. Conf. APLIMAT 2009: 8th Int. Conf. Proc., 2009, pp. 439–444.

[24] Mikeš, J., Sinyukov, N.S., *On quasiplanar mappings of space of affine connection*, Sov. Math. (1983), 63–70, translation from Izv. Vyssh. Uchebn. Zaved., Mat. (1983), 55–61.

[25] Mikeš, J., Vanžurová, A., Hinterleitner, I., *Geodesic Mappings and some Generalizations*, Palacky University Press, Olomouc, 2009.

[26] Otsuki, T., Tashiro, Y., *On curves in Kaehlerian spaces*, Math. J. Okayama Univ. **4** (1954), 57–78.

[27] Petrov, A.Z., *Simulation of physical fields*, Gravitatsiya i Teor. Otnositenosti **4–5** (1968), 7–21.

[28] Prvanović, M., *Holomorphically projective transformations in a locally product space*, Math. Balkanica (N.S.) **1** (1971), 195–213.

[29] Sinyukov, N.S., *Geodesic Mappings of Riemannian Spaces*, Moscow: Nauka, 1979, 256pp.

[30] Škodová, M., Mikeš, J., Pokorná, O., *On holomorphically projective mappings from equiaffine symmetric and recurrent spaces onto Kählerian spaces*, Rend. Circ. Mat. Palermo (2) Suppl., vol. 75, 2005, pp. 309–316.

[31] Stanković, M.S., Zlatanović, M.L., Velimirović, L.S., *Equitorsion holomorphically projective mappings of generalized Kählerian space of the first kind*, Czechoslovak Math. J. **60** (2010), 635–653.

[32] Topalov, P., *Geodesic compatibility and integrability of geodesic flows*, J. Math. Phys. **44** (2) (2003), 913–929.

[33] Yano, K., *Differential geometry on complex and almost complex spaces*, vol. XII, Pergamon Press, Oxford-London-New York-Paris-Frankfurt, 1965, 323pp.

Irena Hinterleitner,
Brno University of Technology, Faculty of Civil Engineering,
Department of Mathematics,
Žižkova 17, 602 00 Brno, Czech Republic
E-mail: `hinterleitner.irena@seznam.cz`

Josef Mikeš,
Department of Algebra and Geometry, Palacky University,
17. listopadu 12, 77146 Olomouc, Czech Republic
E-mail: `josef.mikes@upol.cz`

Patrik Peška,
Department of Algebra and Geometry, Palacky University,
17. listopadu 12, 77146 Olomouc, Czech Republic
E-mail: `patrik_peska@seznam.cz`