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INHOMOGENEOUS PARABOLIC NEUMANN PROBLEMS

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Abstract. Second order parabolic equations on Lipschitz domains subject to inhomogeneous Neumann (or, more generally, Robin) boundary conditions are studied. Existence and uniqueness of weak solutions and their continuity up to the boundary of the parabolic cylinder are proved using methods from the theory of integrated semigroups, showing in particular the well-posedness of the abstract Cauchy problem in spaces of continuous functions. Under natural assumptions on the coefficients and the inhomogeneity the solutions are shown to converge to an equilibrium or to be asymptotically almost periodic.

Keywords: parabolic initial-boundary value problem; inhomogeneous Robin boundary conditions; existence of weak solution; continuity up to the boundary; asymptotic behavior; asymptotically almost periodic solution

MSC 2010: 35K20, 35B15, 35B65

1. Introduction

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Our model problem is the heat equation

$$\begin{cases} u_t(t,x) - \Delta u(t,x) = f(t,x), & t > 0,\ x \in \Omega \\ \dfrac{\partial u(t,z)}{\partial \nu} = g(t,z), & t > 0,\ z \in \partial\Omega \\ u(0,x) = u_0(x), & x \in \Omega \end{cases}$$

subject to inhomogeneous Neumann boundary conditions. The above problem has a unique weak solution in an L^2-sense if f, g and u_0 are square-integrable. We are interested in its regularity on the boundary and its asymptotic behavior. Such problems appear in a natural way for example in control theory [7], [8] or thermal imaging [9].

We show the following: if u_0 is continuous and f and g satisfy some integrability conditions, then the solution u is continuous up to the boundary of the parabolic cylinder; if f and g converge to zero in a time-averaged sense, then u converges to zero uniformly on $\overline{\Omega}$; finally, if f and g are almost periodic functions, then u is asymptotically almost periodic with essentially the same frequencies.

For the particular case $f = 0$ and $g = 0$ the regularity assertion states that for all initial values $u_0 \in \mathrm{C}(\overline{\Omega})$ there exists a unique mild solution u in the space $\mathrm{C}(\overline{\Omega})$, i.e., that the realization of Δ in $\mathrm{C}(\overline{\Omega})$ with Neumann boundary conditions generates a strongly continuous semigroup. In this sense our results continue the recent struggle to study well-posedness of parabolic equations in the space of continuous functions [23], [22], [14], [5], [26], [3].

Even though the heat equation will be our model case, we admit general strongly elliptic operators with bounded coefficients subject to Robin boundary conditions in all of our results, imposing only some additional structure conditions for the analysis of the asymptotic behavior in order to prevent exponential blow-up and decay. For homogeneous boundary conditions, i.e., if $g = 0$, these problems are well understood and can be studied by semigroup methods. Inhomogeneous boundary conditions, however, are more delicate. For smooth data, some existence and regularity results can be found in [20], Theorem 5.18, or [12]. Existence of a weak solution is shown in [21], §4.15.3. Here we proceed in the spirit of [1], where regularity and asymptotic almost periodicity of the inhomogeneous Dirichlet problem have been studied.

In order to study the asymptotic behavior we follow a semigroup approach by considering the equation as an abstract Cauchy problem in a suitable space, which is adapted to the boundary data. To this end one could use spaces of distributions that contain functionals arising from boundary integrals, a strategy which has been pursued with negative exponent Sobolev spaces [18] and Sobolev-Morrey spaces [17]. This approach, however, has the disadvantage that a priori the solutions are no more regular than generic elements of these spaces, whereas it would be favorable to have continuous functions as solutions. The parabolic structure of the equation does not immediately help because a gain in regularity is not obvious in presence of the inhomogeneities. The regularity matters in particular in the limits $t \to 0$ and $t \to \infty$ since semigroup methods provide convergence in the norm of the underlying space.

In view of these considerations we aim towards results in the space $\mathrm{C}(\overline{\Omega})$. Existence is however much more convenient in $L^2(\Omega)$, which is why we will start out by considering L^2-solution. By using $\mathrm{C}(\overline{\Omega})$ we are able to obtain uniform convergence of u on $\overline{\Omega}$ as $t \to 0$ and as $t \to \infty$, or more generally asymptotic almost periodicity. This seems to be new for Neumann boundary conditions and is our main result.

Our strategy is the following. When formulating the initial-boundary value problem as an abstract Cauchy problem on $L^2(\Omega)$ or $\mathrm{C}(\overline{\Omega})$, we switch to a product space. More precisely, we regard the inhomogeneous heat equation as an inhomogeneous abstract Cauchy problem for the operator A given by $A(u, 0) = (\Delta u, -\partial u/\partial \nu)$ in the space $L^2(\Omega) \times L^2(\partial\Omega)$. This operator A is not densely defined and hence it is not the generator of a strongly continuous semigroup. In fact, it turns out that A does not even satisfy the Hille-Yosida estimates. Still, the operator is resolvent positive and hence generates a once integrated semigroup. This implies existence and uniqueness of solutions for regular right hand sides f and g and gives information about the asymptotic behavior of solutions. These results can be extended to a larger class of less regular right hand sides once we obtain suitable a priori estimates.

The idea to consider a non-densely defined operator A on a product space in order to treat inhomogeneous boundary conditions was first used by Arendt for the study of the heat equation with inhomogeneous Dirichlet boundary conditions [1]. Here we copy the skeleton of his proofs. The details are however quite different, the main aspects being the following:

(1) We restrict ourselves to Lipschitz domains, which is the usual framework for Neumann problems, whereas one of Arendt's main points are the optimal boundary regularity assumptions.

(2) In [1] the a priori estimate is a consequence of a version of the parabolic maximum principle, which is proved there. In our situation, on the other hand, we do not have contractivity properties and thus need more sophisticated estimates.

(3) The Neumann problem has a smoothing effect with respect to the boundary conditions, which allows us to obtain continuous solutions even for non-smooth functions g, whereas for Dirichlet problems the boundary function has to be continuous. This also explains why for the Neumann problem the solution is asymptotically almost periodic in the sense of Bohr even if the right hand side is almost periodic only in the sense of Stepanoff.

The article is organized as follows. In Section 2 we introduce the initial-boundary value problem. We show existence and uniqueness of solutions and discuss the relationship between three different notions of solutions. Section 3 contains results and pointwise estimates for the solutions as well as their continuity. The most technical part of this section is however postponed to Appendix A in the hope that this improves the readability of the article as a whole. In Section 4 we study the convergence of solutions. More precisely, we give natural sufficient conditions for the solution to be bounded or to converge to a constant function. Finally, in Section 5 we show that for asymptotically almost period right hand sides in the sense of Stepanoff, the solution is asymptotically almost periodic in the sense of Bohr.

2. Solutions

Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, $N \geqslant 2$. For convenience we assume throughout that Ω is connected; otherwise we could consider each connected component separately. Let $a_{ij} \in L^\infty(\Omega)$, $b_j, c_i \in L^q(\Omega)$, $d \in L^{q/2}(\Omega)$ and $\beta \in L^{q-1}(\partial\Omega)$ be given, where $q > N$ is arbitrary, and assume that there exists $\mu > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geqslant \mu|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N. \tag{2.1}$$

Throughout the article we will always refer to the inhomogeneous Robin problem

$$(P_{u_0,f,g}) \quad \begin{cases} u_t(t,x) - Au(t,x) = f(t,x), & t > 0,\ x \in \Omega, \\ \dfrac{\partial u(t,z)}{\partial \nu_A} + \beta u(t,z) = g(t,z), & t > 0,\ z \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{2.2}$$

with given $u_0 \in L^2(\Omega)$, $f \in L^2((0,\infty); L^2(\Omega))$ and $g \in L^2((0,\infty); L^2(\partial\Omega))$. Here, at least on a formal level,

$$\begin{aligned} Au &:= \sum_{j=1}^{N} D_j\Big(\sum_{i=1}^{N} a_{ij}D_iu + b_ju\Big) - \Big(\sum_{i=1}^{N} c_iD_iu + du\Big), \\ \frac{\partial u}{\partial \nu_A} &:= \sum_{j=1}^{N}\Big(\sum_{i=1}^{N} a_{ij}D_iu + b_ju\Big)\nu_j, \end{aligned}$$

where $\nu = (\nu_j)_{j=1}^N$ denotes the outer unit normal of Ω at the boundary $\partial\Omega$. It is convenient to introduce also the bilinear forms

$$a_0(u,v) := \int_\Omega \sum_{j=1}^{N}\Big(\sum_{i=1}^{N} a_{ij}D_iu + b_ju\Big)D_jv \,\mathrm{d}x + \int_\Omega \Big(\sum_{i=1}^{N} c_iD_iu + du\Big)v \,\mathrm{d}x \tag{2.3}$$

and

$$a_\beta(u,v) := a_0(u,v) + \int_{\partial\Omega} \beta uv \,\mathrm{d}\sigma \tag{2.4}$$

for u and v in $H^1(\Omega)$, where $H^1(\Omega)$ refers to the Sobolev space of all functions in $L^2(\Omega)$ whose first derivative also lies in $L^2(\Omega)$.

We introduce and compare various notions for a solution of $(P_{u_0,f,g})$, which are based on the observation that on a formal level the divergence theorem gives

$$a_0(u,v) = \int_{\partial\Omega} \frac{\partial u}{\partial \nu_A} v \,\mathrm{d}\sigma - \int_\Omega Auv \,\mathrm{d}x \tag{2.5}$$

for all $v \in H^1(\Omega)$. A *weak solution* is now defined by testing against a smooth function and formally integrating by parts.

Definition 2.1. We say that a function $u \in \mathrm{C}([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ is a *weak solution of* $(P_{u_0,f,g})$ *on* $[0,T]$ for some $T > 0$ if

$$\tag{2.6} \begin{aligned} -\int_0^T\int_\Omega u(s)\psi_t(s)\,\mathrm{d}x\,\mathrm{d}s + \int_0^T a_\beta(u(s),\psi(s))\,\mathrm{d}s \\ = \int_\Omega u_0\psi(0)\,\mathrm{d}x + \int_0^T\int_\Omega f(s)\psi(s)\,\mathrm{d}x\,\mathrm{d}s + \int_0^T\int_{\partial\Omega} g(s)\psi(s)\,\mathrm{d}\sigma\,\mathrm{d}s \end{aligned}$$

for all $\psi \in H^1(0,T;H^1(\Omega))$ that satisfy $\psi(T) = 0$.

We say that a function $u\colon [0,\infty) \to L^2(\Omega)$ is a *weak solution of* $(P_{u_0,f,g})$ *on* $[0,\infty)$ if for every $T > 0$ its restriction to $[0,T]$ is a weak solution on $[0,T]$.

In order to give two further definitions of a solution, we first introduce the L^2-realization A_2 of A with Robin boundary conditions, which is also based on (2.5).

Definition 2.2.

(a) Let $u \in H^1(\Omega)$. We say that $Au \in L^2(\Omega)$ if there exists a (necessarily unique) function $f \in L^2(\Omega)$ satisfying $a_0(u,\eta) = -\int_\Omega f\eta\,\mathrm{d}x$ for all $\eta \in H^1_0(\Omega)$. In this case we define $Au := f$.

(b) Let $u \in H^1(\Omega)$ satisfy $Au \in L^2(\Omega)$. We say that $\partial u/\partial\nu_A \in L^2(\Omega)$ if there exists a (necessarily unique) function $g \in L^2(\partial\Omega)$ which satisfies the relation $a_0(u,\eta) = \int_{\partial\Omega} g\eta\,\mathrm{d}\sigma - \int_\Omega Au\eta\,\mathrm{d}x$ for all $\eta \in H^1(\Omega)$. In this case we define $\partial u/\partial\nu_A := g$.

(c) We define the operator A_2 on the space $L^2(\Omega) \times L^2(\partial\Omega)$ by

$$\begin{aligned} D(A_2) &:= \Big\{(u,0)\colon\ u \in H^1(\Omega),\ Au \in L^2(\Omega),\ \frac{\partial u}{\partial\nu_A} \in L^2(\partial\Omega)\Big\} \\ A_2(u,0) &:= \Big(Au,\ -\frac{\partial u}{\partial\nu_A} - \beta u|_{\partial\Omega}\Big). \end{aligned}$$

Remark 2.3. It is easily checked that $(u,0) \in D(A_2)$ with $-A_2(u,0) = (f,g)$ if and only if

$$a_\beta(u,v) = \int_\Omega fv\,\mathrm{d}x + \int_{\partial\Omega} gv\,\mathrm{d}\sigma$$

for all $v \in H^1(\Omega)$.

It is an exercise in applying Hölder's inequality, the Sobolev embedding theorems and Young's inequality to prove that there exists $\omega \geqslant 0$ such that

$$\tag{2.7} a_\beta(u,u) \geqslant \frac{\mu}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x - \omega\int_\Omega |u|^2\,\mathrm{d}x$$

for all $u \in H^1(\Omega)$. We leave the verification to the reader.

Next we collect a few facts about A_2.

Lemma 2.4. *The operator A_2 is resolvent positive. More precisely, the operator $\lambda - A_2\colon D(A_2) \to L^2(\Omega) \times L^2(\partial\Omega)$ is invertible for all $\lambda > \omega$, where ω is as in (2.7), and if $A_2(u,0) = (f,g)$ with non-negative functions $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$, then $u \geqslant 0$ almost everywhere. Moreover, if $D(A_2)$ is equipped with the graph norm, then $D(A_2)$ is continuously embedded into $H^1(\Omega) \times \{0\}$.*

Proof. Let ω be as in (2.7) and fix $\lambda > \omega$. Then

$$\lambda \int_\Omega |u|^2 \, \mathrm{d}x + a_\beta(u,u) \geqslant \alpha \|u\|^2_{H^1(\Omega)} \tag{2.8}$$

for all $u \in H^1(\Omega)$ with $\alpha := \min\{\lambda - \omega, \mu/2\} > 0$. Hence by the Lax-Milgram theorem [16], §5.8, for every $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ there exists a unique function $u \in H^1(\Omega)$ such that

$$\lambda \int_\Omega uv \, \mathrm{d}x + a_\beta(u,v) = \int_\Omega fv \, \mathrm{d}x + \int_{\partial\Omega} gv \, \mathrm{d}\sigma \tag{2.9}$$

for all $v \in H^1(\Omega)$. By Remark 2.3 this means precisely that there is a unique function $u \in H^1(\Omega)$ with $(u,0) \in D(A_2)$ and

$$(\lambda - A_2)(u,0) = (\lambda u, 0) - A_2(u,0) = (f,g).$$

We have seen that $\lambda - A_2\colon D(A_2) \to L^2(\Omega) \times L^2(\partial\Omega)$ is a bijection for $\lambda > \omega$. Assume now that $f \leqslant 0$ and $g \leqslant 0$. Let $(u,0) := (\lambda - A_2)^{-1}(f,g)$ and set $v := u^+ = u\mathbb{1}_{\{u>0\}}$. Then

$$D_j v = D_j u \mathbb{1}_{\{u>0\}} \qquad \text{and} \qquad v|_{\partial\Omega} = u|_{\partial\Omega}\mathbb{1}_{\{u|_{\partial\Omega}>0\}}$$

and hence

$$0 \geqslant \int_\Omega fv \, \mathrm{d}x + \int_{\partial\Omega} gv \, \mathrm{d}\sigma = \lambda \int_\Omega uv \, \mathrm{d}x + a_\beta(u,v) = \lambda \int_\Omega |v|^2 \, \mathrm{d}x + a_\beta(v,v) \geqslant 0$$

by (2.9). By (2.8) this shows that $v = 0$, i.e., $u \leqslant 0$ almost everywhere. We have shown that the resolvent $(\lambda - A_2)^{-1}$ is a positive operator. Since every positive operator is continuous [4] we deduce that $\lambda - A_2$ is in fact invertible.

In particular we have proved that A_2 is closed. Hence $D(A_2)$ is a Banach space for the graph norm of A_2, and by definition of A_2 we have $D(A_2) \subset H^1(\Omega) \times \{0\}$. Since both of these spaces are continuously embedded into $L^2(\Omega) \times L^2(\partial\Omega)$, we deduce from the closed graph theorem that $D(A_2)$ is continuously embedded into $H^1(\Omega) \times \{0\}$. $\square$

We always equip $D(A_2)$ with the graph norm.

Now we can define mild and classical solutions of $(P_{u_0,f,g})$. The definition of a *classical solution* is obtained by writing $(P_{u_0,f,g})$ in terms of A_2 in a straightforward way, assuming smoothness in the time variable. The definition of a *mild solution* is similar, but uses an integrated form of the equation. These two notions are the most common ones in the study of abstract Cauchy problems.

Definition 2.5. Let $I = [0, T]$ for some $T > 0$, or let $I = [0, \infty)$.

(a) We say that a function u is a *classical L^2-solution of $(P_{u_0,f,g})$ on I* if u is in $\mathrm{C}^1(I; L^2(\Omega))$, $u(0) = u_0$, the mapping $t \mapsto (u(t), 0)$ is in $\mathrm{C}(I; D(A_2))$ and the relation

$$(u_t(t), 0) - A_2(u(t), 0) = (f(t), g(t)) \tag{2.10}$$

holds for all $t \in I$.

(b) We say that a function u is a *mild L^2-solution of $(P_{u_0,f,g})$ on I* if $u \in \mathrm{C}(I; L^2(\Omega))$, $(\int_0^t u(s)\,\mathrm{d}s, 0) \in D(A_2)$ for all $t \geqslant 0$ and

$$(u(t) - u_0, 0) - A_2\Big(\int_0^t u(s)\,\mathrm{d}s, 0\Big) = \Big(\int_0^t f(s)\,\mathrm{d}s, \int_0^t g(s)\,\mathrm{d}s\Big) \tag{2.11}$$

for all $t \geqslant 0$.

It will turn out later that weak solutions and mild L^2-solutions are in fact the same. Let us start with an easy relationship between the three notions of a solution.

Theorem 2.6. *Let either $I = [0, T]$ with $T > 0$ or $I = [0, \infty)$.*

(a) *Every classical L^2-solution of $(P_{u_0,f,g})$ on I is a weak solution on I.*

(b) *Every weak solution of $(P_{u_0,f,g})$ on I is a mild L^2-solution on I.*

Proof. All three definitions depend only on the behavior of u on bounded intervals, so it suffices to consider the case $I = [0, T]$.

(a) Let u be a classical L^2-solution. Then $u \in \mathrm{C}([0, T]; H^1(\Omega))$ by Lemma 2.4, which shows that u has the regularity requested in Definition 2.1. Let ψ be in $H^1(0, T; H^1(\Omega))$ and satisfy $\psi(T) = 0$. From (2.10) and Remark 2.3 we obtain that

$$\int_\Omega u_t(t)\psi(t) + a_\beta(u(t), \psi(t)) = \int_\Omega f(t)\psi(t)\,\mathrm{d}x + \int_{\partial\Omega} g(t)\psi(t)\,\mathrm{d}\sigma$$

for all $t \in [0, T]$. Integrating over $[0, T]$ and integrating the first summand by parts gives (2.6).

(b) Let u be a weak solution. Fix functions $\varphi \in H^1(0,T)$ and $\eta \in H^1(\Omega)$, where $\varphi(T) = 0$. Define $\psi(t) := \varphi(t) \cdot \eta$. Then $\psi \in H^1(0,T; H^1(\Omega))$ with $\psi(T) = 0$ and hence

$$
\begin{aligned}
-\int_0^T \Big(\int_\Omega u(s)\eta \,\mathrm{d}x\Big)\varphi_t(s)\,\mathrm{d}s = \Big(\int_\Omega u_0\eta\,\mathrm{d}x\Big)\varphi(0) \\
+ \int_0^T \Big(-a_\beta(u(s),\eta) + \int_\Omega f(s)\eta\,\mathrm{d}x + \int_{\partial\Omega} g(s)\eta\,\mathrm{d}\sigma\Big)\varphi(s)\,\mathrm{d}s
\end{aligned}
$$

by (2.6). Hence $t \mapsto \int_\Omega u(t)\eta\,\mathrm{d}x$ is weakly differentiable for all $\eta \in H^1(\Omega)$ with weak derivative

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t)\eta\,\mathrm{d}x = -a_\beta(u(s),\eta) + \int_\Omega f(s)\eta\,\mathrm{d}x + \int_{\partial\Omega} g(t)\eta\,\mathrm{d}\sigma
$$

and initial value $\int_\Omega u(0)\eta\,\mathrm{d}x = \int_\Omega u_0\eta\,\mathrm{d}x$, hence $u(0) = u_0$. We deduce that

$$
\int_\Omega u(t)\eta\,\mathrm{d}x = \int_\Omega u_0\eta\,\mathrm{d}x + \int_0^t \Big(-a_\beta(u(s),\eta) + \int_\Omega f(s)\eta\,\mathrm{d}x + \int_{\partial\Omega} g(s)\eta\,\mathrm{d}\sigma\Big)\,\mathrm{d}s
$$

for all $t \in [0,T]$ and all $\eta \in H^1(\Omega)$. Since $u \in L^2(0,T;H^1(\Omega))$ and $v \mapsto a_\beta(v,\eta)$ is a continuous linear functional on $H^1(\Omega)$, this implies that

$$
\begin{aligned}
\int_\Omega (u(t) - u_0)\eta\,\mathrm{d}x + a_\beta\Big(\int_0^t u(s)\,\mathrm{d}s, \eta\Big) \\
= \int_\Omega \Big(\int_0^t f(s)\,\mathrm{d}s\Big)\eta\,\mathrm{d}x + \int_{\partial\Omega}\Big(\int_0^t g(s)\,\mathrm{d}s\Big)\eta\,\mathrm{d}\sigma
\end{aligned}
$$

for all $\eta \in H^1(\Omega)$. Hence by Remark 2.3 the function u is a mild solution. □

We want to establish the existence of a weak solution via the theory of resolvent positive operators. Since $L^2(\Omega) \times L^2(\partial\Omega)$ is a Banach lattice with order continuous norm, the resolvent positive operator A_2 generates a once integrated semigroup, see [2], Theorem 3.11.7. This yields the following existence, uniqueness and comparison results for L^2-solutions.

Proposition 2.7. *Let $u_0 \in L^2(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$ and $g \in L^2(0,T;L^2(\partial\Omega))$ for some $T > 0$.*

(a) *Problem $(P_{u_0,f,g})$ has at most one mild L^2-solution.*

(b) *Assume that*

  (i) $(u_0, 0) \in D(A_2)$,

  (ii) $A_2(u_0,0) + (f(0), g(0)) \in D(A_2)$,

(iii) $f \in \mathrm{C}^2([0,T]; L^2(\Omega))$ *and*

(iv) $g \in \mathrm{C}^2([0,T]; L^2(\partial\Omega))$.

Then $(P_{u_0,f,g})$ *has a classical* L^2*-solution.*

(c) *Assume that* $u_0 \geqslant 0$, $f(t) \geqslant 0$ *and* $g(t) \geqslant 0$ *almost everywhere for almost every* $t \in (0,T)$. *If* u *is a mild* L^2*-solution of* $(P_{u_0,f,g})$, *then* $u(t) \geqslant 0$ *almost everywhere for every* $t \in (0,T)$.

P r o o f. By Definition 2.5 a function u is a mild (classical) L^2-solution of $(P_{u_0,f,g})$ if and only if the mapping $t \mapsto (u(t),0)$ is a mild (classical) solution of the abstract Cauchy problem associated with A_2 with inhomogeneity (f,g), confer [2], §3.1. Hence, part (b) follows from [2], Corollary 3.2.11 b.

Let $(S_1(t))_{t\geqslant 0}$ and $(S_2(t))_{t\geqslant 0}$ denote, respectively, the once and twice integrated semigroups generated by A_2. As in the proof of [2], Theorem 3.11.11, the function S_2 is convex. Since S_1 is the strong derivative of S_2, the function S_1 is non-decreasing. Under the assumptions of part (c), by [2], Lemma 3.2.9 a,

$$v(t) := S_1(t)(u_0,0) + \int_0^t S_1(s)(f(t-s), g(t-s))\,\mathrm{d}s \tag{2.12}$$

defines a function v in $\mathrm{C}^1([0,\infty); L^2(\Omega)\times L^2(\partial\Omega))$ with $(u(t),0) = v'(t)$ for all $t \geqslant 0$. Thus in order to show (c) it suffices to show that v is non-decreasing in t. Since S_1 is non-decreasing and $u_0 \geqslant 0$, the first summand on the right hand side of (2.12) is non-decraesing. For the second summand, note that with the convention that $S_1(s) := 0$ for $s < 0$ the function S_1 is non-decreasing on all of $\mathbb{R}$ since $S_1(0) = 0$, hence

$$\int_0^t S_1(s)(f(t-s), g(t-s))\,\mathrm{d}s = \int_0^\infty S_1(t-s)(f(s), g(s))\,\mathrm{d}s$$

is also non-decreasing in t. This concludes the proof of part (c).

Part (a) follows from (c) by linearity. □

Remark 2.8. The hypothesis in part (b) of Proposition 2.7 can be equivalently stated as follows: the function $u_0 \in L^2(\Omega)$ satisfies $Au_0 \in L^2(\Omega)$, $\partial u_0/\partial\nu_A \in L^2(\partial\Omega)$ and $\partial u_0/\partial\nu_A + \beta u_0 = g(0)$. Moreover, $v := Au_0 + f(0) \in L^2(\Omega)$ satisfies $Av \in L^2(\Omega)$ and $\partial v/\partial\nu_A \in L^2(\partial\Omega)$.

We want to show that for all square-integrable functions u_0, f and g we have a unique weak solution. As the first step we prove a bound for classical L^2-solutions in the norm of $\mathrm{C}([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$.

Lemma 2.9. *If u is a classical L^2-solution of $(P_{u_0,f,g})$ on $[0,T]$ for some $T>0$, then*

$$\sup_{0\leqslant t\leqslant T}\int_\Omega |u(t)|^2\,\mathrm{d}x+\int_0^T\!\!\int_\Omega|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}s \leqslant c\int_\Omega|u_0|^2\,\mathrm{d}x+c\int_0^T\!\!\int_\Omega|f(t)|^2\,\mathrm{d}x\,\mathrm{d}s+c\int_0^T\!\!\int_{\partial\Omega}|g(t)|^2\,\mathrm{d}\sigma\,\mathrm{d}s \tag{2.13}$$

for a constant $c\geqslant 0$ that depends only on T, Ω and the values μ and ω in (2.7).

Proof. Let $t\in[0,T]$ be arbitrary. Then

$$\begin{aligned}
&\frac12\int_\Omega|u(t)|^2\,\mathrm{d}x-\frac12\int_\Omega|u_0|^2\,\mathrm{d}x\\
&\quad=\frac12\int_0^t\frac{\mathrm{d}}{\mathrm{d}s}\int_\Omega|u(s)|^2\,\mathrm{d}x\,\mathrm{d}s=\int_0^t\!\!\int_\Omega u(s)u_t(s)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad=\int_0^t\!\!\int_\Omega u(s)(Au(s)+f(s))\,\mathrm{d}x\,\mathrm{d}s\\
&\quad=\int_0^t\!\!\int_{\partial\Omega}\frac{\partial u(s)}{\partial\nu_A}u(s)\,\mathrm{d}\sigma\,\mathrm{d}s-\int_0^t a_0(u(s),u(s))\,\mathrm{d}s+\int_0^t\!\!\int_\Omega f(s)u(s)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad=\int_0^t\!\!\int_\Omega f(s)u(s)\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\!\!\int_{\partial\Omega}g(s)u(s)\,\mathrm{d}\sigma\,\mathrm{d}s-\int_0^t a_\beta(u(s),u(s))\,\mathrm{d}s\\
&\quad\leqslant\frac12\int_0^t\!\!\int_\Omega|f(s)|^2\,\mathrm{d}x\,\mathrm{d}s+\frac{1}{4\varepsilon}\int_0^t\!\!\int_{\partial\Omega}|g(s)|^2\,\mathrm{d}\sigma\,\mathrm{d}s\\
&\qquad-\Big(\frac{\mu}{2}-\varepsilon c_1^2\Big)\int_0^t\!\!\int_\Omega|\nabla u(s)|^2\,\mathrm{d}x\,\mathrm{d}s+\Big(\omega+\frac12+\varepsilon c_1^2\Big)\int_0^t\!\!\int_\Omega|u(s)|^2\,\mathrm{d}x\,\mathrm{d}s,
\end{aligned}$$

where we have used Young's inequality and (2.7). Here $c_1\geqslant 0$ is the norm of the trace operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$. We pick $\varepsilon:=\mu/(4c_1^2)$ and vary over t to deduce that

$$\begin{aligned}
&\sup_{0\leqslant s\leqslant t}\int_\Omega|u(s)|^2\,\mathrm{d}x+\int_0^t\!\!\int_\Omega|\nabla u(s)|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad\leqslant c_2\int_\Omega|u_0|^2\,\mathrm{d}x+c_2\int_0^t\!\!\int_\Omega|f(s)|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad+c_2\int_0^t\int_{\partial\Omega}|g(s)|^2\,\mathrm{d}\sigma\,\mathrm{d}s+c_2\int_0^t\!\!\int_\Omega|u(s)|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad\leqslant c_2\int_\Omega|u_0|^2\,\mathrm{d}x+c_2\int_0^t\!\!\int_\Omega|f(s)|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad+c_2\int_0^t\!\!\int_{\partial\Omega}|g(s)|^2\,\mathrm{d}\sigma\,\mathrm{d}s+tc_2\sup_{0\leqslant s\leqslant t}\int_\Omega|u(s)|^2\,\mathrm{d}x
\end{aligned}$$

for all $t \in [0,T]$ with a constant $c_2 \geqslant 0$ that depends only on c_1, μ and ω. This shows that with $t_0 := 1/(2c_2)$ we have

$$\begin{aligned}\sup_{0\leqslant s\leqslant t}\int_\Omega |u(s)|^2\,\mathrm{d}x + \int_0^t\!\!\int_\Omega |\nabla u(s)|^2\,\mathrm{d}x\,\mathrm{d}s \\ \leqslant 2c_2\int_\Omega |u_0|^2\,\mathrm{d}x + 2c_2\int_0^t\!\!\int_\Omega |f(s)|^2\,\mathrm{d}x\,\mathrm{d}s + 2c_2\int_0^t\!\!\int_{\partial\Omega} |g(s)|^2\,\mathrm{d}\sigma\,\mathrm{d}s\end{aligned}$$

for all $t \in [0,t_0]$. We split $[0,T]$ into finitely many intervals of length at most s_0 and apply the last inequality successively on these intervals. This gives (2.13). $\square$

We also collect some results about the homogeneous problem $(P_{u_0,0,0})$ for later use. To this end we introduce the generator $A_{2,h}$ for the homogeneous problem, which is the part of A_2 in $L^2(\Omega)\times\{0\}$. All of the following results stem from the semigroup theory.

Proposition 2.10. *The operator $A_{2,h}$ given by*

$$\begin{aligned}D(A_{2,h}) &= \Big\{u \in H^1(\Omega)\colon\ Au \in L^2(\Omega),\ \frac{\partial u}{\partial \nu_A} + \beta u = 0\Big\},\\ A_{2,h}u &= Au\end{aligned}$$

is the generator of an analytic C_0*-semigroup* $(T_{2,h}(t))_{t\geqslant 0}$ *on* $L^2(\Omega)$*. Given* $u_0 \in L^2(\Omega)$*, the function* u *defined by* $u(t) := T_{2,h}(t)u_0$ *is the unique mild* L^2*-solution of* $(P_{u_0,0,0})$*, and we have the following properties:*

(i) *There exist* $M \geqslant 0$ *and* $\omega \in \mathbb{R}$ *depending only on* N, Ω *and the coefficients of the equation such that* $\|u(t)\|_{L^\infty(\Omega)} \leqslant M\mathrm{e}^{\omega t}\|u_0\|_{L^\infty(\Omega)}$ *for all* $t \geqslant 0$.

(ii) *For every* $t > 0$ *we have* $u(t) \in \mathrm{C}(\overline{\Omega})$.

(iii) *If* $u_0 \in \mathrm{C}(\overline{\Omega})$*, then* $u \in \mathrm{C}([0,\infty);\mathrm{C}(\overline{\Omega}))$ *for all* $T > 0$.

Proof. The operator $-A_{2,h}$ is associated with the bounded, $L^2(\Omega)$-elliptic bilinear form $a_\beta\colon H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$ defined in (2.4). Hence $A_{2,h}$ generates an analytic C_0-semigroup on $L^2(\Omega)$, see [11], Proposition XVII.A.6.3. By construction a function u is a mild solution for the abstract Cauchy problem associated with $A_{2,h}$ if and only if it is a mild L^2-solution of $(P_{u_0,0,0})$, which proves the assertion about the mild L^2-solutions [2], Theorem 3.1.12. Property (i) follows from [10], Proposition 7.1. Properties (ii) and (iii) were proved in [23], Theorem 4.3, for bounded coefficients. The same arguments work here, but compare also [24], [22], where unbounded (and nonlinear) coefficients are considered. $\square$

The following is our main existence theorem.

Theorem 2.11. *Let $u_0 \in L^2(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$ and $g \in L^2(0,T;L^2(\partial\Omega))$ be given, where $T > 0$ is arbitrary. Then there exists a weak solution u of $(P_{u_0,f,g})$ on $[0,T]$, which is unique even within the class of mild L^2-solutions.*

Proof. Pick sequences $(f_n) \subset \mathrm{C}^2([0,T];L^2(\Omega))$ and $(g_n) \subset \mathrm{C}^2([0,T];L^2(\partial\Omega))$ that satisfy $f_n(0) = 0$, $g_n(0) = 0$, $f_n \to f$ in $L^2(0,T;L^2(\Omega))$ and $g_n \to g$ in $L^2(0,T;L^2(\partial\Omega))$. Since $A_{2,h}$ is the generator of a C_0-semigroup, there exists a sequence $(u_{n,0}) \subset D(A_{2,h}^2)$ satisfying $u_{n,0} \to u_0$ in $L^2(\Omega)$, see [15], Proposition II.1.8. By Proposition 2.7 there exists a classical L^2-solution u_n of $(P_{u_{n,0},f_n,g_n})$.

By Lemma 2.9 the sequence (u_n) is Cauchy in $\mathrm{C}([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$. Denote its limit by u. Using that u_n is a weak solution of $(P_{u_{n,0},f_n,g_n})$ by Theorem 2.6, we can pass in (2.6) to the limit and obtain that u is a weak solution of $(P_{u_0,f,g})$. Uniqueness has already been asserted in Proposition 2.7. $\square$

Since being a solution is a local concept, we obtain the following corollary.

Corollary 2.12. *For given functions $u_0 \in L^2(\Omega)$, $f \in L^2_{\mathrm{loc}}([0,\infty);L^2(\Omega))$ and $g \in L^2_{\mathrm{loc}}([0,\infty);L^2(\partial\Omega))$, equation $(P_{u_0,f,g})$ has a weak solution on $[0,\infty)$, which is unique even within the class of mild solutions.*

We deduce the following from Theorem 2.6 and Theorem 2.11 or Corollary 2.12.

Corollary 2.13. *For problem $(P_{u_0,f,g})$ the notions of weak and mild solutions coincide.*

Let us have a glance on the regularity of the weak solution that exists by Theorem 2.11. We can ask whether for all $u_0 \in L^2(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$ and $g \in L^2(0,T;L^2(\partial\Omega))$ the weak solution u of $(P_{u_0,f,g})$ is in fact a strong solution, i.e., $u \in H^1(0,T;L^2(\Omega))$ or, equivalently, $t \mapsto (u(t),0)$ is in $L^2(0,T;D(A_2))$. In other words, we ask whether $(P_{u_0,f,g})$ has maximal parabolic regularity.

One might expect maximal regularity at first because for $g = 0$ all weak solutions are strong solutions, see [13], Theorem 4.1. We show, however, that for general g the solution u is not in $H^1(0,T;L^2(\Omega))$.

Proposition 2.14. *For every $T > 0$ there exists $g \in L^2(0,T;L^2(\partial\Omega))$ such that the weak solution u of $(P_{0,0,g})$ is not in $H^1(0,T;L^2(\Omega))$.*

Proof. Assume the converse. Since $(P_{0,f,0})$ has a strong solution for all $f \in L^2(0,T;L^2(\Omega))$ by [13], Theorem 4.1, we thus deduce that $(P_{0,f,g})$ has a unique strong solution for every $f \in L^2(0,T;L^2(\Omega))$ and every $g \in L^2(0,T;L^2(\partial\Omega))$. Now the proof of [13], Theorem 2.2, shows that

$$\sup_{\lambda \geqslant \lambda_0} \|\lambda(\lambda - A_2)^{-1}\| < \infty \tag{2.14}$$

for some $\lambda_0 > 0$, for which we only have to note that the proof still works for non-densely defined operators. But by [2], Proposition 3.3.8, estimate (2.14) contradicts the fact that A_2 is not densely defined. $\square$

Remark 2.15. We can be more precise in Proposition 2.14 by relating the existence of strong solutions to the membership of g in some trace space. Namely, $(P_{u_0,f,g})$ has a strong solution if and only if there exists $G \in H^1(0,T;L^2(\Omega))$ such that $t \mapsto (G(t),0)$ is in $L^2(0,T;D(A_2))$ and satisfies $A_2(G(t),0) = (AG(t),-g(t))$. To see this, let G be a function with this property and let u denote the weak solution of $(P_{u_0,f,g})$. Then $u-G$ is a weak solution of $(P_{u_0-G(0),f+AG-G_t,0})$ and hence a strong solution of this problem thanks to maximal regularity in the case of homogeneous boundary conditions. Thus u is in $H^1(0,T;L^2(\Omega))$ and $t \mapsto (u(t),0)$ is in $L^2(0,T;D(A_2))$, i.e., u is a strong solution of $(P_{u_0,f,g})$. On the other hand, if u is a strong solution of $(P_{u_0,f,g})$, we may set $G := u$.

3. Regularity

The goal of this section is to show that for $u_0 \in \mathrm{C}(\overline{\Omega})$ the weak solution of $(P_{u_0,f,g})$ is continuous on the parabolic cylinder $[0,\infty) \times \overline{\Omega}$, so in particular continuous up to the boundary. The main tool is the following pointwise a priori estimate, which we will use also for the study of the asymptotic behavior.

Proposition 3.1. *Fix $T > 0$. Let $r_1, r_2, q_1, q_2 \in [2,\infty)$ satisfy*

$$\frac{1}{r_1} + \frac{N}{2q_1} < 1 \qquad \text{and} \qquad \frac{1}{r_2} + \frac{N-1}{2q_2} < \frac{1}{2}. \tag{3.1}$$

Let $u_0 \in L^2(\Omega)$, $f \in L^{r_1}(0,T;L^{q_1}(\Omega))$ and $g \in L^{r_2}(0,T;L^{q_2}(\partial\Omega))$ be given and denote by u the weak solution of $(P_{u_0,f,g})$. Then

$$\begin{aligned} \|u\|^2_{L^\infty(T/2,T;L^\infty(\Omega))} \leqslant{}& c\|u\|^2_{L^2(0,T;L^2(\Omega))} \\ &+ c\|f\|^2_{L^{r_1}(0,T;L^{q_1}(\Omega))} + c\|g\|^2_{L^{r_2}(0,T;L^{q_2}(\partial\Omega))}, \end{aligned} \tag{3.2}$$

where c depends only on T, N, Ω, r_1, q_1, r_2, q_2 and the coefficients of the equation.

If we have $u_0 = 0$, then we obtain the global estimate

$$\begin{aligned} \|u\|^2_{L^\infty(0,T;L^\infty(\Omega))} \leqslant{}& c\|u\|^2_{L^2(0,T;L^2(\Omega))} \\ &+ c\|f\|^2_{L^{r_1}(0,T_0;L^{q_1}(\Omega))} + c\|g\|^2_{L^{r_2}(0,T_0;L^{q_2}(\partial\Omega))}. \end{aligned} \tag{3.3}$$

The proof of Proposition 3.1 is lengthy and technical. We postpone it to Appendix A in order not to interrupt the train of thought. We will use mainly the following consequence of Proposition 3.1, which arises from combining it with Proposition 2.10.

Theorem 3.2. *Let $T > 0$ be arbitrary, let f and g satisfy the conditions of Proposition* 3.1 *and let $u_0 \in L^\infty(\Omega)$ be given. Then the weak solution u of $(P_{u_0,f,g})$ satisfies*

$$\|u\|_{L^\infty(0,T;L^\infty(\Omega))} \leqslant c\|u_0\|_{L^\infty(\Omega)} + c\|f\|_{L^{r_1}(0,T;L^{q_1}(\Omega))} + c\|g\|_{L^{r_2}(0,T;L^{q_2}(\Omega))}, \tag{3.4}$$

where c depends on the same parameters as in Proposition 3.1.

Proof. By linearity we have $u(t) = T_{2,h}(t)u_0 + v(t)$, where $(T_{2,h}(t))_{t\geqslant 0}$ has been introduced in Proposition 2.10 and v is the weak solution of $(P_{0,f,g})$. Hence we deduce from (3.3) and Proposition 2.10 that

$$\begin{aligned}\|u\|^2_{L^\infty(0,T;L^\infty(\Omega))} &\leqslant 2\sup_{0\leqslant t\leqslant T}\|T_{2,h}(t)u_0\|^2_{L^\infty(\Omega)} + 2\|v\|^2_{L^\infty(0,T;L^\infty(\Omega))}\\ &\leqslant 2M^2\mathrm{e}^{2|\omega|T}\|u_0\|^2_{L^\infty(\Omega)} + 2c\|v\|^2_{L^2(0,T;L^2(\Omega)}\\ &\quad + 2c\|f\|^2_{L^{r_1}(0,T;L^{q_1}(\Omega))} + 2c\|g\|^2_{L^{r_2}(0,T;L^{q_2}(\Omega))}.\end{aligned}$$

In addition, by Lemma 2.9 and Hölder's inequality we have

$$\|v(s)\|^2_{L^2(\Omega)} \leqslant c\|u_0\|^2_{L^\infty(\Omega)} + c\|f\|^2_{L^{r_1}(0,T;L^{q_1}(\Omega))} + c\|g\|^2_{L^{r_2}(0,T;L^{q_2}(\Omega))}$$

for all $s \in [0,T]$, where we note that by the proof of Theorem 2.11 the lemma is valid for all weak solutions, not only classical solutions. Combining these two estimates we have proved (3.4). $\square$

We use (3.4) to deduce continuity of the solution up to the boundary of the parabolic cylinder, which is our main regularity result.

Theorem 3.3. *Let $T > 0$ be arbitrary, let f and g satisfy the conditions of Proposition 3.1 and let $u_0 \in \mathrm{C}(\overline{\Omega})$ be given. Then the weak solution u of $(P_{u_0,f,g})$ is in $\mathrm{C}([0,T];\mathrm{C}(\overline{\Omega}))$. So in particular $u(t) \to u_0$ uniformly on $\overline{\Omega}$ as $t \to 0$.*

Proof. Let A_X denote the realization of A in $X := L^{q_1}(\Omega) \times L^{q_2}(\partial\Omega)$ with the same boundary conditions as A_2, i.e.,

$$\begin{aligned}D(A_X) &:= \Big\{(u,0) \in D(A_2)\colon\ Au \in L^{q_1}(\Omega),\ \frac{\partial u}{\partial\nu_A} \in L^{q_2}(\partial\Omega)\Big\},\\ A_X(u,0) &:= \Big(Au,\ -\frac{\partial u}{\partial\nu_A} - \beta u|_{\partial\Omega}\Big).\end{aligned}$$

Thus $(u,0) \in D(A_X)$ if and only if there exist $f \in L^{q_1}(\Omega)$ and $g \in L^{q_2}(\partial\Omega)$ such that u solves

$$\begin{cases} Au = f \text{ on } \Omega, \\ \dfrac{\partial u}{\partial \nu_A} + \beta u = g \text{ on } \partial\Omega \end{cases}$$

in the weak sense. Since by (3.1) we have in particular $q_1 > N/2$ and $q_2 > (N-1)/2$, the elliptic regularity theory shows that in this case $u \in \mathrm{C}(\overline{\Omega})$, compare [23], Theorem 3.14, for bounded coefficients or [24], Example 4.2.7, for the general case. Hence $D(A_X) \subset \mathrm{C}(\overline{\Omega}) \times \{0\}$ and in particular $D(A_X) \subset X$. Hence A_X is the part of the resolvent positive operator A_2 in X, and hence is resolvent positive. Thus A_X generates a once integrated semigroup on X by [2], Theorem 3.11.7.

Pick sequences $(f_n) \subset \mathrm{C}^2([0,T]; L^\infty(\Omega))$ and $(g_n) \subset \mathrm{C}^2([0,T]; L^\infty(\partial\Omega))$ that satisfy $f_n(0) = 0$, $g_n(0) = 0$, $f_n \to f$ in $L^{r_1}(0,T; L^{q_1}(\Omega))$ and $g_n \to g$ in $L^{r_2}(0,T; L^{q_2}(\partial\Omega))$, and let v_n denote the weak solution of (P_{0,f_n,g_n}).

By [2], Corollary 3.2.11, the abstract Cauchy problem

$$\begin{cases} \dot{W}_n(t) = A_X W_n(t) + (f_n(t), g_n(t)), \\ W(0) = (0,0) \end{cases}$$

has a unique solution $W_n = (w_n, 0) \in \mathrm{C}^1([0,T]; X) \cap \mathrm{C}([0,T]; D(A_X))$, and in particular we have $w_n \in \mathrm{C}([0,T]; \mathrm{C}(\overline{\Omega}))$; we could call w_n a *classical X-solution of* (P_{0,f_n,g_n}) in analogy to Definition 2.5. The function w_n is in particular a classical L^2-solution of (2.5), hence $w_n = v_n$ by uniqueness. We have shown that $v_n \in \mathrm{C}([0,T]; \mathrm{C}(\overline{\Omega}))$.

Now, since by Theorem 3.2 we have $v_n \to v$ uniformly on $[0,T] \times \overline{\Omega}$, where v denotes the weak solution of $(P_{0,f,g})$, we deduce that $v \in \mathrm{C}([0,T]; \mathrm{C}(\overline{\Omega}))$. Hence, since $u(t) = T_{2,h}(t)u_0 + v(t)$ with $(T_{2,h}(t))_{t \geqslant 0}$ defined in Proposition 2.10, continuity of u follows from Proposition 2.10. $\square$

Remark 3.4. If in Theorem 3.3 we only have $u_0 \in L^2(\Omega)$ instead of $u_0 \in \mathrm{C}(\overline{\Omega})$, we still obtain that $u|_{[t_0,T]} \in \mathrm{C}([t_0,T]; \mathrm{C}(\overline{\Omega}))$ for all $t_0 \in (0,T)$. In fact, this can be seen easily from the proof since by Proposition 2.10, $t \mapsto T_{2,h}(t)u_0$ is continuous from $[t_0,\infty)$ to $\mathrm{C}(\overline{\Omega})$ for every $t_0 > 0$.

In particular, $u_0 \in \mathrm{C}(\overline{\Omega})$ is a necessary condition for the convergence $u(t) \to u_0$ as $t \to 0$ to be uniform on $\overline{\Omega}$. Theorem 3.3 shows that it is also sufficient if f and g do not behave too badly.

We close this section by a comparison with the situation for Dirichlet boundary conditions.

Remark 3.5. For the Dirichlet initial-boundary value problem studied in [1] one has to work with a realization $A_{c,D}$ of A with Dirichlet boundary conditions in a space of continuous functions because L^p-regularity conditions on the boundary data do not suffice in order to obtain continuous solutions, which contrasts the situation in Theorem 3.3 for Neumann boundary data. This leads to a minor difficulty. More precisely, since $\mathrm{C}(\partial\Omega)$ does not have order continuous norm, it is not immediately clear that $A_{c,D}$ is the generator of a once integrated semigroup. In fact, this is even false since if $A_{c,D}$ were the generator of a once integrated semigroup, then by [2], Corollary 3.2.11, there would exist a mild solution of the corresponding abstract Cauchy problem

$$\begin{cases} u_t(t) = \Delta u(t), \\ u(t)|_{\partial\Omega} = \varphi(t), \\ u(0) = u_0, \end{cases}$$

regardless of any compatibility assumptions between $\varphi \in \mathrm{C}^1([0,\infty);\mathrm{C}(\partial\Omega))$ and $u_0 \in \mathrm{C}(\overline{\Omega})$. This contradicts the simple observation that the existence of a mild solution enforces the condition $\varphi(0) = u_0|_{\partial\Omega}$, see [1], Proposition 3.2. Still, $A_{c,D}$ generates a twice integrated semigroup, see [2], Theorem 3.11.5, which is sufficient for the results in [1].

The situation is different for Neumann boundary conditions, as we can already expect from the fact that no compatibility condition appears in Theorem 3.3. In fact, we have a once integrated semigroup in that case. In order to see this, consider the realization A_c in $\mathrm{C}(\overline{\Omega}) \times \mathrm{C}(\partial\Omega)$ of A with Robin boundary conditions and set $Z := \mathrm{C}(\overline{\Omega}) \times \{0\}$. Then $D(A_c) \subset Z$, the space Z is invariant under the resolvent of A_c, and the part of A_c in Z is the generator of a strongly continuous semigroup, see [23], Theorem 4.3. Hence by [2], Theorem 3.10.4, the operator A_c generates a once integrated semigroup on $\mathrm{C}(\overline{\Omega}) \times \mathrm{C}(\partial\Omega)$.

4. Convergence

In this section we study boundedness of the solution u of $(P_{u_0,f,g})$ as $t \to \infty$. We are not interested in (exponential) blow-up or decay, but want to consider the border case only. Inspired by our model case, i.e., $A = \Delta$ and $\beta = 0$, a natural condition that helps with this issue is to assume conservation of total energy, i.e.,

$$(4.1) \qquad \int_\Omega u(t)\,\mathrm{d}x = \int_\Omega u_0\,\mathrm{d}x + \int_0^t\int_\Omega f(s)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\partial\Omega} g(s)\,\mathrm{d}\sigma\,\mathrm{d}s$$

for all $t > 0$. We restrict ourselves to this situation, which can be characterized as follows.

Proposition 4.1. *The following assertions are equivalent:*

(i) *for every $T > 0$, $f \in L^2(0,T;L^2(\Omega))$, $g \in L^2(0,T;L^2(\partial\Omega))$ and $u_0 \in L^2(\Omega)$ relation* (4.1) *holds for all $t \in [0,T]$, where u is the weak solution of $(P_{u_0,f,g})$;*

(ii) *for every $u_0 \in L^2(\Omega)$ we have $\int_\Omega u(t)\,\mathrm{d}x = \int_\Omega u_0\,\mathrm{d}x$ for all $t > 0$, where u is the weak solution of $(P_{u_0,0,0})$;*

(iii) *the relation*

$$\begin{cases} \operatorname{div} c = d & \text{on } \Omega, \\ c\cdot\nu = -\beta & \text{on } \partial\Omega \end{cases} \tag{4.2}$$

holds in the weak sense, i.e.,

$$\sum_{i=1}^N \int_\Omega c_i D_i\eta\,\mathrm{d}x + \int_\Omega d\eta\,\mathrm{d}x + \int_{\partial\Omega} \beta\eta\,\mathrm{d}\sigma = 0 \quad \text{for all } \eta \in H^1(\Omega).$$

Proof. Assume (iii) and let u be the weak solution of $(P_{u_0,f,g})$, which is a mild L^2-solution by Theorem 2.6. By Remark 2.3 we have

$$a_\beta\Big(\int_0^t u(s)\,\mathrm{d}s, v\Big) = \int_0^t\!\!\int_\Omega f(s)v\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_{\partial\Omega} g(s)v\,\mathrm{d}\sigma\,\mathrm{d}s - \int_\Omega (u(t)-u_0)v\,\mathrm{d}x \tag{4.3}$$

for all $v \in H^1(\Omega)$. Picking $v := \mathbb{1}_\Omega|_\Omega$ and using that by (4.2) we have $a_\beta(\eta, \mathbb{1}_\Omega|_\Omega) = 0$ for all $\eta \in H^1(\Omega)$, we obtain (4.1).

It is trivial that (i) implies (ii). So now assume that (ii) holds, i.e., that $\int_\Omega T_{2,h}(t)u_0\,\mathrm{d}x = \int_\Omega u_0\,\mathrm{d}x$ for all $t \geqslant 0$ and all $u_0 \in L^2(\Omega)$, where $(T_{2,h}(t))_{t\geqslant 0}$ is defined in Proposition 2.10. Then $\mathbb{1}_\Omega|_\Omega$ is a fixed point of the adjoint semigroup $(T^*_{2,h}(t))_{t\geqslant 0}$, which implies $A^*_{2,h}\mathbb{1}_\Omega|_\Omega = 0$, i.e., $a_\beta(\eta, \mathbb{1}_\Omega|_\Omega) = 0$ for all $\eta \in H^1(\Omega)$. This is (4.2). $\square$

We aim at a bound of the solution of $(P_{u_0,f,g})$ in $L^\infty(0,\infty;L^\infty(\Omega))$. As the first step, we consider the homogeneous problem $(P_{u_0,0,0})$.

Lemma 4.2. *Under condition* (4.2) *we have $\|u\|_{L^\infty(0,\infty;L^\infty(\Omega))} \leqslant \|u_0\|_{L^\infty(\Omega)}$ for the weak solution u of $(P_{u_0,0,0})$ if and only if*

$$\begin{cases} \sum\limits_{j=1}^N b_j = d & \text{on } \Omega, \\ \sum\limits_{j=1}^N b_j\nu_j = -\beta & \text{on } \partial\Omega \end{cases} \tag{4.4}$$

in the weak sense.

P r o o f. Relation (4.4) is equivalent to $a_\beta(\mathbb{1}_\Omega|_\Omega, \eta) = 0$ for all $\eta \in H^1(\Omega)$, i.e., $A_{2,h}\mathbb{1}_\Omega = 0$. Hence (4.4) is equivalent to $\mathbb{1}_\Omega|_\Omega$ being a fixed point of $(T_{2,h}(t))_{t\geqslant 0}$, where $(T_{2,h}(t))_{t\geqslant 0}$ is defined in Proposition 2.10.

Since $(T_{2,h}(t))_{t\geqslant 0}$ is positive, $T_{2,h}(t)\mathbb{1}_\Omega|_\Omega = \mathbb{1}_\Omega|_\Omega$ for all $t \geqslant 0$ implies that the semigroup is contractive with respect to the norm of $L^\infty(\Omega)$, which is precisely the bound for u. On the other hand, if $(T_{2,h}(t))_{t\geqslant 0}$ is $L^\infty(\Omega)$-contractive and $\int_\Omega T_{2,h}(t)u_0\,\mathrm{d}x = \int_\Omega u_0\,\mathrm{d}x$ for all $t \geqslant 0$, which is satisfied by Proposition 4.1, then $\mathbb{1}_\Omega|_\Omega$ is a fixed point of $(T_{2,h}(t))_{t\geqslant 0}$. $\square$

We will see in Corollary 4.8 that (4.4) implies that also the inhomogeneous problem $(P_{u_0,f,g})$ has bounded solutions under the additional assumption that $\int_\Omega f(t)\,\mathrm{d}x + \int_{\partial\Omega} g(t)\,\mathrm{d}\sigma = 0$ for all $t \geqslant 0$ and that the functions f and g are not too irregular. The first step in this direction is an L^2-bound on bounded time intervals, Proposition 4.4, for which we need the following lemma.

Lemma 4.3. *If* (4.2) *and* (4.4) *hold, then* $a_\beta(v,v) \geqslant \mu\int_\Omega |\nabla v|^2\,\mathrm{d}x$ *for all* $v \in H^1(\Omega)$.

P r o o f. By virtue of continuity of a_β it suffices to prove the estimate for all $v \in H^1(\Omega) \cap L^\infty(\Omega)$. For such v we have by (2.1) and the chain rule that

$$\begin{aligned} a_\beta(v,v) \geqslant \mu\int_\Omega |\nabla v|^2\,\mathrm{d}x + \frac{1}{2}\sum_{j=1}^N \int_\Omega b_j D_j(v^2)\,\mathrm{d}x + \frac{1}{2}\sum_{i=1}^N \int_\Omega c_i D_i(v^2)\,\mathrm{d}x \\ + \int_\Omega dv^2\,\mathrm{d}x + \int_{\partial\Omega} \beta v^2\,\mathrm{d}\sigma = \mu\int_\Omega |\nabla v|^2\,\mathrm{d}x, \end{aligned}$$

where in the second step we used the weak formulations of (4.2) and (4.4) with $\eta := v^2 \in H^1(\Omega)$. $\square$

Proposition 4.4. *Let u be the weak solution of $(P_{u_0,f,g})$ on $[0,T]$ for $u_0 \in L^2(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$ and $g \in L^2(0,T;L^2(\partial\Omega))$. Assume that $\int_\Omega u_0\,\mathrm{d}x = 0$ and*

$$\int_\Omega f(t)\,\mathrm{d}x + \int_{\partial\Omega} g(t)\,\mathrm{d}\sigma = 0 \quad \textit{for all } t \geqslant 0. \tag{4.5}$$

If (4.2) *and* (4.4) *hold, then there exist $\tau > 0$ and $c \geqslant 0$ depending only on μ and Ω such that*

$$\begin{aligned} \int_\Omega |u(t)|^2\,\mathrm{d}x \leqslant \mathrm{e}^{-t/\tau}\int_\Omega |u_0|^2\,\mathrm{d}x \\ + c\int_0^t \mathrm{e}^{(s-t)/\tau}\Big(\int_\Omega |f(s)|^2\,\mathrm{d}x + \int_{\partial\Omega} |g(s)|^2\,\mathrm{d}\sigma\Big)\,\mathrm{d}s \end{aligned} \tag{4.6}$$

for all $t \in [0,T]$.

P r o o f. Since u can be approximated by classical L^2-solutions of equations with right hand sides close to f and g, compare the proof of Theorem 2.11, we can assume without loss of generality that u is a classical L^2-solution of $(P_{u_0,f,g})$.

By (4.5) and Proposition 4.1 we have $\int_\Omega u(t)\,\mathrm{d}x = \int_\Omega u_0\,\mathrm{d}x = 0$ for all $t \in [0,T]$. Recall that Ω was assumed to be connected throughout the article. Hence by Poincaré's inequality and the Sobolev embedding theorems there exists $c_1 \geqslant 0$ depending only on Ω such that

$$\int_\Omega |u(t)|^2\,\mathrm{d}x + \int_{\partial\Omega} |u(t)|^2\,\mathrm{d}\sigma \leqslant c_1 \int_\Omega |\nabla u(t)|^2\,\mathrm{d}x \tag{4.7}$$

for all $t \geqslant 0$. Using Remark 2.3, Lemma 4.3, Young's inequality and estimate (4.7) we obtain that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |u(t)|^2\,\mathrm{d}x &= \int_\Omega u(t)u_t(t)\,\mathrm{d}x = \int_\Omega u(t)(Au(t)+f(t))\,\mathrm{d}x \\
&= \int_\Omega f(t)u(t)\,\mathrm{d}x + \int_{\partial\Omega} g(t)u(t)\,\mathrm{d}\sigma - a_\beta(u(t),u(t)) \\
&\leqslant \frac{c_1}{2\mu}\Big(\int_\Omega |f(t)|^2\,\mathrm{d}x + \int_{\partial\Omega} |g(t)|^2\,\mathrm{d}\sigma\Big) \\
&\quad + \frac{\mu}{2c_1}\Big(\int_\Omega |u(t)|^2\,\mathrm{d}x + \int_{\partial\Omega} |u(t)|^2\,\mathrm{d}x\Big) - \mu\int_\Omega |\nabla u(t)|^2\,\mathrm{d}x \\
&\leqslant c_2\Big(\int_\Omega |f(t)|^2\,\mathrm{d}x + \int_{\partial\Omega} |g(t)|^2\,\mathrm{d}\sigma\Big) - \frac{\mu}{2}\int_\Omega |\nabla u(t)|^2\,\mathrm{d}x,
\end{aligned}$$

with $c_2 := c_1(2\mu)$. Define $\tau := c_2/\mu$. Then by (4.7) and the above inequality

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |u(t)|^2\,\mathrm{d}x - \mathrm{e}^{-t/\tau}\frac{1}{2}\int_\Omega |u_0|^2\,\mathrm{d}x = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathrm{e}^{(s-t)/\tau}\frac{1}{2}\int_\Omega |u(s)|^2\,\mathrm{d}x\Big)\,\mathrm{d}s \\
&\quad\leqslant \frac{1}{2\tau}\int_0^t \mathrm{e}^{(s-t)/\tau}\int_\Omega |u(s)|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad + \int_0^t \mathrm{e}^{(s-t)/\tau}\Big(c_2\int_\Omega |f(s)|^2\,\mathrm{d}x + c_2\int_{\partial\Omega} |g(s)|^2\,\mathrm{d}\sigma - \frac{\mu}{2}\int_\Omega |\nabla u(s)|^2\,\mathrm{d}x\Big)\,\mathrm{d}s \\
&\quad\leqslant c_2\int_0^t \mathrm{e}^{(s-t)/\tau}\Big(\int_\Omega |f(s)|^2\,\mathrm{d}x + \int_{\partial\Omega} |g(s)|^2\,\mathrm{d}\sigma\Big)\,\mathrm{d}s,
\end{aligned}$$

where in the last step we have used that $c_2/(2\tau) = \mu/2$. $\square$

We want to find a condition on f and g which would ensure that the right hand side of (4.6) remains bounded as $t \to \infty$. To this end we introduce some function spaces.

Definition 4.5. Let r_1 and q_1 be in $[1,\infty)$, and let $T>0$. For a strongly measurable function $f\colon (0,\infty)\to L^{q_1}(\Omega)$ we define

$$R_{f,T}^{r_1,q_1}(t) := \|f|_{(t,t+T)}\|_{L^{r_1}(t,t+T;L^{q_1}(\Omega))} = \left(\int_0^\infty \|f(s)\|_{L^{q_1}(\Omega)}^{r_1} \mathbb{1}_{(t,t+T)}(s)\,\mathrm{d}s\right)^{1/r_1}$$

and introduce the spaces

$$L_m^{r_1,q_1}(\Omega) := \{f\colon (0,\infty)\to L^{q_1}(\Omega);\ R_{f,T}^{r_1,q_1}\in L^\infty(0,\infty)\}$$

and

$$L_{m,0}^{r_1,q_1}(\Omega) := \{f\in L_m^{r_1,q_1};\ \lim_{t\to\infty} R_{f,T}^{r_1,q_1}(t)=0\}$$

of uniformly mean integrable functions, where we identify functions that coincide almost everywhere. Similarly, for r_2 and q_2 in $[1,\infty)$ and $g\colon (0,\infty)\to L^{q_2}(\partial\Omega)$ we set

$$\begin{aligned}
R_{g,T}^{r_1,q_1}(t) &:= \|g|_{(t,t+T)}\|_{L^{r_2}(t,t+T;L^{q_2}(\partial\Omega))},\\
L_m^{r_2,q_2}(\partial\Omega) &:= \{g\colon (0,\infty)\to L^{q_2}(\partial\Omega);\ R_{g,T}^{r_2,q_2}\in L^\infty(0,\infty)\},\\
L_{m,0}^{r_2,q_2}(\partial\Omega) &:= \{g\in L_m^{r_2,q_2};\ \lim_{t\to\infty} R_{g,T}^{r_2,q_2}(t)=0\}.
\end{aligned}$$

Let us collect a few properties of the spaces introduced in Definition 4.5.

Lemma 4.6. *Let r_1 and q_1 be in $[1,\infty)$. Then*

(a) *for every $T>0$, the expression $\|f\|_{L_m^{r_1,q_1}(\Omega)} := \sup\limits_{t\geqslant 0} R_{f,T}^{r_1,q_1}(t)$ defines a complete norm on $L_m^{r_1,q_1}(\Omega)$;*

(b) *the norms in* (a) *are pairwise equivalent for different values of T;*

(c) *for every $f\in L_m^{r_1,q_1}(\Omega)$ and every $T>0$ the function $R_{f,T}^{r_1,q_1}$ is continuous on $[0,\infty)$;*

(d) *the space $L_{m,0}^{r_1,q_1}(\Omega)$ is a closed subspace of $L_m^{r_1,q_1}(\Omega)$;*

(e) *if $1\leqslant r_1'\leqslant r_1$ and $1\leqslant q_1'\leqslant q_1$, then*

$$L_m^{r_1,q_1}(\Omega)\subset L_m^{r_1',q_1'}(\Omega) \quad \text{and} \quad L_{m,0}^{r_1,q_1}(\Omega)\subset L_{m,0}^{r_1',q_1'}(\Omega)$$

with continuous embeddings;

(f) *we have $L^\infty(0,\infty;L^{q_1}(\Omega))\subset L_m^{r_1,q_1}(\Omega)$ and $\mathrm{C}_0([0,\infty);L^{q_1}(\Omega))\subset L_{m,0}^{r_1,q_1}(\Omega)$ with continuous embeddings;*

(g) *for $f\in L_m^{r_1,q_1}(\Omega)$ and every non-increasing function $h\in L^1(0,\infty)\cap L^\infty(0,\infty)$ we have*

$$\int_0^t h(t-s)\|f(s)\|_{L^{q_1}(\Omega)}^{r_1}\,\mathrm{d}s \leqslant \Big(\|h\|_{L^\infty(0,\infty)} + \frac{2}{T}\|h\|_{L^1(0,\infty)}\Big)\|R_{f,T}^{r_1,q_1}\|_{L^\infty(0,\infty)}^{r_1}$$

for all $T>0$ and $t\geqslant 0$;

(h) *for $f \in L^{r_1,q_1}_{m,0}(\Omega)$ and every non-increasing function $h \in L^1(0,\infty) \cap L^\infty(0,\infty)$ we have*

$$\lim_{t\to 0} \int_0^t h(t-s)\|f(s)\|^{r_1}_{L^{q_1}(\Omega)}\,\mathrm{d}s = 0.$$

Analogous assertions hold for the spaces $L^{r_2,q_2}_m(\partial\Omega)$ and $L^{r_2,q_2}_{m,0}(\partial\Omega)$ with $r_2, q_2 \in [1,\infty)$.

Part (b) justifies that we suppress the dependence on T in the notation for $L^{r_1,q_1}_m(\Omega)$ and its norm.

Proof. Part (a) is routinely checked and we leave the verification to the reader.

Now let $T > 0$ and $T' > 0$ be given and pick a natural number $n \geqslant T'/T$. Then by Hölder's inequality

$$\begin{aligned} R^{r_1,q_1}_{f,T'}(t) \leqslant R^{r_1,q_1}_{f,nT}(t) &= \Big(\sum_{k=0}^{n-1} R^{r_1,q_1}_{f,T}(t+kT)^{r_1}\Big)^{1/r_1} \\ &\leqslant \sum_{k=0}^{n-1} R^{r_1,q_1}_{f,T}(t+kT) \leqslant n \sup_{s\geqslant 0} R^{r_1,q_1}_{f,T}(s) \end{aligned}$$

for all $t \geqslant 0$, which implies (b).

By the reverse triangle inequality we have

$$|R^{r_1,q_1}_{f,T}(t+h) - R^{r_1,q_1}_{f,T}(t)| \leqslant \Big(\int_0^\infty \|f(s)\|^{r_1}_{L^{q_1}(\Omega)} |\mathbb{1}_{(t+h,t+T+h)}(s) - \mathbb{1}_{(t,t+T)}(s)|\,\mathrm{d}s\Big)^{1/r_1}.$$

Since moreover $\mathbb{1}_{(t+h,t+T+h)} \to \mathbb{1}_{(t,t+T)}$ almost everywhere as $h \to 0$, part (c) follows from the dominated convergence theorem, where as dominating function we may take $\|f\|^{r_1}_{L^{q_1}(\Omega)}\mathbb{1}_{(0,t+2T)} \in L^1(0,\infty)$.

By (c) and the definition of the norm the mapping $f \mapsto R^{r_1,q_1}_{f,T}$ is Lipschitz continuous from $L^{r_1,q_1}_m(\Omega)$ to $\mathrm{C}_b([0,\infty))$ for every $T > 0$. Hence the preimage of $\mathrm{C}_0([0,\infty))$ under this function is closed, which proves (d).

For $1 \leqslant r_1' \leqslant r_1$ and $1 \leqslant q_1' \leqslant q_1$ we obtain from Hölder's inequality that

$$R^{r_1',q_1'}_{f,T}(t) \leqslant T^{(r_1-r_1')/(r_1 r_1')}|\Omega|^{(q_1-q_1')/(q_1 q_1')} R^{r_1,q_1}_{f,T}$$

for all $t \geqslant 0$. This implies (e), and (f) is proved similarly.

For (g) let $f \in L^{r_1,q_1}_m(\Omega)$, $t > 0$ and $T > 0$ be fixed and define $n_t \in \mathbb{N}$ by $(n_t-1)T \leqslant t < n_t T$. Let $h \in L^1(0,\infty) \cap L^\infty(0,\infty)$ be non-increasing and assume without loss of generality that $h(0) = \|h\|_{L^\infty(0,\infty)}$. Since for $t \leqslant T$ the estimate in (g) is trivial, we may assume that $t \geqslant T$, i.e., $n_t \geqslant 2$. Then

$$\sum_{k=0}^{n_t-1} h\Big(\frac{(n_t-k)t}{n_t}\Big) \leqslant \frac{n_t}{t}\sum_{k=0}^{n_t-1}\int_{(n_t-k-1)t/n_t}^{(n_t-k)t/n_t} h(s)\,\mathrm{d}s \leqslant \frac{2}{T}\int_0^t h(s)\,\mathrm{d}s. \tag{4.8}$$

Moreover,

$$\int_0^t h(t-s)\|f(s)\|_{L^{q_1}(\Omega)}^{r_1}\,\mathrm{d}s \leqslant \sum_{k=1}^{n_t} h\Big(t-\frac{k}{n_t}t\Big)\int_{(k-1)t/n_t}^{kt/n_t}\|f(s)\|_{L^{q_1}(\Omega)}^{r_1}\,\mathrm{d}s \tag{4.9}$$

$$\leqslant \sum_{k=1}^{n_t} h\Big(\frac{(n_t-k)t}{n_t}\Big)\Big(R_{f,T}^{r_1,q_1}\Big(\frac{(k-1)t}{n_t}\Big)\Big)^{r_1}.$$

The estimate in (g) is an immediate consequence of (4.8) and (4.9).

Now assume in addition that $f \in L_{m,0}^{r_1,q_1}(\Omega)$. Let $\varepsilon > 0$ be given and pick $k_1 \in \mathbb{N}$ so large that $R_{f,T}^{r_1,q_1}(s)^{r_1} \leqslant \varepsilon$ for all $s \geqslant k_1 T$. Let $k_2 \in \mathbb{N}$ be so large that $h(s) \leqslant \varepsilon/(2k_1)$ for all $s \geqslant k_2 T$, set $k_0 := \max\{4k_1, 2k_2\}$ and define $t_0 := k_0 T$. Let $t \geqslant t_0$ be fixed, so $n_t \geqslant k_0$. Then for $k \leqslant 2k_1$ we have

$$\frac{(n_t-k)t}{n_t} = \Big(1-\frac{k}{n_t}\Big)t \geqslant \Big(1-\frac{2k_1}{k_0}\Big)t \geqslant \frac{t}{2} \geqslant k_2 T,$$

whereas for $k \geqslant 2k_1+1$ we have

$$\frac{(k-1)t}{n_t} \geqslant \frac{2k_1 t}{2(n_t-1)} \geqslant k_1 T.$$

Hence from (4.8) and the definitions of k_1 and k_2 we obtain for $t \geqslant k_0 T$ that

$$\sum_{k=1}^{n_t} h\Big(\frac{(n_t-k)t}{n_t}\Big)\Big(R_{f,T}^{r_1,q_1}\Big(\frac{(k-1)t}{n_t}\Big)\Big)^{r_1}$$

$$\leqslant \frac{\varepsilon}{2k_1}\sum_{k=1}^{2k_1}\Big(R_{f,T}^{r_1,q_1}\Big(\frac{(k-1)t}{n_t}\Big)\Big)^{r_1} + \varepsilon\sum_{k=2k_1+1}^{n_t} h\Big(\frac{(n_t-k)t}{n_t}\Big)$$

$$\leqslant \varepsilon(\|R_{f,T}^{r_1,q_1}\|_{L^\infty(0,\infty)}^{r_1} + h(0) + \|h\|_{L^1(0,\infty)}).$$

We have shown that

$$\lim_{t\to 0}\sum_{k=1}^{n_t} h\Big(\frac{(n_t-k)t}{n_t}\Big)\Big(R_{f,T}^{r_1,q_1}\Big(\frac{(k-1)t}{n_t}\Big)\Big)^{r_1} = 0,$$

which by (4.9) implies (h). □

We can now formulate our criterion for boundedness and convergence of solutions of $(P_{u_0,f,g})$, which together with its corollary is the main result of this section.

Theorem 4.7. *If* (4.2) *and* (4.4) *hold, then for all* $u_0 \in L^2(\Omega)$, $f \in L_m^{2,2}(\Omega)$ *and* $g \in L_m^{2,2}(\partial\Omega)$ *that satisfy* (4.5) *the weak solution* u *of* $(P_{u_0,f,g})$ *is bounded in* $L^2(\Omega)$, *and more precisely,*

$$\int_\Omega |u(t)|^2\,\mathrm{d}x \leqslant c\int_\Omega |u_0|^2\,\mathrm{d}x + c\|f\|_{L_m^{2,2}(\Omega)}^2 + c\|g\|_{L_m^{2,2}(\partial\Omega)}^2$$

for all $t \geqslant 0$ *with a constant* $c \geqslant 0$ *that depends only on* Ω *and the coefficients. If even* $f \in L^{2,2}_{m,0}(\Omega)$ *and* $g \in L^{2,2}_{m,0}(\partial\Omega)$*, then* $\lim_{t\to\infty} u(t) = |\Omega|^{-1}\int_\Omega u_0$ *in* $L^2(\Omega)$.

Proof. Write $u_0 = \hat{u}_0 + k$ with $k := |\Omega|^{-1}\int_\Omega u_0$. Then $u(t) = \hat{u}(t) + k$ by Lemma 4.2, where $\hat{u}$ denotes the weak solution of $(P_{\hat{u}_0,f,g})$. Proposition 4.4 and part (g) of Lemma 4.6 applied with $h(r) := \mathrm{e}^{-r/\tau}$ show that

$$\int_\Omega |\hat{u}(t)|^2\,\mathrm{d}x \leqslant c\int_\Omega |\hat{u}_0|^2\,\mathrm{d}x + c\|f\|^2_{L^{2,2}_m(\Omega)} + c\|g\|^2_{L^{2,2}_m(\partial\Omega)},$$

whereas part (h) shows that $\lim_{t\to\infty} \hat{u}(t) = 0$ in $L^2(\Omega)$ if $f \in L^{2,2}_{m,0}(\Omega)$ and $g \in L^{2,2}_{m,0}(\partial\Omega)$. □

Under slightly stronger assumptions on u_0, f and g we obtain even uniform boundedness and uniform convergence.

Corollary 4.8. *Let* r_1*,* q_1*,* r_2 *and* q_2 *be numbers in* $[2,\infty)$ *that satisfy* (3.1). *If* (4.2) *and* (4.4) *hold, then for all* $u_0 \in L^\infty(\Omega)$*,* $f \in L^{r_1,q_1}_m(\Omega)$ *and* $g \in L^{r_2,q_2}_m(\partial\Omega)$ *which satisfy* (4.5) *the weak solution* u *of* $(P_{u_0,f,g})$ *is bounded in* $L^\infty(\Omega)$*, and more precisely,*

$$\|u(t)\|^2_{L^\infty(\Omega)} \leqslant c\|u_0\|^2_{L^\infty(\Omega)} + c\|f\|^2_{L^{r_1,q_1}_m(\Omega)} + c\|g\|^2_{L^{r_2,q_2}_m(\partial\Omega)} \tag{4.10}$$

for all $t \geqslant 0$*. Moreover, if* $f \in L^{r_1,q_1}_{m,0}(\Omega)$ *and* $g \in L^{r_2,q_2}_{m,0}(\partial\Omega)$*, then* $\lim_{t\to\infty} u(t) = |\Omega|^{-1}\int_\Omega u_0\,\mathrm{d}x$ *in* $L^\infty(\Omega)$.

Proof. By Theorem 4.7 and part (e) of Lemma 4.6 we have

$$\|u\|^2_{L^2(\Omega)} \leqslant c\|u_0\|^2_{L^\infty(\Omega)} + c\|f\|^2_{L^{r_1,q_1}_m(\Omega)} + c\|g\|^2_{L^{r_2,q_2}_m(\partial\Omega)}.$$

On the other hand, inequality (3.2) applied to the interval $[t-2,t]$ shows that

$$\|u(t)\|^2_{L^\infty(\Omega)} \leqslant 2c\sup_{s\geqslant t-2}\|u(s)\|^2_{L^2(\Omega)} + c(R^{r_1,q_1}_{f,2}(t-2))^2 + c(R^{r_1,q_1}_{g,2}(t-2))^2 \tag{4.11}$$

for every $t \geqslant 2$. Using in addition Theorem 3.2 to bound u on $[0,2]$, we have shown (4.10).

Let now $f \in L^{r_1,q_1}_{m,0}(\Omega) \subset L^{2,2}_{m,0}(\Omega)$ and $g \in L^{r_2,q_2}_{m,0}(\partial\Omega) \subset L^{2,2}_{m,0}(\partial\Omega)$, see Lemma 4.6. Write $u(t) = \hat{u}(t) + k$ with $k := |\Omega|^{-1}\int_\Omega u_0\,\mathrm{d}x$ as in the proof of Theorem 4.7. Then $\lim_{t\to\infty}\|\hat{u}(t)\|_{L^2(\Omega)} = 0$ by Theorem 4.7. Using the definitions of $L^{r_1,q_1}_{m,0}(\Omega)$ and $L^{r_1,q_1}_{m,0}(\partial\Omega)$, this gives $\lim_{t\to\infty}\|\hat{u}(t)\|_{L^\infty(\Omega)} = 0$ by (4.11) applied to $\hat{u}$. The additional claim is proved. □

Remark 4.9. Remark 3.4 shows that if in the situation of Corollary 4.8 we only have $u_0 \in L^2(\Omega)$ instead of $u_0 \in L^\infty(\Omega)$, the assertions remain valid with the exception that u will not be bounded in $L^\infty(\Omega)$ as $t \to 0$, i.e., estimate (4.10) holds only for $t \geqslant t_0 > 0$ with a constant $c \geqslant 0$ that depends in addition on t_0.

5. Periodicity

We are going to study the periodic behavior of solutions of $(P_{u_0,f,g})$ under periodicity assumptions on f and g. This relies on the spectral theory, which is why in this section (and only in this section) we assume our Banach spaces to be complex. Thus u_0, f and g are complex-valued functions, and hence also the solution u will be complex-valued. For the theory developed in the other sections this makes no difference since we can always treat the real and the imaginary part separately as long as the coefficients of the equation are real-valued, which we still assume. Thus we will neglect this detail in the notation and reuse the symbols for the real spaces for their complex counterparts.

We start this section with a short summary on almost periodic functions in the sense of Harald Bohr, i.e., uniformly almost periodic functions. For further details and proofs we refer to [2], §4.5–4.7, or [6].

Definition 5.1. Let X be a complex Banach space. A function $f\colon (0,\infty) \to X$ is called τ-periodic (for some $\tau > 0$) if $f(t+\tau) = f(t)$ for all $t \geqslant 0$. Set $e_{\mathrm{i}\eta}(t) := \mathrm{e}^{\mathrm{i}\eta t}$ for $\eta \in \mathbb{R}$ and $t \geqslant 0$. The members of the space

$$\mathrm{AP}([0,\infty);X) := \overline{\operatorname{span}}\{e_{\mathrm{i}\eta}x\colon\ \eta \in \mathbb{R},\ x \in X\}$$

are called *uniformly almost periodic functions*, where the closure is taken in the space of bounded, uniformly continuous functions $\mathrm{BUC}([0,\infty);X)$, which is a Banach space for the uniform norm. The direct topological sum

$$\mathrm{AAP}([0,\infty);X) := \mathrm{AP}([0,\infty);X) \oplus \mathrm{C}_0([0,\infty);X) \subset \mathrm{BUC}([0,\infty);X)$$

is called the space of *uniformly asymptotically almost periodic functions.* For all $f \in \mathrm{AAP}([0,\infty);X)$ and $\eta \in \mathbb{R}$ the Cesàro limit

$$C_\eta f := \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{e}^{-\mathrm{i}\eta s} f(s)\,\mathrm{d}s$$

exists in X. We let

$$\mathrm{Freq}(f) := \{\eta \in \mathbb{R}\colon\ C_\eta f \neq 0\}$$

denote the *set of frequencies of* f. For $f \in \mathrm{AAP}([0,\infty);X)$ the set $\mathrm{Freq}(f)$ is countable. The function f can be decomposed into its frequencies in the sense that

$$f \in \overline{\operatorname{span}}\{e_{\mathrm{i}\eta}x\colon\ \eta \in \mathrm{Freq}(f),\ x \in X\} \oplus \mathrm{C}_0([0,\infty);X).$$

In particular, $f \in \mathrm{C}_0([0,\infty);X)$ if and only if $\mathrm{Freq}(f) = \emptyset$. Moreover, $\mathrm{Freq}(f) \subset 2\pi\tau^{-1}\mathbb{Z}$ if and only if there exists a τ-periodic function g such that $f - g \in \mathrm{C}_0([0,\infty);X)$.

We show that for uniformly asymptotically almost periodic data, the solution is uniformly asymptotically almost periodic with essentially the same frequencies. In fact, this is a general phenomenon for mild solutions of abstract Cauchy problems and we merely have to check the assumptions of [2], Corollary 5.6.9. We are going to improve this result later, which is why we call this preliminary result a lemma.

Lemma 5.2. *Assume* (4.2) *and* (4.4) *and let* $u_0 \in L^2(\Omega)$, $f \in \mathrm{AAP}([0,\infty);L^2(\Omega))$ *and* $g \in \mathrm{AAP}([0,\infty);L^2(\partial\Omega))$ *satisfy* (4.5). *Then the weak solution* u *of* $(P_{u_0,f,g})$ *is in* $\mathrm{AAP}([0,\infty);L^2(\Omega))$.

P r o o f. Define $u_h(t) := u(t+h)$, $f_h(t) := f(t+h)$ and $g_h(t) := g(t+h)$ for $h \geqslant 0$ and $t \geqslant 0$. Then by uniform continuity of f, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|f_h - f\|_{L^{2,2}_m(\Omega)} \leqslant \varepsilon$ holds whenever $0 \leqslant h < \delta$, see part (f) of Lemma 4.6. A similar assertion holds for g. Applying Theorem 4.7 to u and $u_h - u$, which is the weak solution of $(P_{u(h)-u(0),f_h-f,g_h-g})$, and using in addition that u is continuous by Definition 2.1 we thus obtain that $u \in \mathrm{BUC}([0,\infty);L^2(\Omega))$.

Let A_2 be as in Definition 2.2. By Lemma 2.4 the operator A_2 generates a once integrated semigroup $(S(t))_{t\geqslant 0}$ on $L^2(\Omega) \times L^2(\partial\Omega)$, see [2], Theorem 3.11.7, which by [2], Lemma 3.2.9, satisfies $S(t)(v,0) = (\int_0^t T_{2,h}(s)v\,\mathrm{d}s, 0)$ for all $v \in L^2(\Omega)$, where $(T_{2,h}(t))_{t\geqslant 0}$ is defined in Proposition 2.10. By Proposition 4.1 the closed subspace

$$X_0 := \left\{(v,0)\colon\ v \in L^2(\Omega),\ \int_\Omega v\,\mathrm{d}x = 0\right\}$$

of $L^2(\Omega) \times L^2(\partial\Omega)$ is invariant under the action of $(S(t))_{t\geqslant 0}$, which by [2], Definition 3.2.1, implies that X_0 is invariant under the resolvent of A_2. Hence for the part $A_2|_{X_0}$ of A_2 in X_0 we have $\sigma(A_2|_{X_0}) \subset \sigma(A_2)$ and in particular $\varrho(A_2|_{X_0}) \neq \emptyset$. We obtain from Lemma 2.4 and the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ that $A_2|_{X_0}$ has a compact resolvent.

We now show that $\sigma(A_2|_{X_0}) \cap \mathrm{i}\mathbb{R} = \emptyset$. Assume to the contrary that there exists $\eta \in \mathbb{R}$ such that $\mathrm{i}\eta \in \sigma_p(A_2|_{X_0}) = \sigma(A_2|_{X_0})$. Then there exists $0 \neq v_0 \in L^2(\Omega)$ satisfying $\int_\Omega v_0\,\mathrm{d}x = 0$ and $A_2(v_0,0) = (\mathrm{i}\eta v_0, 0)$. Then $v(t) := \mathrm{e}^{\mathrm{i}\eta t}v_0$ defines a classical

L^2-solution of $(P_{v_0,0,0})$. This contradicts Proposition 4.4 because $\|v(t)\|^2_{L^2(\Omega)} \not\to 0$ as $t \to \infty$.

Write $u_0 = \hat{u}_0 + k$ with $k := |\Omega|^{-1} \int_\Omega u_0 \,\mathrm{d}x$. Then $u(t) = \hat{u}(t) + k$ by Lemma 4.2, where $\hat{u}$ is the weak (and hence mild) solution of $(P_{\hat{u}_0,f,g})$. Since in addition $\int_\Omega u(t)\,\mathrm{d}x = 0$ for all $t \geqslant 0$ by Proposition 4.1, we deduce that $(u, 0)$ is a mild solution of the abstract Cauchy problem associated with $A_2|_{X_0}$ for the inhomogeneity (f, g). Since $\hat{u} \in \mathrm{BUC}([0,\infty); L^2(\Omega))$ we now obtain from [2], Corollary 5.6.9, that $\hat{u} \in \mathrm{AAP}([0,\infty); L^2(\Omega))$, which shows $u \in \mathrm{AAP}([0,\infty); L^2(\Omega))$. $\square$

Via an approximation argument we can relax the assumptions of Lemma 5.2. For this we introduce Stepanoff almost periodic functions. We omit the proofs of the implicit statements about this class of functions, which are similar to the ones for uniformly almost periodic functions. The interested reader may consult [6], §99, and [25] for the scalar-valued case.

Definition 5.3. Let X be a complex Banach space. For $r \in [1, \infty)$ the members of the space

$$\mathrm{AP}^r([0,\infty); X) := \overline{\operatorname{span}}\{e_{\mathrm{i}\eta} x\colon\ \eta \in \mathbb{R},\ x \in X\}$$

are called *Stepanoff almost periodic functions (with the exponent r)*, where the closure is taken with respect to the norm

$$\|f\|_{L^r_m(X)} := \sup_{t \geqslant 0} \left(\int_t^{t+1} \|f(s)\|_X^r \,\mathrm{d}s \right)^{1/r}.$$

The space of *Stepanoff asymptotically almost periodic functions* is defined as

$$\mathrm{AAP}^r([0,\infty); X) := \mathrm{AP}^r([0,\infty); X) \oplus L^r_{m,0}(X),$$

where we set $L^r_{m,0}(X) := \{f \in L^r_m(X)\colon\ \lim_{t\to\infty} \int_t^{t+1} \|f(s)\|^r \,\mathrm{d}s \to 0\}$. The Cesàro limit

$$C_\eta := \lim_{T\to\infty} \frac{1}{T} \int_0^T f(s)\,\mathrm{d}s$$

exists for all $\eta \in \mathbb{R}$ and $f \in \mathrm{AAP}^r([0,\infty); X)$. We define the set of frequencies of f as

$$\mathrm{Freq}(f) := \{\eta \in \mathbb{R}\colon\ C_\eta f \neq 0\}$$

and remark that $\mathrm{Freq}(f) \subset 2\pi\tau^{-1}\mathbb{Z}$ if and only if there exists a τ-periodic function g such that $f - g \in L^r_{m,0}(X)$.

Now we improve the statement of Lemma 5.2 by showing that for Stepanoff asymptotically almost periodic data we obtain uniformly asymptotically almost periodic solutions with a precise description of their frequencies. We start with the result in the L^2-framework.

Theorem 5.4. *Assume that* (4.2) *and* (4.4) *hold. We assume that* $u_0 \in L^2(\Omega)$, $f \in \mathrm{AAP}^2([0,\infty); L^2(\Omega))$ *and* $g \in \mathrm{AAP}^2([0,\infty); L^2(\partial\Omega))$ *satisfy* (4.5). *Then the weak solution* u *of* $(P_{u_0,f,g})$ *is in* $\mathrm{AAP}([0,\infty); L^2(\Omega))$. *For* $\eta \neq 0$ *we have* $\eta \in \mathrm{Freq}(u)$ *if and only if* $\eta \in \mathrm{Freq}(f) \cup \mathrm{Freq}(g)$. *Moreover,* $0 \in \mathrm{Freq}(u)$ *if and only if* $0 \in \mathrm{Freq}(f) \cup \mathrm{Freq}(g)$ *or* $\int_\Omega u_0 \,\mathrm{d}x \neq 0$.

Proof. Write $f = f_P + f_C$ with $f_P \in \mathrm{AP}([0,\infty); L^2(\Omega))$ and $f_C \in L^2_{m,0}(L^2(\Omega))$, $g = g_P + g_C$ with $g_P \in \mathrm{AP}([0,\infty); L^2(\partial\Omega))$ and $g_C \in L^2_{m,0}(L^2(\partial\Omega))$ and $u_0 = \hat{u}_0 + k$ with $k := |\Omega|^{-1} \int_\Omega u_0 \,\mathrm{d}x$. Then $u = u_P + u_C + k$ by Lemma 4.2, where u_P denotes the solution of $(P_{\hat{u}_0,f_P,g_P})$ and u_C is the solution of (P_{0,f_C,g_C}).

Pick $f_n \in \mathrm{span}\{e_{\mathrm{i}\eta} v \colon \eta \in \mathbb{R},\ v \in L^2(\Omega)\}$ and $g_n \in \mathrm{span}\{e_{\mathrm{i}\eta} w \colon \eta \in \mathbb{R},\ w \in L^2(\partial\Omega)\}$ such that $f_n \to f$ in the norm of $L^2_m(L^2(\Omega)) = L^{2;2}_m(\Omega)$ and $g_n \to g$ in the norm of $L^2_m(L^2(\partial\Omega)) = L^{2;2}_m(\partial\Omega)$. Let u_n denote the weak solution of $(P_{\hat{u}_0,f_n,g_n})$. Then $u_n \to u_P$ in $L^\infty(0,\infty; L^2(\Omega))$ by Theorem 4.7 and $u_n \in \mathrm{AAP}([0,\infty); L^2(\Omega))$ by Lemma 5.2. Hence $u_P \in \mathrm{AAP}([0,\infty); L^2(\Omega))$. Since $(u_n, 0)$ is a mild solution of the abstract Cauchy problem associated with $A_2|_{X_0}$ for the inhomogeneity (f_n, g_n), see the proof of Lemma 5.2, we obtain from [2], Proposition 5.6.7 that $C_\eta u_n = (\mathrm{i}\eta - A_2|_{X_0})^{-1}(C_\eta f_n, C_\eta g)$ for all $\eta \in \mathbb{R}$. Passing to the limit we have the relation $C_\eta u_P = (\mathrm{i}\eta - A_2|_{X_0})^{-1}(C_\eta f, C_\eta g)$. Thus $\mathrm{Freq}(u_P) = \mathrm{Freq}(f) \cup \mathrm{Freq}(g)$.

Since $u_C \in \mathrm{C}_0([0,\infty); L^2(\Omega))$ by Theorem 4.7 and $u_P(t) \perp k$ for all $t \geqslant 0$ by Proposition 4.1, we deduce that $u \in \mathrm{AAP}([0,\infty); L^2(\Omega))$ and

$$\mathrm{Freq}(u) = \mathrm{Freq}(u_P) + \mathrm{Freq}(k) = \mathrm{Freq}(f) \cup \mathrm{Freq}(g) \cup \mathrm{Freq}(k),$$

which is a different way to write down the description of $\mathrm{Freq}(u)$. $\square$

We can also obtain an analogue of Theorem 5.4 in the more regular setting of continuous solutions.

Theorem 5.5. *Let* r_1, q_1, r_2 *and* q_2 *be numbers in* $[2,\infty)$ *that satisfy relation* (3.1). *Assume that* (4.2) *and* (4.4) *hold and let* $u_0 \in L^\infty(\Omega)$, $f \in \mathrm{AAP}^{r_1}([0,\infty); L^{q_1}(\Omega))$ *and* $g \in \mathrm{AAP}^{r_2}([0,\infty); L^{q_2}(\partial\Omega))$ *satisfy* (4.5). *Then the weak solution* u *of* $(P_{u_0,f,g})$ *is in* $\mathrm{AAP}([0,\infty); L^\infty(\Omega))$. *For* $\eta \neq 0$ *we have* $\eta \in \mathrm{Freq}(u)$ *if and only if* $\eta \in \mathrm{Freq}(f) \cup \mathrm{Freq}(g)$. *Moreover,* $0 \in \mathrm{Freq}(u)$ *if and only if* $0 \in \mathrm{Freq}(f) \cup \mathrm{Freq}(g)$ *or* $\int_\Omega u_0 \,\mathrm{d}x \neq 0$. *If* $u_0 \in \mathrm{C}(\overline{\Omega})$, *then* $u \in \mathrm{AAP}([0,\infty); \mathrm{C}(\overline{\Omega}))$.

P r o o f. This theorem can be proved in precisely the same way as Theorem 5.4. We have to use Corollary 4.8 instead of Theorem 4.7 and the realization of A in $L^{q_1}(\Omega) \times L^{q_2}(\partial\Omega)$ instead of A_2 like in Theorem 3.3, where from we also obtain the continuity of u if $u_0 \in \mathrm{C}(\overline{\Omega})$. We leave the details to the reader. $\square$

As an immediate consequence of the previous two theorems, we see that for periodic data the solution is asymptotically periodic. This formulation is simpler, but we lose the precise information about the frequencies.

Corollary 5.6. *Assume that* (4.2) *and* (4.4) *hold. Fix functions* $u_0 \in L^2(\Omega)$, $f \in L^2(0,\tau;L^2(\Omega))$ *and* $g \in L^2(0,\tau;L^2(\partial\Omega))$ *for some* $\tau > 0$. *We identify* f *and* g *with their* τ*-periodic extensions to* $(0,\infty)$. *Then there exists a* τ*-periodic function* u_P *such that the weak solution* u *of* $(P_{u_0,f,g})$ *satisfies* $\lim\limits_{t\to\infty} \|u(t) - u_P(t)\|_{L^2(\Omega)} = 0$. *If* $u_0 \in \mathrm{C}(\overline{\Omega})$, $f \in L^\infty(0,\tau;L^\infty(\Omega))$ *and* $g \in L^\infty(0,\tau;L^\infty(\partial\Omega))$, *then* u *and* u_P *are in* $\mathrm{C}_b([0,\infty);\mathrm{C}(\overline{\Omega}))$ *and* $\lim\limits_{t\to\infty} \|u(t) - u_P(t)\|_{L^\infty(\Omega)} = 0$.

Appendix A. Pointwise estimates via De Giorgi's techniques

In this section we prove Proposition 3.1. The proof is similar to what can be found in [19], §III.7–8, which in turn is a refined version of De Giorgi's famous technique. We need, however, the following improvements over [19]:

(i) the presence of the inhomogeneity g in $(P_{u_0,f,g})$, makes it necessary to keep track of the measure of the sublevel sets of $u|_{\partial\Omega}$;

(ii) we need a precise dependence of the constants on f and g. More precisely, these quantities have to enter linearly into the right hand side. This is not obvious from the proofs in [19], but can be asserted after some small modifications;

(iii) we need an estimate that is local in time but global in space, whereas the results in [19] are either global in both variables or local. This requires only trivial modifications.

Another motivation to give the details is that the relevant parts in [19] contain some misprints, for example the relations between n, $\hat{r}$ and $\hat{q}$ in the proof of [19], Theorem III.7.1, as can be seen by taking $n = 2$, $r = q = 4$ and $\kappa = 1/2$.

This is another subtle mistake in the claim that the constant in [19], II.6.11, does not depend on τ_0 and ϱ_0. In fact, the explicit constant given in [19], II.6.25, still contains $\theta = \tau_0\varrho_0^{-2}$. And indeed, otherwise, we could apply estimate [19], II.6.11, to the solution u of the heat equation with initial datum $u_0 \in L^2(\mathbb{R}^N) \setminus L^\infty(\mathbb{R}^N)$ like in [19], §III.8, and deduce that given a ball $B \subset \mathbb{R}^N$ we have

$$\sup_{T/2 \leqslant t \leqslant T} \|u(t)\|^2_{L^\infty(B)} \leqslant c\|u_0\|^2_{L^2(\mathbb{R}^N)}$$

for all $T > 0$ with a constant $c \geqslant 0$ that depends only on the radius of the ball. This contradicts that $u(t) \to u_0$ in $L^2(\mathbb{R}^N)$.

For these reasons, we give a complete proof of Proposition 3.1. The only part of the argument that we copy from [19] without change is the following lemma, which is easily proved by induction.

Lemma A.1 ([19], Lemma II.5.7)**.** *Let $(y_n)_{n\in\mathbb{N}_0}$ and $(z_n)_{n\in\mathbb{N}_0}$ be sequences of non-negative real numbers such that*

$$y_{n+1} \leqslant cb^n(y_n^{1+\delta} + z_n^{1+\varepsilon} y_n^{\delta}) \quad \text{and} \quad z_{n+1} \leqslant cb^n(y_n + z_n^{1+\varepsilon})$$

for all $n \in \mathbb{N}_0$ with positive constants c, b, ε and δ, where $b \geqslant 1$. Define

$$d := \min\left\{\delta,\ \frac{\varepsilon}{1+\varepsilon}\right\} \quad \text{and} \quad \lambda := \min\{(2c)^{-1/\delta} b^{-1/(\delta d)},\ (2c)^{-(1+\varepsilon)/\varepsilon} b^{-1/(\varepsilon d)}\}$$

and assume that

$$y_0 \leqslant \lambda \quad \text{and} \quad z_0 \leqslant \lambda^{1/(1+\varepsilon)}.$$

Then

$$y_n \leqslant \lambda b^{-n/d} \quad \text{and} \quad z_n \leqslant (\lambda b^{-n/d})^{1/(1+\varepsilon)}$$

for all $n \in \mathbb{N}_0$.

We partially adopt the notation of [19] here. More precisely, let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and $T > 0$. It will be convenient to work with functions defined for negative times, so we will always assume that $u \in L^\infty(-T, 0; L^2(\Omega)) \cap L^2(-T, 0; H^1(\Omega))$. In that case we write

$$\|u\|_{Q(\tau)}^2 := \sup_{-\tau \leqslant t \leqslant 0} \int_\Omega |u(t)|^2 \,\mathrm{d}x + \int_{-\tau}^0 \int_\Omega |\nabla u(t)|^2 \,\mathrm{d}x,$$

and for $k \geqslant 0$ we define

$$u^{(k)}(t) := (u(t) - k)^+.$$

For a fixed function u, we set

$$A_k(t) := \{x \in \Omega\colon\ u(t) > k\}$$

and

$$B_k(t) := \{x \in \partial\Omega\colon\ u(t) > k\}$$

and denote by $|A_k(t)|$ and $|B_k(t)|$ their volumes with respect to the Lebesgue measure or the surface measure of $\partial\Omega$, respectively.

In what follows we will frequently need that for $r_1 \in [2,\infty]$, $q_1 \in [2, 2N/(N-2)]$, $r_2 \in [2,\infty]$ and $q_2 \in [2, 2(N-1)/(N-2)]$ satisfying

$$\frac{1}{r_1} + \frac{N}{2q_1} = \frac{N}{4} \quad \text{and} \quad \frac{1}{r_2} + \frac{N-1}{2q_2} = \frac{N}{4}$$

we have

$$\|u\|_{L^{r_1}(-\tau,0;L^{q_1}(\Omega))} + \|u\|_{L^{r_2}(-\tau,0;L^{q_2}(\partial\Omega))} \leqslant c\|u\|_{Q(\tau)}, \tag{A.1}$$

where $c \geqslant 0$ depends only on Ω, r_1, q_1, r_2 and q_2. This anisotropic Sobolev inequality follows from the multiplicative Sobolev inequalities on Ω, see [19], §II.3.

We start with a modified version of [19], Theorem II.6.2.

Theorem A.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, $N \geqslant 2$. Fix $T > 0$ and $u \in L^\infty(-T,0;L^2(\Omega)) \cap L^2(-T,0;H^1(\Omega))$. Let $r_{1,l} \in [2,\infty)$, $q_{1,l} \in [2, 2N/(N-2)]$, $r_{2,l} \in [2,\infty)$ and $q_{2,l} \in [2, 2(N-1)/(N-2)]$ satisfy*

$$\frac{1}{r_{1,l}} + \frac{N}{2q_{1,l}} = \frac{N}{4} \quad (1 \leqslant l \leqslant L_1) \quad \text{and} \quad \frac{1}{r_{2,l}} + \frac{N-1}{2q_{2,l}} = \frac{N}{4} \quad (1 \leqslant l \leqslant L_2). \tag{A.2}$$

Assume that there exist $\hat{k} \geqslant 0$, $\gamma \geqslant 0$ and numbers $\kappa_{1,l} > 0$ and $\kappa_{2,l} > 0$ such that for all $\tau \in (0,T]$, $\sigma \in (0,1/2)$ and $k \geqslant \hat{k}$ we have

$$\begin{aligned} \|u^{(k)}\|^2_{Q((1-\sigma)\tau)} \leqslant{}& \frac{\gamma}{\sigma\tau} \int_{-\tau}^0 \int_\Omega |u^{(k)}(t)|^2 \,\mathrm{d}x \,\mathrm{d}t \\ &+ \gamma k^2 \sum_{l=1}^{L_1} \Bigl(\int_{-\tau}^0 |A_k(t)|^{r_{1,l}/q_{1,l}} \,\mathrm{d}t \Bigr)^{2(1+\kappa_{1,l})/r_{1,l}} \\ &+ \gamma k^2 \sum_{l=1}^{L_2} \Bigl(\int_{-\tau}^0 |B_k(t)|^{r_{2,l}/q_{2,l}} \,\mathrm{d}t \Bigr)^{2(1+\kappa_{2,l})/r_{2,l}}. \end{aligned} \tag{A.3}$$

Then

$$\operatorname*{ess\,sup}_{(t,x)\in[-T/2,0]\times\Omega} u(t,x) \leqslant c \Bigl(\int_{-T}^0 \int_\Omega |u(t)|^2 \,\mathrm{d}x \,\mathrm{d}t + \hat{k}^2 \Bigr)^{1/2}, \tag{A.4}$$

where the constant $c \geqslant 0$ is independent of u and $\hat{k}$.

Proof. In the proof the constants c, c_0, c_1 and c_2 never depend on u and $\hat{k}$. Moreover, c is a generic constant in the sense that it may change its numeric value between occurrences.

Since $|A_k(t)| \leqslant |\Omega|$ and $|B_k(t)| \leqslant |\partial\Omega|$, estimate (A.3) remains valid if we replace all the $\kappa_{1,l}$ and $\kappa_{2,l}$ by their least member

$$\kappa := \min\{\kappa_{1,1}, \dots, \kappa_{1,L_1}, \kappa_{1,L_1}, \kappa_{2,1}, \kappa_{2,L_2}\} > 0$$

provided we replace γ by a larger constant γ' that depends on $\kappa_{1,l}$, $\kappa_{2,l}$, $r_{1,l}$, $q_{1,l}$ $r_{2,l}$, $q_{2,l}$, T, γ, $|\Omega|$ and $|\partial\Omega|$. Thus we may assume without loss of generality that $\kappa_{1,l} = \kappa$ for all $1 \leqslant l \leqslant L_1$ and $\kappa_{2,l} = \kappa$ for all $1 \leqslant l \leqslant L_2$.

Let $M \geqslant \hat{k}$ be arbitrary and define

$$\begin{aligned}
\tau_n &:= (1 + 2^{-(n+1)})\frac{T}{2} \in \Big[\frac{T}{2}, T\Big],\\
k_n &:= (2 - 2^{-n})M \geqslant \hat{k},\\
y_n &:= \frac{1}{M^2}\int_{-\tau_n}^0 \int_\Omega |u^{(k_n)}(t)|^2 \,\mathrm{d}x\,\mathrm{d}t,\\
z_n &:= \sum_{l=1}^{L_1}\Big(\int_{-\tau_n}^0 |A_{k_n}(t)|^{r_{1,l}/q_{1,l}}\,\mathrm{d}t\Big)^{2/r_{1,l}} + \sum_{l=1}^{L_2}\Big(\int_{-\tau_n}^0 |B_{k_n}(t)|^{r_{2,l}/q_{2,l}}\,\mathrm{d}t\Big)^{2/r_{2,l}}
\end{aligned}$$

for all $n \in \mathbb{N}_0$. We prove that the sequences (y_n) and (z_n) satisfy the inequalities in Lemma A.1.

To this end, let $n \in \mathbb{N}_0$ be fixed. From (A.1) and the trivial estimate

$$|u^{(k_n)}(t)|^2 \geqslant (k_{n+1} - k_n)^2 \mathbb{1}_{A_{k_{n+1}}(t)}$$

we obtain that

$$\begin{aligned}
M^2 y_{n+1} &\leqslant c\Big(\int_{-\tau_{n+1}}^0 |A_{k_{n+1}}(t)|\,\mathrm{d}t\Big)^{2/(N+2)} \|u^{(k_{n+1})}\|^2_{Q(\tau_{n+1})}\\
&\leqslant c((k_{n+1} - k_n)^{-2} M^2 y_n)^{2/(N+2)} \|u^{(k_{n+1})}\|^2_{Q(\tau_{n+1})}\\
&\leqslant c 2^{2(n+1)} y_n^{2/(N+2)} \|u^{(k_{n+1})}\|^2_{Q(\tau_{n+1})}.
\end{aligned} \tag{A.5}$$

Similarly,

$$\begin{aligned}
2^{-2(n+1)} M^2 z_{n+1} &= (k_{n+1} - k_n)^2 z_{n+1}\\
&\leqslant \sum_{l=1}^{L_1}\Big(\int_{-\tau_{n+1}}^0 \Big(\int_\Omega |u^{(k_n)}(t)|^{q_{1,l}}\,\mathrm{d}x\Big)^{r_{1,l}/q_{1,l}}\,\mathrm{d}t\Big)^{2/r_{1,l}}\\
&\quad + \sum_{l=1}^{L_2}\Big(\int_{-\tau_{n+1}}^0 \Big(\int_{\partial\Omega} |u^{(k_n)}(t)|^{q_{2,l}}\,\mathrm{d}\sigma\Big)^{r_{2,l}/q_{2,l}}\,\mathrm{d}t\Big)^{2/r_{2,l}}\\
&\leqslant c\|u^{(k_n)}\|^2_{Q(\tau_{n+1})}.
\end{aligned} \tag{A.6}$$

Moreover, from (A.3) applied with $\tau = \tau_n$ and $\sigma = 1 - \tau_{n+1}/\tau_n \geqslant 2^{-(n+3)}$ we get that

$$\begin{aligned}\|u^{(k_{n+1})}\|^2_{Q(\tau_{n+1})} \leqslant \|u^{(k_n)}\|^2_{Q(\tau_{n+1})} &\leqslant \frac{\gamma}{\sigma\tau_n} M^2 y_n + \gamma k_n^2 z_n^{1+\kappa} \\ &\leqslant \gamma M^2 2^{n+4}(T^{-1}+1)(y_n + z_n^{1+\kappa}).\end{aligned} \tag{A.7}$$

Combining (A.5), (A.6) and (A.7) we obtain with $\delta := 2/(N+2)$ that

$$\begin{cases} y_{n+1} \leqslant c_0 2^{3n}(y_n^{1+\delta} + z_n^{1+\kappa} y_n^{\delta}), \\ z_{n+1} \leqslant c_0 2^{3n}(y_n + z_n^{1+\kappa}) \end{cases} \tag{A.8}$$

for all $n \in \mathbb{N}_0$.

Next we want to estimate y_0 and z_0 for large M. On the one hand, we have

$$y_0 \leqslant \frac{1}{M^2}\int_{-T}^{0}\int_\Omega |u(t)|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{A.9}$$

On the other hand, similarly to (A.6) and (A.7), we have

$$\begin{aligned}(M-\hat{k})^2 z_0 &\leqslant \sum_{l=1}^{L_1}\biggl(\int_{-\tau_0}^{0}\biggl(\int_\Omega |u^{(\hat{k})}(t)|^{q_{1,l}}\,\mathrm{d}x\biggr)^{r_{1,l}/q_{1,l}}\mathrm{d}t\biggr)^{2/r_{1,l}} \\ &\quad + \sum_{l=1}^{L_2}\biggl(\int_{-\tau_0}^{0}\biggl(\int_{\partial\Omega} |u^{(\hat{k})}(t)|^{q_{2,l}}\,\mathrm{d}\sigma\biggr)^{r_{2,l}/q_{2,l}}\mathrm{d}t\biggr)^{2/r_{2,l}} \\ &\leqslant c\|u^{(\hat{k})}\|^2_{Q(\tau_0)} \leqslant \frac{4\gamma}{T}\int_{-T}^{0}\int_\Omega |u^{(\hat{k})}(t)|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \gamma\hat{k}^2(T|\Omega|^{r_1/q_1})^{2(1+\kappa)/r_1} + \gamma\hat{k}^2(T|\partial\Omega|^{r_2/q_2})^{2(1+\kappa)/r_2},\end{aligned}$$

so that

$$z_0 \leqslant \frac{c_1}{(M-\hat{k})^2}\biggl(\int_{-T}^{0}\int_\Omega |u(t)|^2\,\mathrm{d}x\,\mathrm{d}t + \hat{k}^2\biggr) \tag{A.10}$$

for all $M \geqslant \hat{k}$. Define $d := \min\{\delta, \kappa/(1+\kappa)\}$ and

$$\lambda := \min\{(2c_0)^{-1/\delta}2^{-3/(\delta d)}, (2c_0)^{-(1+\kappa)/\kappa}2^{-3/(\kappa d)}\}.$$

Then for

$$\begin{aligned}M := \max\biggl\{&\lambda^{-1/2}\biggl(\int_{-T}^{0}\int_\Omega |u(t)|^2\,\mathrm{d}x\,\mathrm{d}t\biggr)^{1/2}, \\ &\hat{k} + \lambda^{-1/(2(1+\kappa))}c_1^{1/2}\biggl(\int_{-T}^{0}\int_\Omega |u(t)|^2\,\mathrm{d}x\,\mathrm{d}t + \hat{k}^2\biggr)^{1/2}\biggr\} \\ \leqslant c_2\biggl(&\int_{-T}^{0}\int_\Omega |u(t)|^2\,\mathrm{d}x\,\mathrm{d}t + \hat{k}^2\biggr)^{1/2}\end{aligned} \tag{A.11}$$

we obtain from (A.9) and (A.10) that

$$y_0 \leqslant \lambda, \quad z_0 \leqslant \lambda^{1/(1+\kappa)}. \tag{A.12}$$

Estimates (A.8) and (A.12) show in view of Lemma A.1 that $z_n \to 0$ as $n \to \infty$, which implies that $u(t) \leqslant \lim\limits_{n\to\infty} k_n = 2M$ almost everywhere on Ω for almost every $t \in \bigcap\limits_{n\in\mathbb{N}} [-\tau_n, 0] = [-T/2, 0]$ if we define M as in (A.11). This is (A.4). □

Theorem A.2 is a local estimate in time, hence it allows us to estimate the solution of $(P_{u_0,f,g})$ independently of the initial value u_0. The price is that we obtain estimates only away from $t = 0$. We also need the following modification of Theorem A.2 that gives good estimates for small t.

Corollary A.3. *In the situation of Theorem* A.2, *assume that instead of* (A.3) *we even have*

$$\begin{aligned}\|u^{(k)}\|^2_{Q(T)} \leqslant \gamma \int_{-T}^0 \int_\Omega |u^{(k)}(t)|^2 \,\mathrm{d}x \,\mathrm{d}t + \gamma k^2 \sum_{l=1}^{L_1} \Big(\int_{-T}^0 |A_k(t)|^{r_{1,l}/q_{1,l}} \,\mathrm{d}t \Big)^{2(1+\kappa_{1,l})/r_{1,l}} \\ + \gamma k^2 \sum_{l=1}^{L_2} \Big(\int_{-T}^0 |B_k(t)|^{r_{2,l}/q_{2,l}} \,\mathrm{d}t \Big)^{2(1+\kappa_{2,l})/r_{2,l}}\end{aligned}$$

for all $k \geqslant \hat{k}$. Then

$$\operatorname{ess\,sup}_{t\in[-T,0], x\in\Omega} u(t,x) \leqslant c \Big(\int_0^T \int_\Omega |u(t)|^2 \,\mathrm{d}x \,\mathrm{d}t + \hat{k}^2 \Big)^{1/2}$$

for all $t \in [-T, 0]$, where the constant $c \geqslant 0$ is independent of u and $\hat{k}$.

Proof. The proof is very similar to the one of Theorem A.2. In fact, we only have to notice that after changing the definition of τ_n to $\tau_n := T$ for all $n \in \mathbb{N}$ the rest of the proof is carried over verbatim with the mere exception that this time we have $\bigcap\limits_{n\in\mathbb{N}} [-\tau_n, 0] = [-T, 0]$, which gives the result. □

Before we can check that Theorem A.2 applies to the solutions of $(P_{u_0,f,g})$, we have to supply the following tool for the calculations.

Lemma A.4. *Let $T > 0$, $u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ and $k \geqslant 0$. Define $u^{(k)}(t) := (u(t)-k)^+$ for $t \geqslant 0$. Then $u^{(k)} \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ with derivative $(u^{(k)})_t(t) = u_t(t)\mathbb{1}_{\{u(t)>k\}}$ and $\nabla u^{(k)}(t) = \nabla u(t)\mathbb{1}_{\{u(t)>k\}}$. Moreover, $u^{(k)}(t)|_{\partial\Omega} = (u|_{\partial\Omega}(t) - k)^+$.*

Proof. After identifying $H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ with $H^1((0,T)\times\Omega)$ up to equivalent norms in the obvious way, the formulas for the derivatives follow from the chain rule for weakly differentiable functions, see for example [16], Theorem 7.8. The assertion about the trace is true for continuous functions and thus by approximation for all functions under consideration. $\square$

We now prove Proposition 3.1 for classical L^2-solutions. Basically, we will check that every solution of $(P_{u_0,f,g})$ satisfies (A.3).

Lemma A.5. *Proposition 3.1 holds if in addition we assume that u is a classical L^2-solution and $T \leqslant T_0$, where $T_0 > 0$ depends only on N, Ω, r_1, q_1, r_2, q_2 and the coefficients of the equation.*

Proof. After a linear substitution in the time variable we may consider problem $(P_{u_0,f,g})$ on $[-T,0]$ instead of $[0,T]$, the initial value now being $u_0 = u(-T)$. We check the conditions of Theorem A.2 with

$$\hat{k}^2 := \|f\|^2_{L^{r_1}(-T,0;L^{q_1}(\Omega))} + \|g\|^2_{L^{r_2}(-T,0;L^{q_2}(\Omega))}. \tag{A.13}$$

Fix $0 < \tau \leqslant T$ and let ζ be a function in $H^1(-\tau,0)$ satisfying $0 \leqslant \zeta(t) \leqslant 1$ for all $t \in [-\tau,0]$. Assume either that $\zeta(-\tau) = 0$ or that $\tau = T$ and $u^{(k)}(-T) = 0$. Then for $t \in [-\tau,0]$ we have

$$\begin{aligned}\zeta(t)^2 \cdot \frac{1}{2}\int_\Omega |u^{(k)}(t)|^2 \,\mathrm{d}x &= \int_{-\tau}^t \frac{\mathrm{d}}{\mathrm{d}s}\Big(\zeta(s)^2 \cdot \frac{1}{2}\int_\Omega |u^{(k)}(s)|^2 \,\mathrm{d}x\Big)\,\mathrm{d}s \\ &= \int_{-\tau}^t \zeta(s)\zeta'(s)\int_\Omega |u^{(k)}(s)|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_{-\tau}^t \zeta(s)^2\int_\Omega u_t^{(k)}(s)u^{(k)}(s)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{A.14}$$

From Lemma A.4 and the fact that u is a classical L^2-solution of $(P_{u_0,f,g})$ we obtain that for all $s \in [-\tau,0]$ we have

$$\begin{aligned}\int_\Omega u_t^{(k)}(s)u^{(k)}(s)\,\mathrm{d}x &= \int_\Omega u_t(s)u^{(k)}(s)\,\mathrm{d}x = \int_\Omega (Au(s) + f(s))u^{(k)}(s)\,\mathrm{d}x \\ &= \int_\Omega f(s)u^{(k)}(s)\,\mathrm{d}x + \int_{\partial\Omega}\big(g(s)u^{(k)}(s) - a_\beta(u(s),u^{(k)}(s))\big)\,\mathrm{d}\sigma.\end{aligned} \tag{A.15}$$

$\square$

We now estimate the right hand side of (A.15). From Lemma A.4, (2.7) and Young's inequality we obtain that

$$
\begin{aligned}
a_\beta(u(s), u^{(k)}(s)) &= a_\beta(u^{(k)}(s), u^{(k)}(s)) + \sum_{j=1}^{N} \int_\Omega b_j k D_j u^{(k)}(s)\,\mathrm{d}x \\
&\quad + \int_\Omega d k u^{(k)}(s)\,\mathrm{d}x + \int_{\partial\Omega} \beta k u^{(k)}(s)\,\mathrm{d}\sigma \\
&\geqslant \frac{\mu}{2} \int_\Omega |\nabla u^{(k)}(s)|^2\,\mathrm{d}x - \omega \int_\Omega |u^{(k)}(s)|^2\,\mathrm{d}x \\
&\quad - \frac{k^2}{\mu} \sum_{j=1}^{N} \int_{A_k(s)} |b_j|^2\,\mathrm{d}x - \frac{\mu}{4} \int_\Omega |\nabla u^{(k)}(s)|^2\,\mathrm{d}x \\
&\quad - \int_{A_k(s)} |d|(|u^{(k)}(s)|^2 + k^2)\,\mathrm{d}x - \int_{B_k(s)} |\beta|(|u^{(k)}(s)|^2 + k^2)\,\mathrm{d}\sigma.
\end{aligned}
$$

Using (A.15) and again Young's inequality this gives

$$
\begin{aligned}
\text{(A.16)} \qquad \int_\Omega u_t^{(k)}(s) u^{(k)}(s)\,\mathrm{d}x \leqslant &-\frac{\mu}{4} \int_\Omega |\nabla u^{(k)}(s)|^2\,\mathrm{d}x \\
&+ \int_{A_k(s)} \Big(\frac{1}{k}|f(s)| + \mathcal{D}_0\Big)(|u^{(k)}(s)|^2 + k^2)\,\mathrm{d}x \\
&+ \int_{B_k(s)} \Big(\frac{1}{k}|g(s)| + |\beta|\Big)(|u^{(k)}(s)|^2 + k^2)\,\mathrm{d}\sigma
\end{aligned}
$$

with

$$
\mathcal{D}_0 := \omega + \frac{1}{\mu} \sum_{j=1}^{N} |b_j|^2 + |d| \in L^{q/2}(\Omega),
$$

where $q > N$. Plugging (A.16) into (A.14) and varying over t we arrive at the estimate

$$
\begin{aligned}
\text{(A.17)} \qquad &\min\Big\{\frac{1}{2}, \frac{\mu}{4}\Big\} \|\zeta u^{(k)}\|_{Q(\tau)}^2 \\
&\leqslant \sup_{-\tau \leqslant t \leqslant 0} \Big(\zeta(t)^2 \cdot \frac{1}{2} \int_\Omega |u^{(k)}(t)|^2\,\mathrm{d}x\Big) + \frac{\mu}{4} \int_{-\tau}^{0} \zeta(s)^2 \int_\Omega |\nabla u^{(k)}|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant \|\zeta'\|_{L^\infty(-\tau,0)} \int_{-\tau}^{0} \int_\Omega |u^{(k)}(s)|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_{-\tau}^{0} \int_{A_k(s)} \Big(\frac{1}{k}|f(s)| + \mathcal{D}_0\Big) \cdot (\zeta(s)^2 |u^{(k)}(s)|^2 + k^2)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_{-\tau}^{0} \int_{B_k(s)} \Big(\frac{1}{k}|g(s)| + |\beta|\Big) \cdot (\zeta(s)^2 |u^{(k)}(s)|^2 + k^2)\,\mathrm{d}\sigma\,\mathrm{d}s.
\end{aligned}
$$

We estimate the right hand side of (A.17). Define $\kappa_1 > 0$ and $\kappa_2 > 0$ by

$$\frac{1}{r_1} + \frac{N}{2q_1} = 1 - \frac{\kappa_1 N}{2} \qquad \text{and} \qquad \frac{1}{r_2} + \frac{N-1}{2q_2} = \frac{1}{2} - \frac{\kappa_2 N}{2}. \tag{A.18}$$

With $\bar{r}_1 := 2r_1/(r_1-1)$ and $\bar{q}_1 := 2q_1/(q_1-1)$ we obtain from Hölder's inequality that

$$\begin{aligned}
&\int_{-\tau}^{0}\int_{A_k(s)} |f(s)| \cdot \zeta(s)^2 |u^{(k)}(s)|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\quad \leqslant \|f\|_{L^{r_1}(-\tau,0;L^{q_1}(\Omega))} \|\zeta u^{(k)}\|^2_{L^{\bar{r}_1}(-\tau,0;L^{\bar{q}_1}(\Omega))} \\
&\quad \leqslant \hat{k} \|\zeta u^{(k)}\|^2_{L^{(1+\kappa_1)\bar{r}_1}(-\tau,0;L^{(1+\kappa)\bar{q}_1}(\Omega))} \|\mathbb{1}_{A_k}\|^2_{L^{(\kappa_1+1)/\kappa_1 \bar{r}_1}(-\tau,0;L^{(\kappa_1+1)/\kappa_1 \bar{q}_1}(\Omega))}.
\end{aligned}$$

The last factor tends to zero as $\tau \to 0$. As moreover

$$\frac{1}{(1+\kappa_1)\bar{r}_1} + \frac{N}{2(1+\kappa_1)\bar{q}_1} = \frac{N}{4}$$

by (A.18), we deduce from (A.1) that

$$\int_{-\tau}^{0}\int_{A_k(s)} |f(s)| \cdot \zeta(s)^2 |u^{(k)}(s)|^2 \,\mathrm{d}x\,\mathrm{d}s \leqslant \frac{\hat{k}}{8} \min\left\{\frac{1}{2}, \frac{\mu}{4}\right\} \|\zeta u^{(k)}\|^2_{Q(\tau)}$$

if τ is sufficiently small, say $\tau \leqslant T_0$, where T_0 depends on μ, N, Ω, κ_1, r_1, q_1. Similarly, since $2q/(q-2) < 2N/(N-2)$ we obtain that

$$\begin{aligned}
\int_{-\tau}^{0}\int_{A_k(s)} \mathcal{D}_0 \cdot \zeta(s)^2 |u^{(k)}(s)|^2 \,\mathrm{d}x\,\mathrm{d}s &\leqslant \|\mathcal{D}_0\|_{L^{q/2}(\Omega)} \|\zeta u^{(k)}\|^2_{L^2(-\tau,0;L^{2q/(q-2)}(\Omega))} \\
&\leqslant \frac{1}{8} \min\left\{\frac{1}{2}, \frac{\mu}{4}\right\} \|\zeta u^{(k)}\|^2_{Q(\tau)}
\end{aligned}$$

for $\tau \leqslant T_0$ with some possibly smaller $T_0 > 0$ that depends in addition on $\mathcal{D}_0$ and q.

Analogously, with $\bar{r}_2 := 2r_2/(r_2-1)$ and $\bar{q}_2 := 2q_2/(q_2-1)$ we have

$$\begin{aligned}
&\int_{-\tau}^{0}\int_{B_k(s)} |g(s)| \cdot \zeta(s)^2 |u^{(k)}(s)|^2 \,\mathrm{d}\sigma\,\mathrm{d}s \\
&\quad \leqslant \hat{k} \|\zeta u^{(k)}\|^2_{L^{(1+\kappa_2)\bar{r}_2}(-\tau,0;L^{(1+\kappa_2)\bar{q}_2}(\partial\Omega))} \|\mathbb{1}_{B_k}\|^2_{L^{((\kappa_1+1)/\kappa_1)\bar{r}_2}(-\tau,0;L^{((\kappa_1+1)/\kappa_1)\bar{q}_2}(\partial\Omega))} \\
&\quad \leqslant \frac{\hat{k}}{8} \min\left\{\frac{1}{2}, \frac{\mu}{4}\right\} \|\zeta u^{(k)}\|^2_{Q(\tau)}
\end{aligned}$$

and since $2(q-1)/(q-2) < 2(N-1)/(N-2)$ also

$$\int_{-\tau}^{0}\int_{B_k(s)} |\beta|\cdot\zeta(s)^2|u^{(k)}(s)|^2\,\mathrm{d}\sigma\,\mathrm{d}s \leqslant \|\beta\|_{L^{q-1}(\partial\Omega)}\|\zeta u^{(k)}\|^2_{L^2(-\tau,0;L^{2(q-1)/(q-2)}(\partial\Omega))}$$
$$\leqslant \frac{1}{8}\min\Big\{\frac{1}{2},\frac{\mu}{4}\Big\}\|\zeta u^{(k)}\|^2_{Q(\tau)}$$

for $\tau\leqslant T_0$, where this new T_0 depends also on r_2, q_2, κ_2 and β.

Combining the above estimates with (A.17) we obtain that

$$\text{(A.19)}\qquad \|\zeta u^{(k)}\|^2_{Q(\tau)} \leqslant c_\mu\|\zeta'\|_{L^\infty(-\tau,0)}\int_{-\tau}^{0}\int_{\Omega}|u^{(k)}(s)|^2\,\mathrm{d}x\,\mathrm{d}s$$
$$+ c_\mu k^2\int_{-\tau}^{0}\int_{A_k(s)}\Big(\frac{1}{k}|f(s)|+\mathcal{D}_0\Big)\,\mathrm{d}x\,\mathrm{d}s$$
$$+ c_\mu k^2\int_{-\tau}^{0}\int_{B_k(s)}\Big(\frac{1}{k}|g(s)|+|\beta|\Big)\,\mathrm{d}\sigma\,\mathrm{d}s$$

if $\tau\leqslant T_0$ and $k\geqslant\hat{k}$, where c_μ depends only on μ.

Now we estimate for $k\geqslant\hat{k}$

$$\int_{-\tau}^{0}\int_{A_k(s)}\frac{1}{k}|f(s)|\,\mathrm{d}x\,\mathrm{d}s \leqslant \frac{1}{k}\|f\|_{L^{r_1}(-\tau,0;L^{q_1}(\Omega))}\|\mathbb{1}_{A_k}\|_{L^{r_1/(r_1-1)}(-\tau,0;L^{q_1/(q_1-1)}(\Omega))}$$
$$\leqslant \|\mathbb{1}_{A_k}\|_{L^{r_1/(r_1-1)}(-\tau,0;L^{q_1/(q_1-1)}(\Omega))}$$
$$= \|\mathbb{1}_{A_k}\|^{2(1+\kappa_{1,1})}_{L^{r_{1,1}}(-\tau,0;L^{q_{1,1}}(\Omega))},$$

with $\kappa_{1,1} := \kappa_1$, $r_{1,1} := 2(1+\kappa_1)r_1/(r_1-1)$ and $q_{1,1} := 2(1+\kappa_1)q_1/(q_1-1)$, and similarly

$$\int_{-\tau}^{0}\int_{A_k(s)}\mathcal{D}_0\,\mathrm{d}x\,\mathrm{d}s \leqslant \|\mathcal{D}_0\|_{L^{q/2}(\Omega)}\|\mathbb{1}_{A_k}\|_{L^1(-\tau,0;L^{q/(q-2)}(\Omega))}$$
$$= \|\mathcal{D}_0\|_{L^{q/2}(\Omega)}\|\mathbb{1}_{A_k}\|^{2(1+\kappa_{1,2})}_{L^{r_{1,2}}(-\tau,0;L^{q_{1,2}}(\Omega))}$$

with $\kappa_{1,2} := (2(q-N)+(q-2)N)/(qN)$, $r_{1,2} := 2(1+\kappa_{1,2})$ and $q_{1,2} := 2(1+\kappa_{1,2})\times q/(q-2)$. Analogously,

$$\int_{-\tau}^{0}\int_{B_k(s)}\frac{1}{k}|g(s)|\,\mathrm{d}\sigma\,\mathrm{d}s \leqslant \|\mathbb{1}_{B_k}\|^{2(1+\kappa_{2,1})}_{L^{r_{2,1}}(-\tau,0;L^{q_2}(\partial\Omega))}$$

with $\kappa_{2,1} := \kappa_2$, $r_{2,1} := 2(1+\kappa_{2,1})r_2/(r_2-1)$ and $q_{2,1} := 2(1+\kappa_{2,1})q_2/(q_2-1)$, and

$$\int_{-\tau}^{0}\int_{B_k(s)}|\beta|\,\mathrm{d}\sigma\,\mathrm{d}s \leqslant \|\beta\|_{L^{q-1}(\partial\Omega)}\|\mathbb{1}_{B_k}\|^{2(1+\kappa_{2,2})}_{L^{r_{2,2}}(-\tau,0;L^{q_{2,2}}(\partial\Omega))}$$

with $\kappa_{2,2} := (N(q-N)+2(N-1))/((q-1)N)$, $r_{2,2} := 2(1+\kappa_{2,2})$ and $q_{2,2} := 2(1+\kappa_{2,2})(q-1)/(q-2)$. Thus (A.19) yields

$$\tag{A.20} \begin{aligned} \|\zeta u^{(k)}\|^2_{Q(\tau)} \leqslant c_\mu \|\zeta'\|_{L^\infty(-\tau,0)} \int_{-\tau}^0 \int_\Omega |u^{(k)}(s)|^2 \,\mathrm{d}x \,\mathrm{d}s \\ + ck^2 \sum_{l=1}^2 \Bigl(\int_{-\tau}^0 |A_k(s)|^{r_{1,l}/q_{1,l}} \,\mathrm{d}s \Bigr)^{2(1+\kappa_{1,l})/r_{1,l}} \\ + ck^2 \sum_{l=1}^2 \Bigl(\int_{-\tau}^0 |B_k(s)|^{r_{2,l}/q_{2,l}} \,\mathrm{d}s \Bigr)^{2(1+\kappa_{2,l})/r_{2,l}}. \end{aligned}$$

Moreover, (A.18) implies that the parameters $r_{i,l}$ and $q_{i,l}$ satisfy (A.2) for $i = 1,2$ and $l = 1,2$ as elementary calculations show.

If we pick $\zeta(t) := (t+\tau)/(\sigma\tau)$ for $t \in [-\tau, -(1-\sigma)\tau]$ and $\zeta(t) := 1$ for $t \in [-(1-\sigma)\tau, 0]$ with some given $\sigma \in (0, \frac{1}{2})$, we have

$$\|u^{(k)}\|^2_{Q((1-\sigma)\tau)} \leqslant \|\zeta u^{(k)}\|^2_{Q(\tau)}$$

and $\|\zeta'\|_{L^\infty(-\tau,0)} \leqslant 1/(\sigma\tau)$ if $T \leqslant T_0$, where c depends only on μ, $\mathcal{D}_0$ and β. Thus (A.20) implies (A.3). Hence by Theorem A.2 applied to u and $-u$, the latter being a classical solution of $(P_{-u_0,-f,-g})$, we obtain (3.2).

If in addition $u(-T) = 0$, then we can set $\tau := T$ and choose $\zeta(t) := 1$ for all $t \in [-T, 0]$. Now using Corollary A.3 instead of Theorem A.2, we obtain (3.3) from (A.20) like above. □

We finally make the step from classical L^2-solutions to weak solutions and drop the assumption that T be small enough, thus proving Proposition 3.1.

P r o o f of Proposition 3.1. Let u be the weak solution of $(P_{u_0,f,g})$. Pick a sequence $(u_{0,n})$ in $D(A^2_{2,h})$ that satisfies $u_{0,n} \to u_0$ in $L^2(\Omega)$, which exists since by Proposition 2.10 the operator $A_{2,h}$ is a generator of a strongly continuous semigroup and hence densely defined. Pick sequences (f_n) and (g_n) in $\mathrm{C}^2([0,T]; L^\infty(\Omega))$ and $\mathrm{C}^2([0,T]; L^\infty(\partial\Omega))$, respectively, that satisfy $f_n \to f$ in $L^{r_1}(0,T; L^{q_1}(\Omega))$ and $g_n \to g$ in $L^{r_2}(0,T; L^{q_2}(\partial\Omega))$, while $f_n(0) = 0$ and $g_n(0) = 0$ for all $n \in \mathbb{N}$. Then problem $(P_{u_{0,n},f_n,g_n})$ has a unique classical L^2-solution u_n by Proposition 2.7, and as in the proof of Theorem 2.11 we see that $u_n \to u$ in $\mathrm{C}([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$.

Pick $T_0 > 0$ as in Lemma A.5. Shrinking T_0 if necessary, we can assume that $T_0 \leqslant T$. Let $I \subset [T_0/2, T_0]$ be an interval of length at most $T_0/2$. Applying (3.2) to the classical L^2-solutions u_n and $u_n - u_m$ on I, which is allowed by Lemma A.5, we obtain that

$$\text{(A.21)}\qquad \|u_n\|^2_{L^\infty(I;L^\infty(\Omega))} \leqslant c\int_0^T\int_\Omega |u_n(s)|^2\,\mathrm{d}x\,\mathrm{d}s + c\|f_n\|^2_{L^{r_1}(0,T;L^{q_1}(\Omega))} + c\|g_n\|^2_{L^{r_2}(0,T;L^{q_2}(\Omega))}$$

and that $(u_n|_I)$ is a Cauchy sequence in $L^\infty(I;L^\infty(\Omega))$. Hence $u_n \to u$ in $L^\infty(I;L^\infty(\Omega))$, and passing to the limit in (A.21) we have

$$\text{(A.22)}\qquad \|u\|^2_{L^\infty(I;L^\infty(\Omega))} \leqslant c\int_0^T\int_\Omega |u(s)|^2\,\mathrm{d}x\,\mathrm{d}s + c\|f\|^2_{L^{r_1}(0,T;L^{q_1}(\Omega))} + c\|g\|^2_{L^{r_2}(0,T;L^{q_2}(\Omega))}.$$

Covering $[T/2,T]$ by finitely many intervals of length at most $T_0/2$ and using (A.22) for each of these intervals we obtain (3.2).

If in addition $u_0 = 0$, then we can pick $u_{0,n} := 0$ and the same strategy as above yields that

$$\|u\|^2_{L^\infty(0,T_0;L^\infty(\Omega))} \leqslant c\int_0^T\int_\Omega |u(s)|^2\,\mathrm{d}x\,\mathrm{d}s + c\|f\|^2_{L^{r_1}(0,T;L^{q_1}(\Omega))} + c\|g\|^2_{L^{r_2}(0,T;L^{q_2}(\Omega))}.$$

Using in addition (3.2) to estimate $\|u\|_{L^\infty(I;L^\infty(\Omega))}$ for finitely many intervals I of length $T_0/2$ that cover $[T_0,T]$, we have proved also (3.3). $\square$

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