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GEOMETRY OF THE SPECTRAL SEMIDISTANCE IN BANACH ALGEBRAS

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Abstract. Let A be a unital Banach algebra over $\mathbb{C}$, and suppose that the nonzero spectral values of a and $b \in A$ are discrete sets which cluster at $0 \in \mathbb{C}$, if anywhere. We develop a plane geometric formula for the spectral semidistance of a and b which depends on the two spectra, and the orthogonality relationships between the corresponding sets of Riesz projections associated with the nonzero spectral values. Extending a result of Brits and Raubenheimer, we further show that a and b are quasinilpotent equivalent if and only if all the Riesz projections, $p(\alpha, a)$ and $p(\alpha, b)$, correspond. For certain important classes of decomposable operators (compact, Riesz, etc.), the proposed formula reduces the involvement of the underlying Banach space X in the computation of the spectral semidistance, and appears to be a useful alternative to Vasilescu's geometric formula (which requires the knowledge of the local spectra of the operators at each $0 \neq x \in X$). The apparent advantage gained through the use of a global spectral parameter in the formula aside, various methods of complex analysis can then be employed to deal with the spectral projections; we give examples illustrating the usefulness of the main results.

Keywords: asymptotically intertwined; Riesz projection; spectral semidistance; quasinilpotent equivalent

MSC 2010: 46H05, 47A05, 47A10

1. Introduction

Let A denote a complex Banach algebra with identity $\mathbf{1}$. For $a, b \in A$ associate operators L_a, R_b, and $C_{a,b}$, acting on A, by the relations

$$L_a x = ax, \quad R_b x = xb, \quad \text{and} \quad C_{a,b} x = (L_a - R_b)x \quad \text{for each } x \in A.$$

Since L_a and R_b commute, it is easy to show that

$$C_{a,b}^n x = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a^{n-k} x b^k \quad \text{for each } x \in A,$$

with the convention that if $0 \neq a \in A$, then $a^0 = \mathbf{1}$. Using the particular value $x = \mathbf{1}$, define $\varrho\colon A \times A \to \mathbb{R}$ by

$$\varrho(a,b) = \limsup_n \|C_{a,b}^n \mathbf{1}\|^{1/n}, \tag{1.1}$$

and then define

$$\rho(a,b) = \sup\{\varrho(a,b), \varrho(b,a)\}. \tag{1.2}$$

If X is a Banach space, and $A = \mathcal{L}(X)$ is the Banach algebra of bounded linear operators from X into X, then the number $\varrho(S,T)$ is a well-established quantity called the *local spectral radius* [5], page 235, of the commutator $C_{S,T} \in \mathcal{L}(A)$ at I. The number $\rho(S,T)$ is called the *spectral distance* [5], page 251, of the operators S and T. Furthermore, the pair (S,T) is said to be *asymptotically intertwined* [5], page 248, by the identity I, if $\varrho(S,T) = 0$. If each of the pairs (S,T) and (T,S) is asymptotically intertwined by the identity operator (i.e., $\rho(S,T) = 0$), then S and T are called *quasinilpotent equivalent* [5], page 253. A first generalization in the framework of Banach algebras on topics related to the commutator appeared in Section III.4 of the monograph [8]. In the paper [7], ρ is called the *spectral semidistance*, which is perhaps a little more appropriate in view of the fact that ρ is only a semimetric [5], Proposition 3.4.9. One may think of the spectral semidistance as a noncommutative generalization of the distance induced by the spectral radius when a and b do commute. Again, if $\rho(a,b) = 0$, then a and b are said to be *quasinilpotent equivalent.* A good source of results on the topic of spectral (semi)distance is Laursen and Neumann's recent monograph [5]; the reader may also want to look at [2]–[4], [7], [9], [10]. We should mention the following simple but useful property of ϱ and ρ which appears explicitly in [2], Lemma 2.2: If q_a and q_b are quasinilpotent elements of A commuting with a and b, respectively, then $\varrho(a,b) = \varrho(a + q_a, b + q_b)$.

The results in the present paper are related to Vasilescu's geometric formula [10] for the spectral semidistance of decomposable operators $S, T \in \mathcal{L}(X)$:

$$\rho(S,T) = \sup\bigl\{\max\{\operatorname{dist}(\lambda, \sigma_T(x)), \operatorname{dist}(\mu, \sigma_S(x))\}\colon\ x \neq 0,\ \lambda \in \sigma_S(x),\ \mu \in \sigma_T(x)\bigr\},$$

where $\sigma_S(x)$ and $\sigma_T(x)$ are the local spectra of S and T, respectively, at $x \in X$.

The usual spectrum of $a \in A$ will be denoted by $\sigma(a,A)$, the "nonzero" spectrum, $\sigma(a,A) \setminus \{0\}$, by $\sigma'(a,A)$, and the spectral radius of $a \in A$ by $r_\sigma(a,A)$. Whenever there is no ambiguity we shall omit the A in σ and r_σ.

If $a \in A$ and $\alpha \in \mathbb{C}$ is not an accumulation point of $\sigma(a)$, then let Γ_α be a small circle, disjoint from $\sigma(a)$, and isolating α from the remaining spectrum of a. We denote by

$$p(\alpha, a) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\alpha} (\lambda \mathbf{1} - a)^{-1} \, \mathrm{d}\lambda$$

the *Riesz projection associated with* a *and* α. If $\alpha \notin \sigma(a)$, then, by Cauchy's theorem, $p(\alpha, a) = 0$. For Riesz projections $p(\alpha_1, a)$ and $p(\alpha_2, a)$, with $\alpha_1 \neq \alpha_2$, the functional calculus implies that $p(\alpha_1, a)p(\alpha_2, a) = p(\alpha_2, a)p(\alpha_1, a) = 0$.

We recall the following well-known "spectral decomposition" (see [1], page 21) from the theory of Banach algebras:

Lemma 1.1. *Suppose* $a \in A$ *has* $\sigma(a) = \{\lambda_1, \ldots, \lambda_n\}$. *Then* a *has the representation*

$$a = \lambda_1 p_1 + \ldots + \lambda_n p_n + r_a,$$

where $p_i = p(\lambda_i, a)$, $\sum p_i = \mathbf{1}$, *and* r_a *is a quasinilpotent element belonging to the bicommutant of* a.

It is worthwhile to mention here an interesting connection which relates ϱ to the growth characteristics of a certain entire map from $\mathbb{C}$ into A: Let f be an entire A-valued function. Then f has an everywhere convergent power series expansion

$$f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n,$$

with coefficients a_n belonging to A. Define a function $M_f(r) = \sup\limits_{|\lambda| \leqslant r} \|f(\lambda)\|$, $r > 0$. The function f is said to be of *finite order* if there exist $K > 0$ and $R > 0$ such that $M_f(r) < \mathrm{e}^{r^K}$ holds for all $r > R$. The infimum of the set of positive real numbers K such that the preceding inequality holds is called the *order* of f, denoted by ω_f. If $\omega_f = 1$ then f is said to be of *exponential order*. Suppose f is entire, and of finite order $\omega := \omega_f$. Then f is said to be of *finite type* if there exist $L > 0$ and $R > 0$ such that $M_f(r) < \mathrm{e}^{Lr^\omega}$ holds for all $r > R$. The infimum of the set of positive real numbers L such that the preceding inequality holds is called the *type* of f, denoted by τ_f. It it known (see the monograph [6], page 41) that the order and type are given by the formulas

$$\omega_f = \limsup_n \frac{n \log n}{\log \|a_n\|^{-1}} \quad \text{and} \quad \tau_f = \frac{1}{\mathrm{e}\omega_f} \limsup_n \bigl(n \sqrt[n]{\|a_n\|^{\omega_f}}\bigr).$$

Concerning the formula for τ_f, we remark that if f is of order 0 and finite type, then it follows directly from the definition, together with Liouville's theorem, that f must be constant.

Let $a, b \in A$, and define

$$f\colon\ \lambda \mapsto \mathrm{e}^{\lambda a}\mathrm{e}^{-\lambda b}, \quad \lambda \in \mathbb{C}.$$

The corresponding series expansion, valid for all $\lambda \in \mathbb{C}$, is given by

$$f(\lambda) = \mathrm{e}^{\lambda a}\mathrm{e}^{-\lambda b} = \sum_{n=0}^{\infty} \frac{\lambda^n C_{a,b}^n \mathbf{1}}{n!}.$$

Since $\|f(\lambda)\| \leqslant \mathrm{e}^{(\|a\|+\|b\|)|\lambda|}$, for all $\lambda \in \mathbb{C}$, it is immediate from the definition that f is of order at most one. Suppose we know that f is of exponential order (i.e., $\omega_f = 1$). Recall now, using Stirling's formula, that $\lim\limits_n n\bigl(1/n!\bigr)^{1/n} = \mathrm{e}$, from which we subsequently obtain

$$\tau_f = \frac{1}{\mathrm{e}} \limsup_n \left(n\Bigl(\frac{1}{n!}\Bigr)^{1/n} \|C_{a,b}^n \mathbf{1}\|^{1/n}\right) = \varrho(a,b).$$

To start with, we give a brief argument, using these ideas, which quickly leads to (an improvement of) the main result in Section 4 of [2].

Theorem 1.2. *If $\sigma(a)$ and $\sigma(b)$ are finite, then $\varrho(a,b) = 0$ if and only if $a - r_a = b - r_b$, where r_a and r_b are quasinilpotent elements commuting with a and b, respectively.*

Proof. The reverse implication is trivial as in [2]. With Lemma 1.1 we can write $a - r_a = \sum\limits_{j=1}^{n} \lambda_j p_j$ and $b - r_b = \sum\limits_{j=1}^{k} \beta_j q_j$. Denote $\overline{a} = a - r_a$, $\overline{b} = b - r_b$, and define $f(\lambda) = \mathrm{e}^{\lambda\overline{a}}\mathrm{e}^{-\lambda\overline{b}}$. Since $\sum\limits_{j=1}^{n} p_j = \mathbf{1}$ and $\sum\limits_{j=1}^{k} q_j = \mathbf{1}$, and using the orthogonality, we have

$$f(\lambda) = \left[\mathbf{1} + \sum_{j=1}^{n}(\mathrm{e}^{\lambda_j\lambda} - 1)p_j\right]\left[\mathbf{1} + \sum_{j=1}^{k}(\mathrm{e}^{-\beta_j\lambda} - 1)q_j\right] = \sum_{i,j} \mathrm{e}^{(\lambda_i - \beta_j)\lambda} p_i q_j. \tag{1.3}$$

Fix any $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, k\}$ such that $p_i q_j \neq 0$, and define

$$g_{i,j}(\lambda) = p_i f(\lambda) q_j = \mathrm{e}^{(\lambda_i - \beta_j)\lambda} p_i q_j.$$

Let us assume $\lambda_i \neq \beta_j$. If we notice, using Stirling's formula, that $\lim\limits_n n\log n/\log n! = 1$, then the coefficient formula for the order applied to the representation $g_{i,j}(\lambda) =$

$e^{(\lambda_i-\beta_j)\lambda}p_iq_j$ shows that $g_{i,j}$ is of exponential order. But now, on the one hand, using the submultiplicative norm inequality, the representation

$$g_{i,j}(\lambda)=\sum_{n=0}^{\infty}\frac{\lambda^n p_i(C^n_{\overline{a},\overline{b}}\mathbf{1})q_j}{n!}$$

gives the type of $g_{i,j}$ as $\varrho(\overline{a},\overline{b})=\varrho(a,b)=0$, and on the other hand, the representation $g_{i,j}(\lambda)=e^{(\lambda_i-\beta_j)\lambda}p_iq_j$ says the type is equal to $|\lambda_i-\beta_j|\neq 0$. From this contradiction we may conclude that for each pair i,j, either $p_iq_j=0$ or $\lambda_i=\beta_j$. It then follows from (1.3) that f is constant, so $f(\lambda)=e^{\lambda\overline{a}}e^{-\lambda\overline{b}}=\mathbf{1}$ for all $\lambda\in\mathbb{C}$. Differentiation finally gives $\overline{a}=\overline{b}$. □

2. Geometry of ϱ

To obtain the main result, Theorem 2.5, we first need to establish the formula in the case where $\sigma(a)$ and $\sigma(b)$ are finite sets. As in the proof of Theorem 1.2, using Lemma 1.1, we can write $a=\sum\limits_{i=1}^{n}\lambda_ip_i+r_a$ and $b=\sum\limits_{j=1}^{k}\beta_jq_j+r_b$. Setting $\overline{a}=a-r_a$ and $\overline{b}=b-r_b$, we obtain the following:

Lemma 2.1. *Suppose $\sigma(a)$ and $\sigma(b)$ are finite. Then there exists a finite-dimensional Banach space $X\subseteq A$ such that $\varrho(a,b)=r_\sigma(L_{\overline{a}}-R_{\overline{b}},\mathcal{L}(X))$.*

Proof. Let X denote the normed space spanned by the set

$$Y=\{p_i^rq_j^t\colon\ i\in\{1,\dots,n\},\ j\in\{1,\dots,k\},\ r\in\{0,1\},\ t\in\{0,1\}\}.$$

It is elementary that $L_{\overline{a}}$ and $R_{\overline{b}}$ belong to $\mathcal{L}(X)$. Without loss of generality we may assume that Y constitutes a linearly independent set of vectors. Since X has finite dimension, there exist $K_1,K_2>0$ such that if x is a linear combination of elements in Y with coefficients $\gamma_0,\dots,\gamma_s$, then

$$K_1(|\gamma_0|+\ldots+|\gamma_s|)\leqslant\|x\|\leqslant K_2(|\gamma_0|+\ldots+|\gamma_s|).$$

Obviously we may take K_2 as

$$K_2=\sup\{\|p_i\|\|q_j\|+1\colon\ i\in\{1,\dots,n\},\ j\in\{1,\dots,k\}\}.$$

So for $x\in X$ given by, say,

$$x=\gamma_0\mathbf{1}+\gamma_1p_1+\gamma_2q_1+\gamma_3p_1q_1+\ldots+\gamma_sp_nq_k,$$

it follows that

$$C^m_{\overline{a},\overline{b}}x = \gamma_0[C^m_{\overline{a},\overline{b}}\mathbf{1}] + \gamma_1 p_1[C^m_{\overline{a},\overline{b}}\mathbf{1}] + \gamma_2[C^m_{\overline{a},\overline{b}}\mathbf{1}]q_1 + \ldots + \gamma_s p_n[C^m_{\overline{a},\overline{b}}\mathbf{1}]q_k,$$

and thus

$$\begin{aligned}\|C^m_{\overline{a},\overline{b}}x\| &\leqslant \big(|\gamma_0| + |\gamma_1|\|p_1\| + |\gamma_2|\|q_1\| + \ldots + |\gamma_s|\|p_n\|\|q_k\|\big)\|C^m_{\overline{a},\overline{b}}\mathbf{1}\| \\ &\leqslant K_2(|\gamma_0| + |\gamma_1| + |\gamma_2| + \ldots + |\gamma_s|)\|C^m_{\overline{a},\overline{b}}\mathbf{1}\| \\ &\leqslant K_2K_1^{-1}\|x\|\|C^m_{\overline{a},\overline{b}}\mathbf{1}\|.\end{aligned}$$

Taking the supremum over all x of norm 1, we see that

$$\|C^m_{\overline{a},\overline{b}}\| \leqslant K_2K_1^{-1}\|C^m_{\overline{a},\overline{b}}\mathbf{1}\|$$

holds for each m. So it follows that

$$r_\sigma(L_{\overline{a}} - R_{\overline{b}}, \mathcal{L}(X)) = \limsup_m \|C^m_{\overline{a},\overline{b}}\|^{1/m} \leqslant \limsup_m \|C^m_{\overline{a},\overline{b}}\mathbf{1}\|^{1/m} = \varrho(\overline{a},\overline{b}).$$

On the other hand, it follows trivially from $\|C^m_{\overline{a},\overline{b}}\mathbf{1}\| \leqslant \|C^m_{\overline{a},\overline{b}}\|$ that $\varrho(\overline{a},\overline{b}) \leqslant r_\sigma(L_{\overline{a}} - R_{\overline{b}}, \mathcal{L}(X))$, and hence $\varrho(\overline{a},\overline{b}) = r_\sigma(L_{\overline{a}} - R_{\overline{b}}, \mathcal{L}(X))$. But of course $\varrho(\overline{a},\overline{b}) = \varrho(a,b)$. □

Theorem 2.2. *Suppose $\sigma(a)$ and $\sigma(b)$ are finite with $\sigma(a) = \{\lambda_1,\ldots,\lambda_n\}$, $\sigma(b) = \{\beta_1,\ldots,\beta_k\}$. If $\{p_1,\ldots,p_n\}$ and $\{q_1,\ldots,q_k\}$ are the corresponding Riesz projections, then*

$$\varrho(a,b) = \sup\{|\lambda_i - \beta_j|\colon\ p_iq_j \neq 0\}. \tag{2.1}$$

P r o o f. By Lemma 2.1 we have that

$$\varrho(a,b) = r_\sigma\Big(\sum_{i=1}^n \lambda_i L_{p_i} - \sum_{i=1}^k \beta_i R_{q_i}, \mathcal{L}(X)\Big). \tag{2.2}$$

The preceding formula remains valid if we scale down to the commutative unital subalgebra generated by L_{p_i} and R_{q_i}. Notice that $\sum_i L_{p_i} = I$, and $\sum_i R_{q_i} = I$. From this, together with the fact that L_{p_i} are mutually orthogonal and R_{q_i} are mutually orthogonal, we now have the following: Corresponding to each χ belonging to the character space of the algebra, there exists a unique pair, say L_{p_t} and R_{q_s}, such that $\chi(L_{p_t}) = 1 = \chi(R_{q_s})$ and $\chi(L_{p_i}) = 0 = \chi(R_{q_j})$ whenever $i \neq t$, $j \neq s$. Conversely, if the product $p_tq_s \neq 0$, then the projection $L_{p_t}R_{q_s} \neq 0$ and hence there is χ such that $\chi(L_{p_t}R_{q_s}) = 1$. So, for each of the two projections, we have $\chi(L_{p_t}) = 1 = \chi(R_{q_s})$. With these observations, (2.2) gives the formula (2.1). □

It is not obvious from (1.1) that ϱ is not symmetric (see the comments in [5], page 251, regarding this matter). However, Theorem 2.2 prescribes the construction of a, b such that $\varrho(a,b) \neq \varrho(b,a)$; the formula (2.1) suggests that one should look for Riesz projections, say p and q, such that $pq \neq 0$ but $qp = 0$.

Example 2.3. Let A be the free algebra generated by the alphabet $\{\mathbf{1}, x_1, x_2\}$, subject to the conditions $x_1^2 = x_1$, $x_2^2 = x_2$, $x_1x_2 = 0$ and $x_2x_1 \neq 0$. A is a Banach algebra with

$$\|\alpha_0\mathbf{1} + \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_2x_1\| = \sum_j |\alpha_j|.$$

Now take $a = \frac{1}{2}x_1$ and $b = -\frac{1}{2}x_2$. Then

$$C^n_{a,b}\mathbf{1} = \frac{1}{2^n}(x_1 + x_2) \Rightarrow \|C^n_{a,b}\mathbf{1}\| = \frac{1}{2^{n-1}} \Rightarrow \varrho(a,b) = \frac{1}{2}.$$

On the other hand,

$$\begin{aligned} C^n_{b,a}\mathbf{1} &= \Big(-\frac{1}{2}\Big)^n\left[\binom{n}{0}x_2 + \binom{n}{1}x_2x_1 + \ldots + \binom{n}{n-1}x_2x_1 + \binom{n}{n}x_1\right] \\ &\Rightarrow \|C^n_{b,a}\mathbf{1}\| = \frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j} = 1 \Rightarrow \varrho(b,a) = 1. \end{aligned}$$

For a more concrete exposition, notice that A in Example 2.3 is isomorphic to a four-dimensional subalgebra of $M_3(\mathbb{C})$, the algebra of 3×3 complex matrices.

Theorem 2.4. *Suppose $\sigma'(a)$ and $\sigma'(b)$ are discrete sets which cluster at $0 \in \mathbb{C}$, if anywhere. If $\sigma'(a) = \{\lambda_1, \lambda_2, \ldots\}$ and $\sigma'(b) = \{\beta_1, \beta_2, \ldots\}$ denote the nonzero spectral points of a and b, and if $\{p_1, p_2, \ldots\}$ and $\{q_1, q_2 \ldots\}$ are the corresponding Riesz projections, then ϱ takes at least one of the following values:*

(i) $\varrho(a,b) = \sup\{|\lambda_i - \beta_j| \colon p_iq_j \neq 0\}$, *or*
(ii) $\varrho(a,b) = |\lambda_i|$ *for some* $i \in \mathbb{N}$, *or*
(iii) $\varrho(a,b) = |\beta_i|$ *for some* $i \in \mathbb{N}$.

Moreover, $\varrho(a,b) = 0$ if and only if the spectra and the corresponding Riesz projections of a and b coincide.

Proof. We prove the result for the case where both $\sigma(a)$ and $\sigma(b)$ are infinite sets; the other cases follow similarly: For each $n \in \mathbb{N}$, let $a_n = \sum\limits_{i=1}^{n}\lambda_ip_i$ and $b_n = \sum\limits_{i=1}^{n}\beta_iq_i$, and put $p_{0,n} = \mathbf{1} - \sum\limits_{i=1}^{n}p_i$, $q_{0,n} = \mathbf{1} - \sum\limits_{i=1}^{n}q_i$. As $\sigma(a)$, $\sigma(b)$ are assumed to be infinite, we must have $p_{0,n} \neq 0$, $q_{0,n} \neq 0$. Note that $\sigma(a_n) = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ with $\lambda_0 = 0$ and similarly $\sigma(b_n) = \{\beta_0, \beta_1, \ldots, \beta_n\}$ with $\beta_0 = 0$ (because $a_np_{0,n} = 0$ and $b_nq_{0,n} = 0$).

Furthermore, for each n, let $\Gamma_{a,n}$ be a simple closed curve, disjoint from $\sigma(a)$, and surrounding only the subset $\{\lambda_{n+1}, \lambda_{n+2}, \ldots\} \cup \{0\} \subset \sigma(a)$. If we notice that for each n,

$$a = \sum_{i=1}^{n} ap_i + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{a,n}} \lambda(\lambda \mathbf{1} - a)^{-1} \, \mathrm{d}\lambda,$$

and that a_n commutes with a, then it follows that $\sigma(a - a_n) \subseteq \{\lambda_{n+1}, \lambda_{n+2}, \ldots\} \cup \{0\}$, and hence $r_\sigma(a - a_n) \to 0$ as $n \to \infty$. In the same way it follows that $r_\sigma(b - b_n) \to 0$. Using the triangle inequality for ϱ, together with the fact that $\varrho(x, y) = r_\sigma(x - y)$ whenever x and y commute, we then obtain

$$|\varrho(a_n, b_n) - \varrho(a, b)| \leqslant r_\sigma(a - a_n) + r_\sigma(b - b_n),$$

whence it follows that $\varrho(a, b) = \lim_n \varrho(a_n, b_n)$. We now want to use Theorem 2.2 to calculate $\varrho(a_n, b_n)$; this requires the knowledge of the Riesz projections $p(\lambda_i, a_n)$ and $p(\beta_i, b_n)$ for $i = 0, 1, \ldots, n$: Observe, for $\lambda \notin \sigma(a_n)$, that

$$(\lambda \mathbf{1} - a_n)^{-1} = \frac{\mathbf{1}}{\lambda} + \sum_{i=1}^{n} \frac{\lambda_i}{\lambda(\lambda - \lambda_i)} p_i.$$

So it follows from the Cauchy integral formula and the Cauchy integral theorem that for each $0 < i \leqslant n$, $p(\lambda_i, a_n) = p_i$. A similar argument yields $p(\beta_i, b_n) = q_i$ when $0 < i \leqslant n$. It is then obvious that $p(\lambda_0, a_n) = p_{0,n}$ and $p(\beta_0, b_n) = q_{0,n}$. Define, for each $n \in \mathbb{N}$,

$$\begin{aligned} U_{1,n} &= \{|\lambda_i - \beta_j| \colon \ p_i q_j \neq 0, \ i, j = 1, \ldots, n\}, \\ U_{2,n} &= \{|\lambda_i| \colon \ p_i q_{0,n} \neq 0, \ i = 1, \ldots, n\}, \\ U_{3,n} &= \{|\beta_i| \colon \ p_{0,n} q_i \neq 0, \ i = 1, \ldots, n\}, \end{aligned}$$

and $U_n = \bigcup_{j=1}^{3} U_{j,n}$. If we keep n fixed for the moment, writing $p_0 = p_{0,n}$, $q_0 = q_{0,n}$, then, by Theorem 2.2, we obtain

$$\varrho(a_n, b_n) = \sup\{|\lambda_i - \beta_j| \colon \ p_i q_j \neq 0, \ i, j = 0, 1, \ldots, n\} = \sup U_n. \tag{2.3}$$

Notice that $U_n \neq \emptyset$, because if $U_{1,n} = \emptyset$, then, for instance, $p_1 q_j = 0$ for $j = 1, \ldots, n$, so $p_1 q_{0,n} = p_1 \neq 0$, whence $|\lambda_1| \in U_{2,n} \subseteq U_n$. Having established (2.3), we are now in a position to derive the conclusion of Theorem 2.4. We shall first prove the statement that $\varrho(a, b) = 0$ if and only if the spectra and the corresponding Riesz projections of a and b coincide: For the reverse implication notice that we can take

$a_n = b_n$ for each $n \in \mathbb{N}$. Thus $\varrho(a,b) = \lim_n \varrho(a_n, b_n) = 0$. Suppose, conversely, that $\varrho(a,b) = 0$. First let us remark that for each index i_* we can find an index j_* such that $p_{i_*} q_{j_*} \neq 0$; if this was not true, i.e., $p_{i_*} q_j = 0$ for all j, then we may infer that $0 \neq p_{i_*} = p_{i_*} q_{0,n}$ for all $n \geqslant i_*$. But this means that $|\lambda_{i_*}| \in U_{2,n} \subseteq U_n$ for all $n \geqslant i_*$, which in turn implies $\varrho(a,b) = \limsup_n U_n \geqslant |\lambda_{i_*}| > 0$, contradicting $\varrho(a,b) = 0$. We therefore have the implication:

$$\varrho(a,b) = 0 \Rightarrow W := \{|\lambda_i - \beta_j| \colon\ p_i q_j \neq 0\} \neq \emptyset. \tag{2.4}$$

We proceed to prove $\sigma(a) = \sigma(b)$. Since the spectra of both a and b are infinite, the hypothesis implies $0 \in \sigma(a) \cap \sigma(b)$. For a contradiction, suppose that $0 \neq \lambda_{i_*} \in \sigma(a)$ but $\lambda_{i_*} \notin \sigma(b)$. Then, as above, we can find an index j_* such that $p_{i_*} q_{j_*} \neq 0$. If $n \geqslant \max\{i_*, j_*\}$ is arbitrary, then $|\lambda_{i_*} - \beta_{j_*}| \in U_{1,n} \subseteq U_n$ from which

$$\varrho(a,b) = \limsup_n U_n \geqslant |\lambda_{i_*} - \beta_{j_*}| \geqslant \operatorname{dist}(\lambda_{i_*}, \sigma(b)) > 0,$$

giving the required contradiction. Therefore $\sigma(a) \subseteq \sigma(b)$. Similarly $\sigma(b) \subseteq \sigma(a)$, and we have $\sigma(a) = \sigma(b)$. It remains to show that the Riesz projections, $p(\lambda_{i_*}, a) =: p_{i_*}$ and $p(\lambda_{i_*}, b) =: q_{i_*}$, corresponding to a common nonzero spectral value $\lambda_{i_*} \in \sigma(a) = \sigma(b)$, are in fact equal: First observe that $\sup W = 0$; indeed, if for some indices i_*, j_* we have $0 \neq |\lambda_{i_*} - \lambda_{j_*}| \in W$, then $|\lambda_{i_*} - \lambda_{j_*}| \in U_{1,n}$ for all $n \geqslant \max\{i_*, j_*\}$, and hence, as before, $\varrho(a,b) > 0$ which is absurd. If we fix an index i_*, then $p_{i_*} q_j = 0$ whenever $j \neq i_*$, because otherwise $p_{i_*} q_j \neq 0$ implies $|\lambda_{i_*} - \lambda_j| \in W$, forcing $\lambda_{i_*} = \lambda_j$, which is possible only if $j = i_*$ (as the points in the spectrum are distinct). Therefore

$$p_{i_*} - p_{i_*} q_{i_*} = p_{i_*} q_{0,n} = p_{i_*} \left(\mathbf{1} - \sum_{j=1}^{n} q_j \right) \quad \text{for all } n \geqslant i_*.$$

Now if $p_{i_*} \neq p_{i_*} q_{i_*}$, then $\varrho(a_n, b_n) \geqslant |\lambda_{i_*}|$ for all $n \geqslant i_*$, which again leads to $\varrho(a,b) \geqslant |\lambda_{i_*}| > 0$. So we conclude that $p_{i_*} = p_{i_*} q_{i_*}$. A similar argument, using the sets $U_{3,n}$ instead of $U_{2,n}$, gives $q_{i_*} = p_{i_*} q_{i_*}$, and thus $p_{i_*} = q_{i_*}$. We have now shown that $\varrho(a,b) = 0$ if and only if the spectra and the corresponding Riesz projections of a and b coincide.

For the remaining part of the statement: If $\varrho(a,b) = 0$, then (2.4) says $W \neq \emptyset$, and, as we have shown, $\sup W = 0$; hence (i) is valid. Suppose that $\varrho(a,b) > 0$ and that $\sup W < \limsup_n U_n$ (if $W = \emptyset$, we let $\sup W = 0$). If we set $\tau_n = \sup(U_{2,n} \cup U_{3,n})$, then $\limsup_n U_n = \lim_n \tau_n$, whence it follows that there exists $N \in \mathbb{N}$ such that $\tau_n > \sup W$ for all $n \geqslant N$. In particular, we can build either a sequence (λ_{i,n_k}) whose members belong to $\sigma'(a)$, or a sequence (β_{j,n_k}) whose members belong to $\sigma'(b)$, such that

$|\lambda_{i,n_k}| = \tau_{n_k}$ or $|\beta_{j,n_k}| = \tau_{n_k}$, and $\lim_k |\lambda_{i,n_k}| = \limsup_n U_n$ or $\lim_k |\beta_{j,n_k}| = \limsup_n U_n$. To avoid trivial misunderstanding, the notation indicates that these sequences are not subsequences of (λ_i) and (β_j), respectively, but rather sequences constructed by extracting individual members of the sets $\sigma'(a)$ and $\sigma'(b)$ (i.e., repetition of terms may occur). Anyhow, if we assume the existence of the sequence (λ_{i,n_k}) satisfying the aforementioned properties, then, since $\limsup_n U_n > 0$, it follows that the sequence $(|\lambda_{i,n_k}|)$ must eventually be constant (because the spectrum of a clusters only at $0 \in \sigma(a)$). This means there exists an index i_* such that $\limsup_n U_n = |\lambda_{i_*}|$ and hence that $\varrho(a, b) = |\lambda_{i_*}|$, so (ii) holds. If the sequence (λ_{i,n_k}) cannot be found, then a similar argument with the sequence (β_{j,n_k}) shows that (iii) holds. □

For elements $a, b \in A$ satisfying the hypothesis of Theorem 2.4, it follows that $\varrho(a, b) = 0 \Leftrightarrow \varrho(b, a) = 0$, which simplifies the requirement for quasinilpotent equivalence. The proof of Theorem 2.4 also establishes a formula for ϱ: Let us assume the hypothesis of Theorem 2.4, where both $\sigma(a)$ and $\sigma(b)$ are infinite sets. Define, as in the proof of Theorem 2.4,

$$W := \{|\lambda_i - \beta_j| \colon\ p_i q_j \neq 0\}.$$

If $W = \emptyset$, then the proof of Theorem 2.4 shows that for each n, we have $\varrho(a_n, b_n) = \sup\{r_\sigma(a_n), r_\sigma(b_n)\}$. Therefore

$$\varrho(a, b) = \lim_n \varrho(a_n, b_n) = \lim_n \sup\{r_\sigma(a_n), r_\sigma(b_n)\} = \sup\{r_\sigma(a), r_\sigma(b)\}.$$

Suppose now $W \neq \emptyset$. If for some index k we have $|\lambda_k| > \sup W$, then, since $\lim_j \beta_j = 0$, there exists $N > 0$ such that $p_k q_j = 0$ for all $j \geqslant N$; if this was not true, then some subsequence, say (q_{j_m}), of (q_j) satisfies $p_k q_{j_m} \neq 0$ for each m. But then, by definition, $|\lambda_k - \beta_{j_m}| \in W$ for each m. Letting $m \to \infty$, so that $\beta_{j_m} \to 0$, we see that $\sup W \geqslant |\lambda_k|$, contradicting the assumption. So for any index k satisfying $|\lambda_k| > \sup W$, we have that $\lim_n \sum_{j=1}^{n} p_k q_j =: \sum_{j=1}^{\infty} p_k q_j$ exists in A. Moreover, in the same way we can prove that if $|\beta_k| > \sup W$, then $\sum_{j=1}^{\infty} p_j q_k$ exists in A. Thus, if $W \neq \emptyset$, we may define:

$$W_\lambda := \left\{ |\lambda_k| \colon\ |\lambda_k| > \sup W;\ \sum_{j=1}^{\infty} p_k q_j \neq p_k \right\},$$

$$W_\beta := \left\{ |\beta_k| \colon\ |\beta_k| > \sup W;\ \sum_{j=1}^{\infty} p_j q_k \neq q_k \right\}.$$

The arguments leading to Theorem 2.4 now prove the following formula:

Theorem 2.5 (global spectral formula for ϱ)**.** *With the hypothesis of Theorem* 2.4 *(where both* $\sigma(a)$ *and* $\sigma(b)$ *are infinite sets), we have*

$$\varrho(a,b) = \begin{cases} \sup W \cup W_\lambda \cup W_\beta & \textit{if } W \neq \emptyset, \\ \sup\{r_\sigma(a), r_\sigma(b)\} & \textit{if } W = \emptyset. \end{cases}$$

We may remark that if both $\sigma(a)$ and $\sigma(b)$ are finite sets, then the formula in Theorem 2.2 applies. If one spectrum is infinite ($\sigma(a)$), and the other finite ($\sigma(b)$), then one can easily adjust the formula in Theorem 2.4: Specifically, if $\sigma(b)$ is finite, then every spectral value has a corresponding Riesz projection and the set W_λ becomes redundant with its role being taken over by an adjusted version of the set W (where q_0 is the Riesz projection corresponding to $\beta_0 = 0$). To deal with the cluster point $0 \in \sigma(a)$ one needs a limiting process, as in the proof of Theorem 2.4, which necessitates the definition of W_β.

To illustrate the implementation as well as the practical value of Theorem 2.5, consider the following:

Example 2.6. With the usual notation, let X be the Banach space $L^1[1,\infty)$. Given $f \in X$, define noncommuting $T, S \in \mathcal{L}(X)$ by

$$(Tf)(t) = \frac{f(t)}{k} \quad \text{if } t \in [k, k+1),\ k \in \mathbb{N}$$

and

$$(Sf)(t) = \begin{cases} f(t) & \text{if } t \in [1,2), \\ \dfrac{f(t) + f(t-k+1)}{k^2} & \text{if } t \in [k, k+1),\ 1 < k \in \mathbb{N}. \end{cases}$$

It is straightforward to calculate $\sigma(T) = \{1/k\colon\ k \in \mathbb{N}\} \cup \{0\}$, and $\sigma(S) = \{1/k^2\colon\ k \in \mathbb{N}\} \cup \{0\}$. Write $p(1/k, T) =: P_k$ and $p(1/k^2, S) =: Q_k$. If $k \in \mathbb{N}$ and $f \in X$, then it follows readily, by Cauchy's formula, that:

(1) $(P_k f)(t) = \chi_{[k,k+1)}(t) f(t)$,
(2) $Q_1 = P_1$,
(3) $(Q_k f)(t) = \chi_{[k,k+1)}(t)\big[f(t) + f(t-k+1)/(1-k^2)\big]$ if $k \neq 1$.

Then $P_k Q_l = Q_l$ if $k = l$, and $P_k Q_l = 0$ if $k \neq l$. In terms of Theorem 2.5, we observe that $W = \{1/k - 1/k^2\colon\ k \in \mathbb{N}\}$, $W_\lambda = \{1/2, 1/3\}$, and $W_\beta = \emptyset$. Thus $\varrho(T,S) = 1/2$. Also, the fact that $Q_k P_l = P_l$ if $k = l$ and $Q_k P_l = 0$ if $k \neq l$ implies that $\varrho(S,T) = 1/2$. So $\rho(T,S) = 1/2$.

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