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INSERTING MEASURABLE FUNCTIONS PRECISELY

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Abstract. A family of subsets of a set is called a σ-topology if it is closed under arbitrary countable unions and arbitrary finite intersections. A σ-topology is perfect if any its member (open set) is a countable union of complements of open sets. In this paper perfect σ-topologies are characterized in terms of inserting lower and upper measurable functions. This improves upon and extends a similar result concerning perfect topologies. Combining this characterization with a σ-topological version of Katětov-Tong insertion theorem yields a Michael insertion theorem for normal and perfect σ-topological spaces.

Keywords: insertion; σ-topology; σ-ring; perfectness; normality; upper measurable function; lower measurable function; measurable function

MSC 2010: 28A05, 28A20

1. Introduction

Let us recall that Vedenisoff's theorem states that a topological space X is normal and perfect if and only if two disjoint closed sets $F, K \subseteq X$ can precisely be separated by a continuous function $f\colon X \to [0,1]$, meaning: $f = 1$ only on F and $f = 0$ only on K. Equivalently, $\chi_F \leqslant f \leqslant \chi_{X\setminus K}$ and $\chi_F(x) < f(x) < \chi_{X\setminus K}(x)$ if $\chi_F(x) < \chi_{X\setminus K}(x)$. Michael's insertion theorem of [10] arises by replacing the characteristic functions with semicontinuous functions:

Michael insertion theorem. *A topological space X is normal and perfect if and only if, given $u, l\colon X \to \mathbb{R}$ such that $u \leqslant l$, u is upper semicontinuous and l*

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is lower semicontinuous, there is a continuous $f\colon X \to \mathbb{R}$ *such that* $u \leqslant f \leqslant l$ *and* $u(x) < f(x) < l(x)$ *whenever* $u(x) < l(x)$.

Speaking ahistorically, a theorem which dismantles Michael's theorem from perfectness is the Katětov-Tong theorem [7], [13]:

Katětov-Tong insertion theorem. *A topological space* X *is normal if and only if, given* $u, l\colon X \to \mathbb{R}$ *such that* $u \leqslant l$, u *is upper semicontinuous and* l *is lower semicontinuous, there is a continuous* $f\colon X \to \mathbb{R}$ *such that* $u \leqslant f \leqslant l$.

One might choose to act the other way around: instead of avoiding perfectness one might choose to avoid normality. An attempt at doing so has recently been made in [14]: *a topological space* X *is perfect if and only if, given a lower semicontinuous* $l\colon X \to [0,\infty)$, *there exists an upper semicontinuous* $u\colon X \to [0,\infty)$ *such that* $u \leqslant l$ *and* $0 < u(x) < l(x)$ *whenever* $u(x) < l(x)$. One shortcoming of this result is that there is no direct implication from it (using Katětov-Tong theorem) to Michael's theorem.

The purpose of this paper is twofold. First, we improve upon and extend the above characterization of [14]. In contrast to [14], we characterize perfectness in σ-topological spaces (only countable unions are allowed) and we do not assume the involved functions to be non-negative (see Theorem 3.2). Second, by combining our characterization with the insertion theorem of [8] we obtain a σ-topological version of Michael's insertion theorem (see Theorem 4.2). We also show how the related extension and separation theorems look like (see Theorems 3.3 and 4.3).

2. σ-topologies instead of topologies

A family $\mathcal{A}$ of subsets of a set X is called a σ-topology [1], [2] (σ-ring in [3], [6]) if it is closed under countable unions and finite intersections, and $\emptyset, X \in \mathcal{A}$. Then $(X, \mathcal{A})$ is a σ-topological space. If $f\colon X \to \mathbb{R}$ and $t \in \mathbb{R}$, we let $[f > t] = \{x \in X\colon f(x) > t\}$, and similarly for $[f \geqslant t]$, $[f < t]$, etc. Following [3] and [11], an $f\colon X \to \mathbb{R}$ is called lower [upper] $\mathcal{A}$-measurable if $[f > t] \in \mathcal{A}$ [if $[f < t] \in \mathcal{A}$] for all $t \in \mathbb{R}$. It is $\mathcal{A}$-measurable if it is both lower and upper $\mathcal{A}$-measurable. We denote by $\mathrm{Lm}(X)$, $\mathrm{Um}(X)$ and $\mathrm{M}(X)$ the collections of all lower $\mathcal{A}$-measurable, upper $\mathcal{A}$-measurable and $\mathcal{A}$-measurable functions from a σ-topological space $(X, \mathcal{A})$ into $\mathbb{R}$, respectively. Needless to say, $f \in \mathrm{Lm}(X)$ iff $-f \in \mathrm{Um}(X)$. Also, if $f, g \in \mathrm{Lm}(X)$, then $f + g$, $\alpha f \in \mathrm{Lm}(X)$ for all $\alpha > 0$. The characteristic function χ_A of a subset $A \subseteq X$ is in $\mathrm{Lm}(X)$ iff $A \in \mathcal{A}$. We let $\mathcal{A}^c = \{F\colon X \setminus F \in \mathcal{A}\}$. If $S \subseteq X$, then $\mathcal{A}_S = \{A \cap S\colon A \in \mathcal{A}\}$ is a σ-topology on S. For $\mathcal{A}$ a topology on X, lower and upper $\mathcal{A}$-measurability, and $\mathcal{A}$-measurability become, respectively, lower and upper semicontinuity, and continuity. Topological concepts of normality and perfectness extend to σ-topologies: $\mathcal{A}$ is normal if, given any two disjoint members of $\mathcal{A}^c$, there are disjoint members of $\mathcal{A}$ containing them. And $\mathcal{A}$ is perfect if any its member is a countable union of members of $\mathcal{A}^c$.

If $\mathcal{A}$ is understood, the σ-topological space $(X, \mathcal{A})$ is referred to as the space X in which case we also speak about lower measurable, upper measurable and measurable functions.

The concept of a σ-topology is not merely a formal generalization. Even if any topology is a σ-topology, there are many important σ-topologies which are not topologies (see [3] and [11]; also see [8]).

We mention one important example: given a topological space X a subset A is called a cozero set if it is of the form $[f \neq 0]$ for some continuous $f\colon X \to \mathbb{R}$. Then the family $\operatorname{Coz} X$ of all cozero sets is a σ-topology which is always perfect and normal and need not be a topology (see [4], 1.14, 1.15, for details). Lower and upper $\mathcal{A}$-measurable functions with respect to $\mathcal{A} = \operatorname{Coz} X$ have been considered in [2], [9], [12] among others.

3. Perfect σ-topologies

We start with the following lemma.

Lemma 3.1. *Let $\mathcal{A}$ be a σ-topology on X. Let $f\colon X \to [0,\infty)$ be an arbitrary function such that there is a non-decreasing sequence $(F_n)_{n\in\mathbb{N}}$ in $\mathcal{A}^c$ such that $\bigcup\limits_{n\in\mathbb{N}} F_n = [f > 0]$ and $F_n \subseteq [f > 1/n]$ for all n. Then there is an upper $\mathcal{A}$-measurable function $u\colon X \to [0,\infty)$ such that $u \leqslant f$ and $[f > 0] = [u > 0]$.*

Proof. Define $u\colon X \to \mathbb{R}$ by

$$u = \sup_{n\in\mathbb{N}} \min\Big(\frac{1}{n}, \chi_{F_n}\Big).$$

Then $[u \geqslant t] = \emptyset$ if $t > 1$, and $[u \geqslant t] = X$ if $t \leqslant 0$. If $t \in (0,1]$, we have

$$[u \geqslant t] = \bigcup_{1/n \geqslant t} F_n = \bigcup_{1/t \geqslant n} F_n = F_m$$

where m is the integer part of $1/t$. This shows that $u \in \operatorname{Um}(X)$. We also have

$$u \leqslant \sup_{n\in\mathbb{N}} \min\Big(\frac{1}{n}, 1_{[f>1/n]}\Big) \leqslant f$$

and $[f > 0] = \bigcup\limits_{n\in\mathbb{N}} F_n = \bigcup\limits_{n\in\mathbb{N}} [u \geqslant 1/n] = [u > 0]$. $\square$

Theorem 3.2. *For X a σ-topological space the following are equivalent:*

(1) *X is perfect.*

(2) *If $u, l\colon X \to \mathbb{R}$ are such that $u \leqslant l$, u is upper measurable and l is lower measurable, then there exist $u', l'\colon X \to \mathbb{R}$ such that $u \leqslant u' \leqslant l' \leqslant l$, u' is upper and l' is lower measurable, and $u(x) < u'(x) \leqslant l'(x) < l(x)$ whenever $u(x) < l(x)$.*

Proof. (1) $\Rightarrow$ (2): Let $\mathcal{A}$ be the σ-topology on X. If $u \leqslant l$, then $0 \leqslant l - u \in \mathrm{Lm}(X)$. Since $\mathcal{A}$ is perfect and $[l - u > 1/n] \in \mathcal{A}$, there is an infinite matrix $(K_{n,m})_{n,m\in\mathbb{N}}$ of members of $\mathcal{A}^c$ with $[l - u > 1/n] = \bigcup_{m\in\mathbb{N}} K_{n,m}$ for each n. Let

$$F_n = \bigcup_{i,j\leqslant n} K_{i,j}.$$

Then $F_n \subseteq F_{n+1}$ in $\mathcal{A}^c$. Also,

$$\bigcup_{n\in\mathbb{N}} F_n = \bigcup_{n,m\in\mathbb{N}} K_{n,m} = \bigcup_{n\in\mathbb{N}} \Big[l - u > \frac{1}{n}\Big] = [l - u > 0]$$

and

$$F_n \subseteq \bigcup_{i\leqslant n} \Big[l - u > \frac{1}{i}\Big] \subseteq \Big[l - u > \frac{1}{n}\Big]$$

for each n. Now, by Lemma 3.1 there is a $k \in \mathrm{Um}(X)$ such that $0 \leqslant k \leqslant l - u$ and $[l - u > 0] = [k > 0]$. Further, $u' = u + k/2 \in \mathrm{Um}(X)$, $u \leqslant u' \leqslant l$ and $u(x) < u'(x) < l(x)$ whenever $u(x) < l(x)$. By the same argument, there is a $v \in \mathrm{Um}(X)$ such that $-l \leqslant v \leqslant -u'$ and $-l(x) < v(x) < -u'(x)$ whenever $-l(x) < -u'(x)$. Now, u' and $l' = -v$ are as required.

(2) $\Rightarrow$ (1): Let $A \in \mathcal{A}$. Then $u = 0 \leqslant \chi_A = l$ with $l \in \mathrm{Lm}(X)$. Then with u' of (2) one has

$$A = [\chi_A > 0] = [u' > 0] = \bigcup_{n\in\mathbb{N}} \Big[u' \geqslant \frac{1}{n}\Big],$$

a countable union of members of $\mathcal{A}^c$. □

The following may be regarded as counterparts of Tietze's extension theorem and Urysohn's lemma for perfect σ-topological spaces.

Theorem 3.3. *For X a σ-topological space the following are equivalent:*

(1) *X is perfect.*

(2) Precise extension: *For every $F \in \mathcal{A}^c$ each $\mathcal{A}_F$-measurable function $m\colon F \to [0,1]$ has an upper $\mathcal{A}$-measurable extension $u\colon X \to [0,1]$ and a lower $\mathcal{A}$-measurable extension $l\colon X \to [0,1]$ such that $0 < u(x) \leqslant l(x) < 1$ whenever $x \in X \setminus F$.*

(3) Precise separation: *Given disjoint $F, G \in \mathcal{A}^c$, there are $u, l\colon X \to [0,1]$ such that $u \leqslant l$, u is upper and l is lower $\mathcal{A}$-measurable, $u = l = 1$ on F, $u = l = 0$ on G, and $0 < u(x) \leqslant l(x) < 1$ whenever $x \in X \setminus (F \cup G)$.*

Proof. (1) ⇒ (2): Let $F \in \mathcal{A}^c$ and let $m\colon F \to [0,1]$ be $\mathcal{A}_F$-measurable. Let $u, l\colon X \to [0,1]$ be such that $u = m = l$ on F, $u = 0$ and $l = 1$ on $X \setminus F$. Then $u \leqslant l$ and $u, -l \in \mathrm{Um}(X)$. By Theorem 3.2 there are $u', -l' \in \mathrm{Um}(X)$ such that $u \leqslant u' \leqslant l' \leqslant l$ and $u(x) < u'(x) \leqslant l'(x) < l(x)$ whenever $u(x) < l(x)$. Hence $m = u \leqslant u' \leqslant l' \leqslant l = m$ on F, and $0 < u'(x) \leqslant l'(x) < 1$ if $x \in X \setminus F$.

(2) ⇒ (3): Let $F, K \in \mathcal{A}^c$ be disjoint. Let $m\colon F \cup K \to [0,1]$ be given by $m = 1$ on F and $m = 0$ on K. Then m is $\mathcal{A}_{F \cup K}$-measurable. By hypothesis, there are $u', -l' \in \mathrm{Um}(X)$ which extend m to the whole of X and which are such that $0 < u'(x) \leqslant l'(x) < 1$ for $x \in X \setminus (F \cup K)$. Consequently, $u' = l' = 1$ on F, $u' = l' = 0$ on K, and $0 < u'(x) \leqslant l'(x) < 1$ if $x \in X \setminus (F \cup K)$.

(3) ⇒ (1): Let $F \in \mathcal{A}^c$ and $K = \emptyset$. Then there exists an $l \in \mathrm{Lm}(X)$ such that $l' = 1$ on F and $l'(x) \in (0,1)$ if $x \in X \setminus F$. With $A_n = [l' > 1 - 1/n]$ we get $F = \bigcap_{n \in \mathbb{N}} A_n$, so that X is perfect. □

4. Michael's theorem for σ-topologies

We first record a σ-topological version of the Katětov-Tong theorem.

Theorem 4.1 ([8], Theorem 4.4)**.** *A σ-topological space X is normal if and only if, given $u, l\colon X \to \mathbb{R}$ such that $u \leqslant l$, u is upper and l is lower measurable, there is a measurable $m\colon X \to \mathbb{R}$ such that $u \leqslant m \leqslant l$.*

Combining Theorem 4.1 with Theorem 3.2 yields a σ-topological version of Michael's theorem:

Theorem 4.2. *A σ-topological space X is normal and perfect if and only if, given $u, l\colon X \to \mathbb{R}$ such that $u \leqslant l$, u is upper and l is lower measurable, there is a measurable $m\colon X \to \mathbb{R}$ such that $u \leqslant m \leqslant l$ and $u(x) < m(x) < l(x)$ whenever $u(x) < l(x)$.*

When adding normality to Theorem 3.3 we obtain the following (cf. [5]):

Theorem 4.3. *For X a σ-topological space the following are equivalent:*

(1) *X is normal and perfect.*
(2) Precise extension: *For every $F \in \mathcal{A}^c$ each $\mathcal{A}_F$-measurable function $m\colon F \to [0,1]$ has a $\mathcal{A}$-measurable extension $\overline{m}\colon X \to [0,1]$ such that $\overline{m}(X \setminus F) \subseteq (0,1)$.*
(3) Precise separation: *For every $F, G \in \mathcal{A}^c$, there exists an $\mathcal{A}_F$-measurable function $m\colon F \to [0,1]$ such that $m = 1$ on F, $m = 0$ on G, and $m(X \setminus (F \cup G)) \subseteq (0,1)$.*

Remark. Prof. Miklós Laczkovich has read our paper and has made an important observation: the "only if" parts of both Michael's insertion theorem and its σ-topological version (Theorem 4.2) can be deduced from F. Hausdorff's *Grundzüge der Mengenlehre,* 1914 (cf. [6], pages 267, 275, 276).

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