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# A MASCHKE TYPE THEOREM FOR RELATIVE HOM-HOPF MODULES

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*Abstract.* Let $(H,\alpha)$ be a monoidal Hom-Hopf algebra and $(A,\beta)$ a right $(H,\alpha)$-Hom-comodule algebra. We first introduce the notion of a relative Hom-Hopf module and prove that the functor $F$ from the category of relative Hom-Hopf modules to the category of right $(A,\beta)$-Hom-modules has a right adjoint. Furthermore, we prove a Maschke type theorem for the category of relative Hom-Hopf modules. In fact, we give necessary and sufficient conditions for the functor that forgets the $(H,\alpha)$-coaction to be separable. This leads to a generalized notion of integrals.

*Keywords*: monoidal Hom-Hopf algebra; separable functors; Maschke type theorem; total integral; relative Hom-Hopf module

*MSC 2010*: 16T05

## Introduction

The present paper investigates variations on the theme of Hom-algebras, a topic which has recently received much attention from various researchers. The study of Hom-associative algebras originates with the work by Hartwig, Larsson and Silvestrov in the Lie case [9], where a notion of Hom-Lie algebra was introduced in the context of studying deformations of Witt and Virasoro algebras. Later, it was extended to the associative case by Makhlouf and Silverstrov in [10]–[11]. Now the associativity is replaced by Hom-associativity $\alpha(a)(bc)=(ab)\alpha(c)$. Hom-coassociativity for a Hom-coalgebra can be considered in a similar way, see [11]. Caenepeel and Goywaerts [1]

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studied Hom-structures from the point of view of monoidal categories. This leads to the natural definition of monoidal Hom-algebras, Hom-coalgebras, etc. They constructed a symmetric monoidal category, and then introduced monoidal Hom-algebras, Hom-coalgebras, etc. as algebras, coalgebras, etc. in this monoidal category.

The notion of a relative $(H, B)$-Hopf module, where $H$ is a Hopf algebra over a field $k$ and $B$ is a right coideal subalgebra of $H$, was introduced and studied by Takeuchi in [12]. Later, in [5] (see also [4]), Doi noted that the notion of an $(H, B)$-Hopf module works well if $B$ is a right $H$-comodule algebra, Using this module, he proved that the existence of a total integral $\phi\colon H \to B$ is equivalent to $B$ being a relative injective $H$-comodule, and it is also equivalent to any $(H, B)$-Hopf module $M$ being a relative injective $H$-comodule in [3]. Also, in [3], using a commutative assumption for $H$, he deduced a version of the Maschke type theorem for $(H, B)$-Hopf modules which states that every exact sequence of $(H, B)$-Hopf modules which splits $B$-linearly, also splits $(H, B)$-linearly. Afterwards, Doi proved in [3] that the commutative condition can be removed and replaced by some technical conditions involving the center of $B$. Caenepeel et al. [2] proved a Maschke type theorem for the category of relative Hopf modules. In fact, they gave necessary and sufficient conditions for the functor that forgets the $H$-coaction to be separable. This leads to a generalized notion of integrals of Doi [3].

In this paper we study the generalization of the previous results to the Hom-Hopf algebras. In Section 2, we introduce the notion of a relative Hom-Hopf module and prove that the functor $F$ from the category of relative Hom-Hopf modules to the category of right $(A, \beta)$-Hom-modules has a right adjoint (see Proposition 2.3). In Section 3, we introduce the notion of total integrals for Hom-comodule algebras, which is an effective tool for investigating properties of relative Hom-Hopf modules. As an important application, we investigate the injectivity of relative Hom-Hopf modules (see Proposition 3.3), which generalizes the main result in [5]. In Section 4, we obtain the main result of this paper. We give necessary and sufficient conditions for the functor that forgets the $(H, \alpha)$-coaction to be separable (see Theorem 4.2), and we prove a Maschke type theorem for the category of relative Hom-Hopf modules as an application. In fact, let $(A, \beta)$ be a right $(H, \alpha)$-Hom-comodule algebra with a total integral $\phi\colon (H, \alpha) \to (A, \beta)$. If $\phi\colon (H, \alpha) \to (Z(A), \beta)$ (the center of $(A, \beta)$) is a multiplication map, then every short exact sequence of relative Hom-Hopf modules

$$0 \longrightarrow (M, \mu) \xrightarrow{f} (N, \nu) \xrightarrow{g} (P, \pi) \longrightarrow 0$$

which splits as a sequence of $(A, \beta)$-Hom-modules also splits as a sequence of relative Hom-Hopf modules.

## 1. Preliminaries

Throughout this paper we work over a commutative ring $k$ we recall from [1] some information about Hom-structures which are needed in what follows.

Let $\mathcal{C}$ be a category. We introduce a new category $\widetilde{\mathscr{H}}(\mathcal{C})$ as follows: the objects are couples $(M,\mu)$, with $M\in\mathcal{C}$ and $\mu\in \mathrm{Aut}_{\mathcal{C}}(M)$. A morphism $f\colon (M,\mu)\to(N,\nu)$ is a morphism $f\colon M\to N$ in $\mathcal{C}$ such that $\nu\circ f=f\circ\mu$.

Let $\mathscr{M}_k$ denote the category of $k$-modules. $\mathscr{H}(\mathscr{M}_k)$ will be called the Hom-category associated with $\mathscr{M}_k$. If $(M,\mu)\in\mathscr{M}_k$, then $\mu\colon M\to M$ is obviously a morphism in $\mathscr{H}(\mathscr{M}_k)$. It is easy to show that $\widetilde{\mathscr{H}}(\mathscr{M}_k)=(\mathscr{H}(\mathscr{M}_k),\otimes,(I,I),\tilde{a},\tilde{l},\tilde{r}))$ is a monoidal category by Proposition 1.1 in [1]: the tensor product of $(M,\mu)$ and $(N,\nu)$ in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ is given by the formula $(M,\mu)\otimes(N,\nu)=(M\otimes N,\mu\otimes\nu)$.

Assume that $(M,\mu),(N,\nu),(P,\pi)\in\widetilde{\mathscr{H}}(\mathscr{M}_k)$. The associativity and unit constraints are given by the formulas

$$\tilde{a}_{M,N,P}((m\otimes n)\otimes p)=\mu(m)\otimes(n\otimes\pi^{-1}(p)),$$
$$\tilde{l}_M(x\otimes m)=\tilde{r}_M(m\otimes x)=x\mu(m).$$

An algebra in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ will be called a monoidal Hom-algebra.

**Definition 1.1.** A monoidal Hom-algebra is an object $(A,\alpha)\in\widetilde{\mathscr{H}}(\mathscr{M}_k)$ together with a $k$-linear map $m_A\colon A\otimes A\to A$ and an element $1_A\in A$ such that

$$\alpha(ab)=\alpha(a)\alpha(b);\quad \alpha(1_A)=1_A,$$
$$\alpha(a)(bc)=(ab)\alpha(c);\quad a1_A=1_Aa=\alpha(a),$$

for all $a,b,c\in A$. Here we use the notation $m_A(a\otimes b)=ab$.

**Definition 1.2.** A monoidal Hom-coalgebra is an object $(C,\gamma)\in\widetilde{\mathscr{H}}(\mathscr{M}_k)$ together with $k$-linear maps $\Delta\colon C\to C\otimes C$, $\Delta(c)=c_{(1)}\otimes c_{(2)}$ (summation implicitly understood) and $\gamma\colon C\to C$ such that

$$\Delta(\gamma(c))=\gamma(c_{(1)})\otimes\gamma(c_{(2)});\ \varepsilon(\gamma(c))=\varepsilon(c),$$

and

$$\gamma^{-1}(c_{(1)})\otimes c_{(2)(1)}\otimes c_{(2)(2)}=c_{(1)(1)}\otimes c_{(1)(2)}\otimes\gamma^{-1}(c_{(2)}),$$
$$\varepsilon(c_{(1)})c_{(2)}=\varepsilon(c_{(2)})c_{(1)}=\gamma^{-1}(c)$$

for all $c\in C$.

**Definition 1.3.** A monoidal Hom-bialgebra $H = (H, \alpha, m, \eta, \Delta, \varepsilon)$ is a bialgebra in the symmetric monoidal category $\widetilde{\mathscr{H}}(\mathscr{M}_k)$. This means that $(H, \alpha, m, \eta)$ is a Hom-algebra, $(H, \Delta, \alpha)$ is a Hom-coalgebra and that $\Delta$ and $\varepsilon$ are morphisms of Hom-algebras, that is,

$$\Delta(ab) = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)};\ \Delta(1_H) = 1_H \otimes 1_H,$$
$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b),\ \varepsilon(1_H) = 1_H.$$

**Definition 1.4.** A monoidal Hom-Hopf algebra is a monoidal Hom-bialgebra $(H, \alpha)$ together with a linear map $S\colon H \to H$ in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ such that

$$S * I = I * S = \eta\varepsilon, S\alpha = \alpha S.$$

**Definition 1.5.** Let $(A, \alpha)$ be a monoidal Hom-algebra. A right $(A, \alpha)$-Hom-module is an object $(M, \mu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ consisting of a $k$-module and a linear map $\mu\colon M \to M$ together with a morphism $\psi\colon M \otimes A \to M, \psi(m \cdot a) = m \cdot a$ in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ such that

$$(m \cdot a) \cdot \alpha(b) = \mu(m) \cdot (ab);\ m \cdot 1_A = \mu(m)$$

for all $a \in A$ and $m \in M$. The fact that $\psi \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ means that

$$\mu(m \cdot a) = \mu(m) \cdot \alpha(a).$$

A morphism $f\colon (M, \mu) \to (N, \nu)$ in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ is called right $A$-linear if it preserves the $A$-action, that is, $f(m \cdot a) = f(m) \cdot a$. $\widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ will denote the category of right $(A, \alpha)$-Hom-modules and $A$-linear morphisms.

**Definition 1.6.** Let $(C, \gamma)$ be a monoidal Hom-coalgebra. A right $(C, \gamma)$-Hom-comodule is an object $(M, \mu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ together with a $k$-linear map $\varrho_M\colon M \to M \otimes C$ notation $\varrho_M(m) = m_{[0]} \otimes m_{[1]}$ in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ such that

$$m_{[0][0]} \otimes (m_{[0][1]} \otimes \gamma^{-1}(m_{[1]})) = \mu^{-1}(m_{[0]}) \otimes \Delta_C(m_{[1]});\ m_{[0]}\varepsilon(m_{[1]}) = \mu^{-1}(m)$$

for all $m \in M$. The fact that $\varrho_M \in \widetilde{\mathscr{H}}(\mathscr{M}_k)$ means that

$$\varrho_M(\mu(m)) = \mu(m_{[0]}) \otimes \gamma(m_{[1]}).$$

Morphisms of right $(C, \gamma)$-Hom-comodule are defined in the obvious way. The category of right $(C, \gamma)$-Hom-comodules will be denoted by $\widetilde{\mathscr{H}}(\mathscr{M}_k)^C$.

## 2. Adjoint functor

**Definition 2.1.** Let $(H,\alpha)$ be a monoidal Hom-Hopf algebra. A monoidal Hom-algebra $(A,\beta)$ is called a right $(H,\alpha)$-Hom-comodule algebra if $(A,\beta)$ is a right $(H,\alpha)$ Hom-comodule with coaction $\varrho_A\colon A\to A\otimes H$, $\varrho_A(a)=a_{[0]}\otimes a_{[1]}$ such that the conditions

$$\varrho_A(ab)=a_{[0]}b_{[0]}\otimes a_{[1]}b_{[1]},$$
$$\varrho_A(1_A)=1_A\otimes 1_H$$

are satisfied for all $a,b\in A$.

**Definition 2.2.** Let $(H,\alpha)$ be a monoidal Hom-Hopf algebra and $(A,\beta)$ a right $(H,\alpha)$-Hom-comodule algebra. A relative Hom-Hopf module $(M,\mu)$ is a right $(A,\beta)$-Hom-module which is also a right $(H,\alpha)$-Hom-comodule with the coaction structure $\varrho_M\colon M\to M\otimes H$ defined by $\varrho_M(m)=m_{[0]}\otimes m_{[1]}$ such that the following compatible condition holds: for all $m\in M$ and $a\in A$,

$$\varrho_M(ma)=m_{[0]}\cdot a_{[0]}\otimes m_{[1]}a_{[1]}.$$

A morphism between two right relative Hom-Hopf modules is a $k$-linear map which is a morphism in the categories $\widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ and $\widetilde{\mathscr{H}}(\mathscr{M}_k)^H$ at the same time. $\widetilde{\mathscr{H}}(\mathscr{M}_k)_A^H$ will denote the category of right relative Hom-Hopf modules and morphisms between them.

**Proposition 2.3.** *The forgetful functor $F\colon\widetilde{\mathscr{H}}(\mathscr{M}_k)_A^H\to\widetilde{\mathscr{H}}(\mathscr{M}_k)_A$ has a right adjoint $G\colon\widetilde{\mathscr{H}}(\mathscr{M}_k)_A\to\widetilde{\mathscr{H}}(\mathscr{M}_k)_A^H$. $G$ is defined by*

$$G(M)=M\otimes H,$$

*with structure maps*

$$(m\otimes h)\cdot a=m\cdot a_{[0]}\otimes ha_{[1]},$$
$$\varrho_{G(M)}(m\otimes h)=(\mu^{-1}(m)\otimes h_{(1)})\otimes\alpha(h_{(2)})$$

*for all $a\in A$ and $m\in M$, $h\in H$.*

Proof. Let us first show that $G(M)$ is an object of $\widetilde{\mathscr{H}}(\mathscr{M}_k)_A^H$. It is routine to check that $G(M)$ is a right $(H,\alpha)$-Hom-comodule and a right $(A,\beta)$-Hom-module.

Now we only check the compatibility condition, for all $a \in A$. Indeed,

$$\begin{aligned}\varrho_{G(M)}((m \otimes h) \cdot a) &= \varrho_{G(M)}(m \cdot a_{[0]} \otimes h a_{[1]}) \\ &= \mu^{-1}(m) \cdot \beta^{-1}(a_{[0]}) \otimes h_{(1)} a_{[1](1)} \otimes \alpha(h_{(2)} a_{[1](2)}) \\ &= \mu^{-1}(m) \cdot a_{[0][0]} \otimes h_{(1)} a_{[0][1]} \otimes \alpha(h_{(2)}) a_{[1]} \\ &= (m \otimes h)_{[0]} \cdot a_{[0]} \otimes (m \otimes h)_{(1)} a_{[1]} \\ &= \varrho(m \otimes c) \cdot a.\end{aligned}$$

This is exactly what we have to show.

For an $A$-linear map $\varphi\colon (M, \mu) \to (N, \nu)$, we put

$$G(\varphi) = \varphi \otimes \mathrm{id}_H\colon M \otimes H \to N \otimes H.$$

Standard computations show that $G(\varphi)$ is a morphism of right $(A, \beta)$-Hom-modules and right $(H, \alpha)$-Hom-comodules. Let us describe the unit $\eta$ and the counit $\delta$ of the adjunction. The unit is described by the coaction: for $(M, \mu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)_A^H$, we define $\eta_M\colon M \to M \otimes H$ as follows: for all $m \in M$,

$$\eta_M(m) = m_{[0]} \otimes m_{[1]}.$$

We can check that $\eta_M \in \widetilde{\mathscr{H}}(\mathscr{M}_k)_A^H$. For any $(N, \nu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)_A$, we define $\delta_N\colon N \otimes H \to N$ for all $n \in N$ and $h \in H$ by

$$\delta_N(n \otimes h) = \varepsilon(h) n;$$

we can check that $\delta_N$ is $(A, \beta)$-linear. It is easy to check that $\eta_M \in \widetilde{\mathscr{H}}(\mathscr{M}_k)_A^H$. We can check that $\eta$ and $\delta$ defined above are all natural transformations and satisfy

$$\begin{aligned}G(\delta_N) \circ \eta_{G(N)} &= I_{G(N)}, \\ \delta_{F(M)} \circ F(\eta_M) &= I_{F(M)}\end{aligned}$$

for all $M \in \widetilde{\mathscr{H}}(\mathscr{M}_k)_A^H$ and $N \in \widetilde{\mathscr{H}}(\mathscr{M}_k)_A$. □

## 3. Structure type theorem and injective type properties for relative Hom-Hopf modules

**Definition 3.1.** Let $(H, \alpha)$ be a monoidal Hom-Hopf algebra and $(A, \beta)$ a right $(H, \alpha)$-Hom-comodule algebra. The map $\phi\colon (H, \alpha) \to (A, \beta)$ is called a total integral such that the following conditions are satisfied:

$$\varrho_A \phi = (\phi \otimes \mathrm{id}_H) \Delta_H, \quad \phi\alpha = \beta\phi, \quad \phi(1_H) = 1_A.$$

**Lemma 3.2.** *Let $(H,\alpha)$ be a monoidal Hom-Hopf algebra and $(A,\beta)$ a right $(H,\alpha)$-Hom-comodule algebra with a total integral $\phi\colon (H,\alpha)\to(A,\beta)$ and $(M,\mu)\in\widetilde{\mathscr{H}}(\mathscr{M}_k)_A^H$,*

$$\lambda_M\colon\ M\otimes H\to M,\quad m\otimes h\mapsto \mu^{-1}(m_{[0]})\cdot\phi(S(m_{[1]})\alpha(h)).$$

*Then the following assertions hold:*

(1) *$\lambda_M\varrho_M=\mathrm{id}_M$;*

(2) *$\lambda_M$ is a morphism of right $(H,\alpha)$-Hom-comodules, and the right $(H,\alpha)$-Hom-coaction on $M\otimes H$ is given by $\varrho(m\otimes h)=(\mu(m)\otimes h_{(1)})\otimes\alpha^{-1}(h_{(2)})$ for any $m\in M$ and $h\in H$;*

(3) *if $\phi\colon (H,\alpha)\to(Z(A),\beta)$ (the center of $A$) is a multiplication map, then $\lambda_M$ is a morphism in $\widetilde{\mathscr{H}}(\mathscr{M}_k)_A^H$.*

Proof. (1) For any $m\in M$, we have

$$\begin{aligned}\lambda_M\varrho_M(m)&=\lambda_M(m_{[0]}\otimes m_{[1]})=\mu^{-1}(m_{[0][0]})\cdot\phi(S(m_{[0][1]})\alpha(m_{[1]}))\\&=m_{[0]}\cdot\phi(S(m_{[1](1)})m_{[1](2)})=m_{[0]}\cdot\phi(\varepsilon(m_{[1]}))=\mu^{-1}(m)\cdot 1_A=m.\end{aligned}$$

(2) For any $m\in M$ and $h\in H$, we have

$$\begin{aligned}\varrho_M\lambda_M(m\otimes h)&=\varrho_M(\mu^{-1}(m_{[0]})\cdot\phi(S(m_{[1]})\alpha(h)))\\&=\mu^{-1}(m_{[0][0]})\cdot\phi(S(m_{[1](2)})\alpha(h_{(1)}))\otimes\alpha^{-1}(m_{[0][1]})(S(m_{[1](1)})\alpha(h_{(2)}))\\&=\mu^{-2}(m_{[0]})\cdot\phi(\alpha(S(m_{[1](2)(2)}))\alpha(h_{(1)}))\otimes\alpha^{-1}(m_{[1](1)})\alpha(S(m_{[1](2)(1)}))\alpha(h_{(2)})\\&=\mu^{-2}(m_{[0]})\cdot\phi(S(m_{[1](2)})\alpha(h_{(1)}))\otimes m_{[1](1)(1)}(\alpha(S(m_{[1](1)(2)}))\alpha(h_{(2)}))\\&=\mu^{-2}(m_{[0]})\cdot\phi(S(m_{[1](2)})\alpha(h_{(1)}))\otimes(\alpha(m_{[1](1)(1)})\alpha(S(m_{[1](1)(2)})))h_{(2)}\\&=\mu^{-2}(m_{[0]})\cdot\phi(\alpha^{-1}(S(m_{[1]}))\alpha(h_{(1)}))\otimes\alpha(h_{(2)})\\&=(\lambda_M\otimes\mathrm{id}_H)((\mu^{-1}(m)\otimes h_{(1)})\otimes\alpha(h_{(2)}))\\&=(\lambda_M\otimes\mathrm{id}_H)\varrho_{M\otimes H}(m\otimes h).\end{aligned}$$

(3) For any $m\in M$, $h\in H$ and $b\in A$, we have

$$\begin{aligned}\lambda_M((m\otimes h)\cdot b)&=\lambda_M(m\cdot b_{[0]}\otimes hb_{(1)})\\&=\mu^{-1}(m_{[0]}\cdot b_{[0][0]})\cdot\phi(S(m_{[1]}b_{[0][1]})\alpha(hb_{[1]}))\\&=\mu^{-1}(m_{[0]}\cdot b_{[0][0]})\cdot\phi(S(m_{[1]})S(b_{[0][1]})\alpha(hb_{[1]}))\\&=\mu^{-1}(m_{[0]}\cdot b_{[0][0]})\cdot\phi(\alpha(S(m_{[1]})[S(b_{[0][1]})hb_{[1]}]))\\&=\mu^{-1}(m_{[0]}\cdot b_{[0][0]})\cdot\phi(\alpha(S(m_{[1]})[S(b_{[0][1]})(b_{[1]}h)]))\end{aligned}$$

$$
\begin{aligned}
&= \mu^{-1}(m_{[0]} \cdot b_{[0][0]}) \cdot \phi(\alpha(S(m_{[1]})[(\alpha^{-1}(S(b_{[0][1]}))b_{[1]})\alpha(h)])) \\
&= (\mu^{-1}(m_{[0]}) \cdot b_{[0]}) \cdot \phi(\alpha(S(m_{[1]})[(\alpha^{-1}(S(b_{[1](1)}))\alpha^{-1}(b_{[1](2)}))\alpha(h)])) \\
&= (\mu^{-1}(m_{[0]}) \cdot \beta^{-1}(b)) \cdot \phi(\alpha(S(m_{[1]})\alpha^{2}(h))) \\
&= m_{[0]} \cdot (\beta^{-1}(b)\phi(S(m_{[1]})\alpha(h))) \\
&= m_{[0]} \cdot (\phi(\alpha^{-4}(S(m_{[1]}))\alpha^{-3}(h))\beta^{-1}(b)) \\
&= (\mu^{-1}(m_{[0]}) \cdot \phi(S(m_{[1]})\alpha(h))) \cdot b \\
&= \lambda_M(m \otimes h) \cdot b.
\end{aligned}
$$

□

**Proposition 3.3.** *Let $(H,\alpha)$ be a monoidal Hom-Hopf algebra and $(A,\beta)$ a right $(H,\alpha)$-Hom comodule algebra with a total integral $\phi\colon (H,\alpha) \to (A,\beta)$. Then $(M,\mu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)_A^H$ is injective as a right $(H,\alpha)$-Hom-comodule.*

If $H$ is a Hopf algebra, then we obtain the main result of [5], Theorem 1.

**Corollary 3.4.** *Let $H$ be a Hopf algebra and $A$ a right $H$-comodule algebra. If there is a right $H$-comodule map $\phi\colon (H,\alpha) \to (A,\beta)$ such that $\phi(1_H) = 1_A$, then every relative $(H,A)$-Hopf-module is injective as a right $H$-comodule.*

Let $M$ be a relative Hom-Hopf module, and let

$$M_0 = \{m \in M;\ \varrho_M(m) = \mu^{-1}(m) \otimes 1_H\}$$

be an invariant subspace of $M$ and a right $(C,\beta)$-Hom-module, where

$$C = \{b \in A;\ \varrho_A(b) = \beta^{-1}(b) \otimes 1_H\}$$

is a subalgebra of $A$.

**Proposition 3.5.** *Let $(H,\alpha)$ be a monoidal Hom-Hopf algebra and $(A,\beta)$ a right $(H,\alpha)$-Hom-comodule algebra with a total integral $\phi\colon (H,\alpha) \to (A,\beta)$ and $(M,\mu) \in \widetilde{\mathscr{H}}(\mathscr{M}_k)_A^H$. Assume that $\phi$ is a multiplication map and let*

$$\tau_M\colon\ (M,\mu) \to (M,\mu)$$

*be the trace map defined by*

$$m \mapsto m_{[0]} \cdot \phi(S(m_{[1]})).$$

*Then the following assertions hold:*

(1) *$\tau_M(m) \in M_0$ and $\tau|_{M_0} = \mathrm{id}$;*
(2) *$\tau_A\colon (A,\beta) \to (C,\beta)$ defined by $b \mapsto b_{[0]}\phi(S(b_{[1]}))$ is a morphism of left $(C,\beta)$-Hom-modules, so that $(C,\beta)$ is a direct summand of $(A,\beta)$ as a sum of left $(C,\beta)$-Hom-modules;*
(3) *if $\operatorname{Im}\phi \subseteq Z(A)$, then $\tau_M\colon (M,\mu) \to (M,\mu)$ is a morphism of right $(C,\beta)$-Hom-modules.*

*The exact sequence*

$$(M,\mu) \xrightarrow{\tau_M} (M_0,\mu) \longrightarrow 0$$

*thus obtained splits as a sequence of right $(C,\beta)$-Hom-modules.*

Proof. (1) For any $m \in M$, we have

$$\begin{aligned}\varrho(\tau_M(m)) &= \varrho(m_{[0]}\phi(S(m_{[1]})))\\ &= m_{[0][0]}\phi(S(m_{[1](2)})) \otimes m_{[0][1]}\phi(S(m_{[1](1)}))\\ &= \mu^{-1}(m_{[0]})\phi(\alpha(S(m_{[1](2)(2)}))) \otimes m_{[1](1)}\phi(\alpha(S(m_{[1](2)(1)})))\\ &= \mu^{-1}(m_{[0]})\phi(S(m_{[1](2)})) \otimes \alpha(m_{[1](1)(1)})\phi(\alpha(S(m_{[1](1)(2)})))\\ &= \mu^{-1}(m_{[0]})\phi(\alpha^{-1}(S(m_{[1]}))) \otimes 1_H\\ &= \mu^{-1}(\tau_M(m)) \otimes 1_H.\end{aligned}$$

For any $n \in M_0$,

$$\tau_M(n) = n_{[0]}\phi(S(n_{[1]})) = \mu^{-1}(n)1_A = n.$$

(2) For any $c \in C$ and $a \in A$,

$$\begin{aligned}\tau_A(ca) &= (c_{[0]}a_{[0]})\phi(S(c_{[1]}a_{[1]})) = (\beta^{-1}(c)a_{[0]})\phi(\alpha(S(a_{[1]})))\\ &= c(a_{[0]} \cdot \phi(S(a_{[1]}))) = c\tau_A(a),\end{aligned}$$

thus, $\tau_A\colon (A,\beta) \to (C,\beta)$ is a morphism of left $(C,\beta)$-Hom-modules, and by (1), $(C,\beta)$ is a direct summand of $(A,\beta)$ as a sum of left $(C,\beta)$-Hom-modules.

(3) For any $c \in C$ and $m \in M$,

$$\begin{aligned}\tau_M(m \cdot c) &= (m_{[0]} \cdot c_{[0]})\phi(S(m_{[1]}c_{[1]})) = (m_{[0]} \cdot \beta^{-1}(c))\phi(\alpha(S(m_{[1]})))\\ &= \mu(m_{[0]}) \cdot (\beta^{-1}(c))\phi(S(m_{[1]})) = \mu(m_{[0]}) \cdot \phi(S(m_{[1]}))\beta^{-1}(c)\\ &= (m_{[0]} \cdot \phi(S(m_{[1]}))) \cdot c = \tau_M(m) \cdot c.\end{aligned}$$

Thus, $\tau_M$ is a morphism of right $(C,\beta)$-Hom-modules, and by (1), the exact sequence

$$(M,\mu) \xrightarrow{\tau_M} (M_0,\mu) \longrightarrow 0$$

thus obtained splits as a sequence of right $(C,\beta)$-Hom-modules. □

## 4. A Maschke-type theorem for relative Hom-Hopf modules

In this section, we give necessary and sufficient conditions for the functor $F$ which forgets the $(H,\alpha)$-coaction to be separable, and we prove a Maschke type theorem for relative Hom-Hopf modules as an application.

**Definition 4.1.** Let $(H,\alpha)$ be a monoidal Hom-Hopf algebra and $(A,\beta)$ a right $(H,\alpha)$-Hom-comodule algebra. A $k$-linear map

$$\theta\colon\ (H,\alpha)\otimes(H,\alpha)\to(A,\beta)$$

such that $\theta\circ(\alpha\otimes\alpha)=\beta\circ\theta$ is called a normalized $(A,\beta)$-integral, if $\theta$ satisfies the following conditions:

(1) For all $h,g\in H$,

$$\theta(\alpha^{-1}(g)\otimes h_{(1)})\otimes\alpha(h_{(2)})=\beta(\theta(g_{(2)}\otimes\alpha^{-1}(h))_{[0]})\otimes g_{(1)}\theta(g_{(2)}\otimes\alpha^{-1}(h))_{[1]}. \tag{4.1}$$

(2) For all $h\in H$,

$$\theta(h_{(1)}\otimes h_{(2)})=1_A\varepsilon(h). \tag{4.2}$$

(3) For all $a\in A$, $h,g\in H$,

$$\beta^2(a_{[0][0]})\theta(\alpha^{-1}(g)a_{[0][1]}\otimes\alpha^{-1}(h)\alpha^{-1}(a_{[1]}))=\theta(g\otimes h)a. \tag{4.3}$$

**Theorem 4.2.** *Let $(H,\alpha)$ be a monoidal Hom-Hopf algebra and $(A,\beta)$ a right $(H,\alpha)$-Hom-comodule algebra. Then the following assertions are equivalent:*
(1) *The left adjoint $F$ in Proposition 2.3 is separable.*
(2) *There exists a normalized $(A,\beta)$-integral $\theta\colon\ (H,\alpha)\otimes(H,\alpha)\to(A,\beta)$.*

Proof. (2) $\Longrightarrow$ (1). For any relative Hom-Hopf module $(M,\mu)$, we define

$$\nu_M\colon\ M\otimes H\to M,$$
$$\nu_M(m\otimes h)=\mu(m_{[0]})\theta(m_{[1]}\otimes\alpha^{-1}(h)),$$

for all $m\in M$ and $h\in H$. Now, we shall check that $\nu_M\in\widetilde{\mathscr{H}}(\mathscr{M}_k)^H_A$. In fact, for all $m\in M$, $h\in H$ and $a\in A$, it is easy to get that

$$\nu_M(\mu(m)\otimes\alpha(h))=\mu(\nu_M(m\otimes h)).$$

We also have

$$\begin{aligned}
\nu_M((m\otimes h)\cdot a) &= \nu_M(ma_{[0]}\otimes ha_{[1]})\\
&= (\mu(m_{[0]})\cdot\beta(a_{[0][0]}))\theta(m_{[1]}a_{[0][1]}\otimes\alpha^{-1}(h)\alpha^{-1}(a_{[1]}))\\
&= \mu^2(m_{[0]})\cdot(\beta(a_{[0][0]})\beta^{-1}(\theta(m_{[1]}a_{[0][1]}\otimes\alpha^{-1}(h)\alpha^{-1}(a_{[1]})))\\
&= \mu^2(m_{[0]})\cdot(\beta(a_{[0][0]})\theta(\alpha^{-1}(m_{[1]})\alpha^{-1}(a_{[0][1]})\otimes\alpha^{-2}(h)\alpha^{-2}(a_{[1]})))\\
&\overset{(4.3)}{=}\mu^2(m_{[0]})\cdot(\theta(m_{[1]}\otimes\alpha^{-1}(h))\beta^{-1}(a))\\
&= (\mu(m_{[0]})\cdot\theta(m_{[1]}\otimes\alpha^{-1}(h)))\cdot a\\
&= (\nu_M(m\otimes h))\cdot a.
\end{aligned}$$

Hence it is a morphism of $(A,\beta)$-Hom-modules. Next, we shall check that $\nu_M$ is a morphism of Hom-comodules over $(H,\alpha)$. It is sufficient to check that

$$\varrho_M\circ\nu_M = (\nu_M\otimes \mathrm{id}_H)\circ\varrho_M$$

holds. For all $m\in M$ and $h\in H$, we have

$$\begin{aligned}
\varrho_M\circ\nu_M(m\otimes h) &= \varrho_M(\mu(m_{[0]})\theta(m_{[1]}\otimes\alpha^{-1}(h)))\\
&= (\mu(m_{[0]})\theta(m_{(1)}\otimes\alpha^{-1}(h))_{[0]}\otimes(\mu(m_{[0]})\theta(m_{[1]}\otimes\alpha^{-1}(h)))_{[1]}\\
&= \mu(m_{[0][0]})\theta(m_{[1]}\otimes\alpha^{-1}(h))_{[0]}\otimes\alpha(m_{[0][1]})\theta(m_{(1)}\otimes\alpha^{-1}(h))_{[1]}\\
&= m_{[0]}\theta(\alpha(m_{[1](2)})\otimes\alpha^{-1}(h))_{[0]}\otimes\alpha(m_{[1](1)})\theta(\alpha(m_{[1](2)})\otimes\alpha^{-1}(h))_{[1]}\\
&\overset{(4.1)}{=} m_{[0]}\beta^{-1}(\theta(m_{[1]}\otimes h_{(1)}))\otimes\alpha(h_{(2)})\\
&= m_{[0]}\theta(\alpha^{-1}(m_{[1]})\otimes\alpha^{-1}(h_{(1)}))\otimes\alpha(h_{(2)})\\
&= (\nu_M\otimes\mathrm{id}_H)\circ\varrho_M(m\otimes h).
\end{aligned}$$

For all $m\in M$, we have

$$\begin{aligned}
\nu_M\circ\eta_M(m) = \nu_M(m_{[0]}\otimes m_{[1]}) &= \mu(m_{[0][0]})\theta(m_{[0][1]}\otimes\alpha^{-1}(m_{[1]}))\\
&= m_{[0]}\theta(m_{[1](1)}\otimes m_{[1](2)})\overset{(4.2)}{=} m.
\end{aligned}$$

So the left adjoint $F$ in Proposition 2.3 is separable by virtue of Rafael theorem.

(1) $\Longrightarrow$ (2). We consider the relative Hom-Hopf module $A\otimes H$, and the $(A,\beta)$-actions and $(H,\alpha)$-coaction are defined as follows:

$$\begin{cases}(a\otimes h)\cdot b = ab_{[0]}\otimes hb_{[1]};\\ \varrho_{A\otimes H}(a\otimes h) = (\beta^{-1}(a)\otimes h_{(1)})\otimes\alpha(h_{(2)}),\end{cases}$$

for any $a,b\in A$ and $h\in H$.

The retraction $\nu$ of the unit of the adjunction in Proposition 2.3 yields a morphism

$$\nu_{A\otimes H}\colon\ (A\otimes H)\otimes H\to A\otimes H$$

such that, for all $a\in A, h\in H$,

$$\nu_{A\otimes H}((a\otimes h_{(1)})\otimes h_{(2)})=\beta(a)\otimes h.$$

It can be used to construct $\theta$ as follows:

$$\begin{gathered}\theta\colon\ H\otimes H\to A,\\ \theta(h\otimes g)=r_A(\mathrm{id}_A\otimes\varepsilon)\nu_{A\otimes H}((1_A\otimes h)\otimes g),\end{gathered}$$

where $r$ means the right unit constraint. For all $h\in H$ we have

$$\begin{aligned}\theta(h_{(1)}\otimes h_{(2)})&=r_A(\mathrm{id}_A\otimes\varepsilon)\nu_{A\otimes H}((1_A\otimes h_{(1)})\otimes h_{(2)})\\ &=r_A(\mathrm{id}_A\otimes\varepsilon)(1_A\otimes h)=1_A\varepsilon(h).\end{aligned}$$

Hence condition (4.2) follows. It can be seen to obey (4.3) by naturality and the $(A,\beta)$-modules map of $\nu$.

The verification of (4.1) is more involved. For any right $(H,\alpha)$-Hom-comodule $M$, we consider the relative Hom-Hopf module $M\otimes A$, the $(A,\beta)$-action and $(H,\alpha)$-coaction are defined as follows: for all $m\in M$ and $a,b\in A$,

$$\begin{cases}(m\otimes a)\cdot b=\mu^{-1}(m)\otimes a\beta(b),\\ \varrho_{M\otimes A}(m\otimes a)=(m_{[0]}\otimes a_{[0]})\otimes m_{[1]}a_{[1]}.\end{cases}$$

In particular, there is a relative Hom-Hopf module $H\otimes A$ and a map

$$\begin{gathered}\xi\colon\ H\otimes A\to A\otimes H\\ \xi(h\otimes a)=\beta(a_{[0]})\otimes\alpha^{-1}(h)a_{[1]}.\end{gathered}$$

Since $\xi$ is both right $(A,\beta)$-linear and right $(H,\alpha)$-colinear, we have

$$\begin{aligned}\xi(\nu_{H\otimes A}((h\otimes a)\otimes g))&=\nu_{A\otimes H}((\xi\otimes\mathrm{id}_H)((h\otimes a)\otimes g))\\ &=\nu_{A\otimes H}((\beta(a_{[0]})\otimes\alpha^{-1}(h)a_{[1]})\otimes g).\end{aligned}\tag{4.4}$$

It is not hard to check that $GF(H\otimes A)=(H\otimes A)\otimes H\in{}^H\widetilde{\mathscr{H}}(\mathscr{M}_k)_A^H$, and its left $(H,\alpha)$-Hom comodule structure is given by

$$(h\otimes a)\otimes g\mapsto\alpha(h_{(1)})\otimes((h_{(2)}\otimes\beta^{-1}(a))\otimes\alpha^{-1}(g)).$$

Also, $H \otimes A \in {}^H\widetilde{\mathscr{H}}(\mathscr{M}_k)_A^H$, and the left $(H,\alpha)$-coaction of $H \otimes A$ is given by

$$h \otimes a \mapsto \alpha(h_{(1)}) \otimes (h_{(2)} \otimes \beta^{-1}(a)).$$

We also get that $\nu_{H\otimes A}\colon (H \otimes A) \otimes H \to H \otimes A$ is a Hom morphism in ${}^H\widetilde{\mathscr{H}}(\mathscr{M}_k)_A^H$, which means

$$\begin{aligned} \nu_{H\otimes A}((h \otimes a) \otimes g)_{[-1]} \otimes \nu_{H\otimes A}((h \otimes a) \otimes g)_{[0]} \\ = \alpha(h_{(1)}) \otimes \nu_{H\otimes A}((h_{(2)} \otimes \beta^{-1}(a)) \otimes \alpha^{-1}(g)). \end{aligned} \tag{4.5}$$

Thus we conclude that $\nu_{H\otimes A}$ is both left and right $(H,\alpha)$-colinear. Taking $h, g \in H$, and putting

$$\nu_{A\otimes H}((1_A \otimes h) \otimes g) = \sum_i a_i \otimes q_i \in A \otimes H,$$
$$\nu_{H\otimes A}((h \otimes 1_A) \otimes g) = \sum_i p_i \otimes b_i \in H \otimes A,$$

we obtain

$$\begin{aligned} &\beta(\theta(h_{(2)} \otimes \alpha^{-1}(g))_{[0]}) \otimes h_{(1)}\theta(h_{(2)} \otimes \alpha^{-1}(g))_{[1]} \\ &\quad = \beta(r_A(\mathrm{id}_A \otimes\varepsilon)\nu_{A\otimes H}((1_A \otimes h_{(2)}) \otimes \alpha^{-1}(g))_{[0]}) \otimes h_{(1)} \\ &\qquad \times (r_A(\mathrm{id}_A \otimes\varepsilon)\nu_{A\otimes H}((1_A \otimes h_{(2)}) \otimes \alpha^{-1}(g)))_{[1]} \\ &\quad \overset{(4.4)}{=} \beta(r_A(\mathrm{id}_A \otimes\varepsilon)\xi\nu_{H\otimes A}((h_{(2)} \otimes 1_A) \otimes \alpha^{-1}(g))_{[0]}) \otimes h_{(1)} \\ &\qquad \times (r_A(\mathrm{id}_A \otimes\varepsilon)\xi\nu_{H\otimes A}((h_{(2)} \otimes 1_A) \otimes \alpha^{-1}(g))_{[1]}) \\ &\quad \overset{(4.5)}{=} \sum_i \beta(r_A(\mathrm{id}_A \otimes\varepsilon)\xi(p_{i(2)} \otimes \beta^{-1}(b_i))_{[0]}) \\ &\qquad \otimes p_{i(1)}(r_A(\mathrm{id}_A \otimes\varepsilon)\xi(p_{i(2)} \otimes \beta^{-1}(b_i))_{[1]}) \\ &\quad = \sum_i \beta(r_A(\mathrm{id}_A \otimes\varepsilon)(b_{i[0]} \otimes \alpha^{-1}(p_{i(2)})b_{i[1]})_{[0]}) \\ &\qquad \otimes p_{i(1)}(r_A(\mathrm{id}_A \otimes\varepsilon)(b_{i[0]} \otimes \alpha^{-1}(p_{i(2)})b_{i[1]})_{[1]}) \\ &\quad = \sum_i \beta(b_{i[0]}) \otimes p_{i(1)}\varepsilon(p_{i(2)})(b_{i[1]}) \\ &\quad = \sum_i \xi(p_i \otimes b_i) = \xi(\nu_{H\otimes A}((h \otimes 1_A) \otimes g)). \end{aligned}$$

Using the fact that $\nu_{A\otimes H}$ is a morphism of right $(H,\alpha)$-Hom comodules, we also have

$$\begin{aligned} &\theta(\alpha^{-1}(h) \otimes g_{(1)}) \otimes \alpha(g_{(2)}) \\ &\quad = r_A(\mathrm{id}_A \otimes\varepsilon)\nu_{A\otimes H}((1_A \otimes \alpha^{-1}(h)) \otimes g_{(1)}) \otimes \alpha(g_{(2)}) \\ &\quad = \sum_i r_A(\mathrm{id}_A \otimes\varepsilon)(\beta^{-1}(a_i) \otimes q_{i(1)}) \otimes \alpha(q_{i(2)}) \end{aligned}$$

$$= \sum_i a_i \otimes q_i = \nu_{A\otimes H}((1_A \otimes h) \otimes g)$$

$$\overset{(4.4)}{=} \xi(\nu_{H\otimes A}((h \otimes 1_A) \otimes g)).$$

Hence, we can get condition (4.1). □

We will now investigate the relation between the total integrals and the normalized $(A, \beta)$-integrals. This will explain our terminology, and we will also prove that the forgetful functor is separable if and only if there exists a total integral $\phi\colon (H, \alpha) \to (A, \beta)$ such that the image of $\varrho_A \circ \phi$ is contained in the center of $H \otimes A$.

**Proposition 4.3.** *Let $(H, \alpha)$ be a monoidal Hom-Hopf algebra and $(A, \beta)$ a right $(H, \alpha)$-Hom-comodule algebra. If $\theta\colon (H, \alpha)\otimes(H, \alpha) \to (A, \beta)$ is a normalized $(A, \beta)$-integral for $(H, A, H)$, then the $k$-linear map*

$$\phi\colon (H, \alpha) \to (A, \beta),\ \phi(h) = \theta(1_H \otimes h)$$

*for all $h \in H$ is a total integral.*

Proof. Notice first that $\phi(1_H) = \theta(1_H \otimes 1_H) = \varepsilon_H(1_H)1_A = 1_A$. Hence

$$\theta(\alpha^{-1}(g) \otimes \alpha^{-1}(h_{(1)})) \otimes \alpha(h_{(2)})$$
$$= (\theta(\alpha(g_{(2)}) \otimes \alpha^{-1}(h)))_{(0)} \otimes \alpha(g_{(1)})(\theta(\alpha(g_{(2)}) \otimes \alpha^{-1}(h)))_{(1)}.$$

It follows by taking $g = 1_H$ that

$$\theta(1_H \otimes \alpha^{-1}(h_1)) \otimes \alpha(h_2) = \theta(1_H \otimes \alpha^{-1}(h))_{[0]} \otimes \alpha(\theta(1_H \otimes \alpha^{-1}(h))_{[1]}),$$

and applying $\alpha \otimes \alpha^{-1}$ to the above identity, we have

$$\theta(1_H \otimes \alpha^{-1}(h_1)) \otimes h_2 = \theta(1_H \otimes \alpha^{-1}(h))_{[0]} \otimes \theta(1_H \otimes \alpha^{-1}(h))_{[1]}.$$

So we obtain

$$\phi(h_1) \otimes h_2 = \phi(h)_{[0]} \otimes \phi(h)_{[1]}.$$

It is easy to check that $\phi\alpha = \beta\phi$. So $\phi$ is a total integral.

Let $\phi\colon (H, \alpha) \to (A, \beta)$ be a total integral for the right $(H, \alpha)$-Hom-comodule algebra $(A, \beta)$, and define

$$\theta\colon (H, \alpha) \otimes (H, \alpha) \to (A, \beta), \quad \theta(g \otimes h) = \phi(hS^{-1}(g))$$

for all $g, h \in H$. □

**Theorem 4.4.** *Let $(H,\alpha)$ be a monoidal Hom-Hopf algebra and $(A,\beta)$ a right $(H,\alpha)$-Hom-comodule algebra, and $\phi\colon (H,\alpha)\to(A,\beta)$ a total integral. If*

$$g\phi(h)_{[1]}\otimes\phi(h)_{[0]}=\phi(h)_{[1]}g\otimes\phi(h)_{[0]},\quad \phi(h)\in Z(A),$$

*then $\theta$ is a normalized $(A,\beta)$-integral.*

Proof. For any $h,g\in H$ and $a\in A$, we have

$$\begin{aligned}
&\beta^2(a_{[0][0]})\theta(\alpha^{-1}(g)a_{[0][1]}\otimes\alpha^{-1}(h)\alpha^{-1}(a_{[1]}))\\
&\quad=\beta(a_{[0]})\theta(\alpha^{-1}(g)a_{[1](1)}\otimes\alpha^{-1}(h)a_{[1](2)})\\
&\quad=\beta(a_{[0]})\phi(\alpha^{-1}(h)a_{[1](2)}S^{-1}(\alpha^{-1}(g)a_{[1](1)}))\\
&\quad=\beta(a_{[0]})\phi(h[(\alpha^{-1}(a_{[1](2)}S^{-1}(\alpha^{-1}(a_{[1](1)}))))S^{-1}(\alpha^{-1}(g))])\\
&\quad=a\phi(hS^{-1}(g))=\theta(g\otimes h)a,
\end{aligned}$$

and

$$\begin{aligned}
&\beta(\theta(g_{(2)}\otimes\alpha^{-1}(h))_{[0]})\otimes g_{(1)}\theta(g_{(2)}\otimes\alpha^{-1}(h))_{[1]}\\
&\quad=\beta(\phi(\alpha^{-1}(h)S^{-1}(g_{(2)}))_{[0]})\otimes\phi(\alpha^{-1}(h)S^{-1}(g_{(2)}))_{[1]}g_{(1)}\\
&\quad=\phi(h_{(1)}S^{-1}(\alpha(g_{(2)(2)})))\otimes(\alpha^{-1}(h_{(2)})S^{-1}(g_{(2)(1)}))g_{(1)}\\
&\quad=\phi(h_{(1)}S^{-1}(g_{(2)}))\otimes(\alpha^{-1}(h_{(2)})S^{-1}(g_{(1)(2)}))\alpha(g_{(1)(1)})\\
&\quad=\phi(h_{(1)}S^{-1}(g_{(2)}))\otimes h_{(2)}(S^{-1}(g_{(1)(2)})g_{(1)(1)})\\
&\quad=\phi(h_{(1)}S^{-1}(\alpha^{-1}(g)))\otimes\alpha(h_{(2)})\\
&\quad=\theta(\alpha^{-1}(g)\otimes h_{(1)})\otimes\alpha(h_{(2)}),
\end{aligned}$$

$$\theta(h_1\otimes h_2)=\varphi(h_2S^{-1}(h_1))=\varepsilon_H(h)1_A.$$

It is easy to check that $\phi\alpha=\beta\phi$. So $\theta$ is a normalized $(A,\beta)$-integral. □

Since separable functors reflect well the semisimplicity of the objects of a category, by Theorem 4.2, we will prove a Maschke type theorem for relative Hom-Hopf modules as an application.

**Lemma 4.5.** *Let $(H,\alpha)$ be a monoidal Hom-Hopf algebra and $(A,\beta)$ a right $(H,\alpha)$-Hom-comodule algebra with a total integral $\phi\colon (H,\alpha)\to(A,\beta)$ and $(M,\mu)$, $(N,\nu)\in\widetilde{\mathscr{H}}(\mathscr{M}_k)_A^H$ and a Hom-morphism $f\colon (N,\nu)\to(M,\mu)$. Let*

$$f_\phi\colon N\xrightarrow{\varrho_N}N\otimes H\xrightarrow{f\otimes\mathrm{id}_H}M\otimes H\xrightarrow{\tau}M,$$

*that is,*

$$f_\phi(n)=\mu^{-1}(f(n_{[0]})_{[0]})\cdot\phi(S(f(n_{[0]})_{[1]})\alpha(n_{[1]})),$$

*for any $n\in N$. Then the following assertions hold:*

(1) *$f_\phi$ is a morphism of right $(H,\alpha)$-Hom-comodules,*

(2) *if $f\colon (N,\nu)\to(M,\mu)$ is a morphism of right $(A,\beta)$-Hom-modules and $\phi\colon (H,\alpha)\to(Z(A),\beta)$ is a multiplication map, then $f_\phi$ is a morphism of right $(A,\beta)$-Hom-module.*

Proof. (1) For any $n\in N$, we have

$$\begin{aligned}
\varrho_M(f_\phi(n)) &= \varrho_M\big(\mu^{-1}(f(n_{[0]})_{[0]})\cdot\phi(S(f(n_{[0]})_{[1]})\alpha(n_{[1]}))\big)\\
&= \mu^{-1}(f(n_{[0]})_{[0][0]})\cdot\phi\big(S(f(n_{[0]})_{[1](2)})\alpha(n_{[1](1)})\big)\\
&\quad\otimes\alpha^{-1}(f(n_{[0]})_{[0][1]})\big(S(f(n_{[0]})_{[1](1)})\alpha(n_{[1](2)})\big)\\
&= \mu^{-2}(f(n_{[0]})_{[0]})\cdot\phi\big(\alpha(S(f(n_{[0]})_{[1](2)(2)}))\alpha(n_{[1](1)})\big)\\
&\quad\otimes\alpha^{-1}(f(n_{[0]})_{[1](1)})\big(\alpha(S(f(n_{[0]})_{[1](2)(1)}))\alpha(n_{[1](2)})\big)\\
&= \mu^{-2}(f(n_{[0]})_{[0]})\cdot\phi\big(S(f(n_{[0]})_{[1](2)})\alpha(n_{[1](1)})\big)\\
&\quad\otimes f(n_{[0]})_{[1](1)(1)}\big(\alpha(S(f(n_{[0]})_{[1](1)(2)}))\alpha(n_{[1](2)})\big)\\
&= \mu^{-2}(f(n_{[0]})_{[0]})\cdot\phi\big(S(f(n_{[0]})_{[1](2)})\alpha(n_{[1](1)})\big)\\
&\quad\otimes\big(\alpha(f(n_{[0]})_{[1](1)(1)})\alpha(S(f(n_{[0]})_{[1](1)(2)}))\big)n_{[1](2)}\\
&= \mu^{-2}(f(n_{[0]})_{[0]})\cdot\phi\big(\alpha^{-1}(S(f(n_{[0]})_{[1]}))\alpha(n_{[1](1)})\big)\otimes\alpha(n_{[1](2)})\\
&= \mu^{-1}(f(n_{[0][0]})_{[0]})\cdot\phi\big(S(f(n_{[0][0]})_{[1]})\alpha(n_{[0][1]})\big)\otimes n_{[1]}\\
&= (f_\phi\otimes\mathrm{id}_H)\varrho_N(n).
\end{aligned}$$

(2) For any $n\in N$ and $b\in A$, we have

$$\begin{aligned}
f_\phi(n\cdot b) &= \mu^{-1}(f(n_{[0]})_{[0]}\cdot b_{[0][0]})\cdot\phi\big(S(f(n_{[0]})_{[1]}b_{[0][1]})\alpha(n_{[1]}b_{[1]})\big)\\
&= \mu^{-1}(f(n_{[0]})_{[0]}\cdot b_{[0][0]})\cdot\phi\big([S(f(n_{[0]})_{[1]}b_{[0][1]})][\alpha(b_{[1]})\alpha(n_{[1]})]\big)\\
&= \mu^{-1}(f(n_{[0]})_{[0]}\cdot b_{[0][0]})\cdot\phi\big(\alpha(S(f(n_{[0]})_{[1]})[S(b_{[0][1]})(b_{[1]}n_{[1]})])\big)\\
&= \mu^{-1}(f(n_{[0]})_{[0]}\cdot b_{[0][0]})\cdot\phi\big(\alpha(S(f(n_{[0]})_{[1]})[(\alpha^{-1}(S(b_{[0][1]}))b_{[1]})\alpha(n_{[1]})])\big)\\
&= (\mu^{-1}(f(n_{[0]})_{[0]})\cdot b_{[0]})\\
&\quad\times\phi\big(\alpha(S(f(n_{[0]})_{[1]})[(\alpha^{-1}(S(b_{[1](1)}))\alpha^{-1}(b_{[1](2)}))\alpha(n_{[1]})])\big)\\
&= (\mu^{-1}(f(n_{[0]})_{[0]})\cdot\beta^{-1}(b))\cdot\phi\big(\alpha(S(f(n_{[0]})_{[1]}))\alpha^2(n_{[1]})\big)\\
&= f(n_{[0]})_{[0]}\cdot\big(\beta^{-1}(b)\phi\big(S(f(n_{[0]})_{[1]})\alpha(n_{[1]})\big)\big)\\
&= f(n_{[0]})_{[0]}\cdot\big(\phi\big(S(f(n_{[0]})_{[1]})\alpha(n_{[1]})\big)\beta^{-1}(b)\big)\\
&= (\mu^{-1}(f(n_{[0]})_{[0]})\cdot\phi(S(f(n_{[0]})_{[1]})\alpha(n_{[1]})))\cdot b = f_\phi(n)\cdot b.
\end{aligned}$$

□

**Theorem 4.6.** *Let $(H,\alpha)$ be a monoidal Hom-Hopf algebra and $(A,\beta)$ a right $(H,\alpha)$-Hom-comodule algebra with a total integral $\phi\colon (H,\alpha)\to(A,\beta)$. If $\phi\colon (H,\alpha)\to(Z(A),\beta)$ is a multiplication map, then every short exact sequence of relative Hom-Hopf modules*

$$0\longrightarrow(M,\mu)\xrightarrow{f}(N,\nu)\xrightarrow{g}(P,\pi)\longrightarrow 0$$

*which splits as a sequence of $(A,\beta)$-Hom-modules also splits as a sequence of relative Hom-Hopf modules.*

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