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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 53 (2014), No. 2, 85–100

Persistent URL: `http://dml.cz/dmlcz/144041`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **53**, 2 (2014) 85–100

Generalized Boundary Value Problems for Nonlinear Fractional Langevin Equations*

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(Received February 20, 2014)

Abstract

In this paper, generalized boundary value problems for nonlinear fractional Langevin equations is studied. Some new existence results of solutions in the balls with different radius are obtained when the nonlinear term satisfies nonlinear Lipschitz and linear growth conditions. Finally, two examples are given to illustrate the results.

Key words: Nonlinear fractional Langevin equations, boundary value problems, existence, fixed point theorem.

2010 Mathematics Subject Classification: 26A33, 34A12, 34G20.

*The first and third authors acknowledge partially supported by the National Natural Science Foundation of China (11201091), Key Projects of Science and Technology Research in the Chinese Ministry of Education (211169), Key Support Subject (Applied Mathematics), Key Project on the Reforms of Teaching Contents and Course System of Guizhou Normal College and Doctor Project of Guizhou Normal College (13BS010); The second author acknowledge the support by Slovak Research and Development Agency under the contract No. APVV-0134-10 and by the Slovak Grant Agency VEGA No. 1/0071/14.

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1 Introduction

Fractional derivatives provide a good tool for the description of memory and hereditary properties of various materials and processes. In particular, fractional differential equations appear naturally in a number of fields such as electrical circuits, biophysics, blood flow phenomena, physics, polymer rheology, geophysics, aerodynamics, nonlinear oscillation of earthquake, etc. Highly remarkable monographs which provide the main theoretical tools for the qualitative analysis of fractional differential equations, and at the same time, show the interconnection as well as the contrast between integer differential models and fractional differential models, are [1, 2, 3, 4, 5, 6, 7, 8]. Many research on the existence, limit properties, stability, periodicity of solutions and optimal controls for all kinds of fractional differential equations have been reported, see for instance the contrbutions of Agarwal, O'Regan and Staněk [9, 10, 11, 12, 13] and other researchers [14, 15, 16, 17, 18, 19, 20, 21, 22, 23]) and the references therein.

On the other hand, all kinds of Langevin equations are widely used to describe the evolution of physical phenomena in fluctuating environments. For instance, Brownian motion is well described by the Langevin equation (or generalized Langevin equation) when the random fluctuation force is assumed to be white noise (or not white noise). For systems in complex media, ordinary Langevin equation does not provide the correct description of the dynamics. As a results, various generalizations of Langevin equations have been offered to describe dynamical processes in a fractal medium. One such generalization is the generalized Langevin equation which incorporates the fractal and memory properties with a dissipative memory kernel into the Langevin equation. This gives rise to study fractional Langevin equation, see for instance [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37] and the references therein.

Motivated by [31, 32, 33, 34, 35], we study the existence of solutions for the following generalized boundary value problems for nonlinear fractional Langevin equations:

$$\begin{cases} {}^cD_t^\beta({}^cD_t^\alpha+\lambda)x(t)=f(t,x(t)), \quad t\in J:=[0,1],\ 1<\alpha<2,\ 0<\beta<1, \\ ax(0)+bx(1)=c, \quad x'(0)=x'(1)=0, \end{cases} \tag{1}$$

where ${}^cD_t^\alpha$ (or ${}^cD_t^\beta$) is the Caputo fractional derivative of order α (or β) with the lower limit 0, $f\colon J\times R\to R$ is a given continuous function, R denotes the set of real numbers, a,b,λ are real numbers and $a+b\neq 0$. We mention that the solution of the problem (1) is understood in the classical sense, i.e., a solution x is taken to mean a function in $C(J,R)$ for which the corresponding fractional derivatives exist and which satisfies the first equation in the problem (1) and prescribed boundary conditions identically.

In the present paper, we show existence and uniqueness results of solutions for the problem (1) by virtue of fractional calculus and fixed point method. Compared with the earlier results appeared in [31, 32, 33, 35], there are at least three differences: (i) the boundary value conditions in the current model has more precise for physical measurements; (ii) the nonlinear term f satisfies nonlinear D-contraction (see Definition 2.3) and linear growth conditions; (iii) our assumptions are weakened and easy to check. Compared with the model in [34], the constants a, b and c are important and will guarantee us to extend the linear Lipschitz condition to the nonlinear D-contraction condition. In particular, we point that there exists an error in the Lemma 1 of [35] since the author assume that $0 < \alpha \le 1$ in the equation (3.1), which can not derive the formula (3.2) in the Lemma 1 of [35] in anyway. In fact, the formula (3.2) in the Lemma 1 of [35] holds for $1 < \alpha \le 2$.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts.

Throughout this paper, let $C(J, R)$ be the Banach space of all continuous functions from J into R with the norm $\|x\|_C := \sup\{|x(t)| : t \in J\}$, for $x \in C(J, R)$. For Lebesgue measurable functions $l\colon J \to R$, define the norm

$$\|l\|_{L^p(J,R)} := \left(\int_J |l(t)|^p \, dt\right)^{\frac{1}{p}}, \quad 1 \le p < \infty.$$

We denote $L^p(J, R)$ the Banach space of all Lebesgue measurable functions l with $\|l\|_{L^p(J,R)} < \infty$.

Let us recall the following known definitions. For more details, see Kilbas et al. [3].

Definition 2.1 The fractional integral of order γ with the lower limit zero for an integrable function $f\colon [0, \infty) \to R$ is defined as

$$I_t^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} \, ds, \quad t > 0, \ \gamma > 0,$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 For a n-times differentiable function $f\colon [0, \infty) \to R$, the Caputo derivative of fractional order γ is defined as

$${}^cD_t^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t (t-s)^{n-\gamma-1} f^{(n)}(s) \, ds, \quad n-1 < \gamma \le n.$$

Definition 2.3 Let X be a Banach space with the norm $\|\cdot\|$. A mapping $T\colon X \to X$ is called D-Lipschitzian if there exists a continuous nondecreasing function $\phi_T\colon R^+ \to R^+$ satisfying

$$\|Tx - Ty\| \le \phi_T(\|x - y\|)$$

for all $x, y \in X$ with $\phi_T(0) = 0$. Here R^+ denotes the set of nonnegative real numbers. Sometime we call the function ϕ_T a D-function of T on X. If

$\phi_T(r) = \alpha r$ for some constant $\alpha > 0$, then T is called a Lipschitzian with a Lipschitz constant α and further if $\alpha < 1$, then T is called a contraction with the contraction constant α. Again if ϕ_T satisfies $\phi_T(r) < r, r > 0$, then T is called a nonlinear D-contraction on X.

Define a class Φ of function $\phi\colon R^+ \to R^+$ satisfying properties: (i) ϕ is continuous, (ii) ϕ is nondecreasing, and (iii) $\phi(x+y) \le \phi(x) + \phi(y)$ for all $x, y \in R^+$. Obviously, if $\phi \in \Phi$, then $|\phi(x) - \phi(y)| \le \phi(|x-y|)$ for all $x, y \in R^+$.

Now consider the function $f\colon J \times R \to R$ defined by

$$f(t,x) = \frac{H\phi(|x|)}{N + \phi(|x|)}. \tag{2}$$

where $\phi \in \Phi$ satisfying $\phi(r) < r$, and N is a positive constant such that $H \le N$. It is clear that

$$|f(t,x) - f(t,y)| \le \frac{H\,|\phi(|x|) - \phi(|y|)|}{N + \phi(|x|) + \phi(|y|)} \le \frac{H|x-y|}{N + |x-y|}, \tag{3}$$

which implies f is nonlinear D-contraction on the second variable. There do exist function ϕ given in (2), such as:

$$\begin{aligned} \phi(r) &= kr, \\ \phi(r) &= \ln(2+r), \\ \phi(r) &= \sqrt{1+r} - 1, \\ \phi(r) &= \arctan r. \end{aligned}$$

In order to study the problem (1), we introduce the following linear problem

$$\begin{cases} {}^cD_t^\beta({}^cD_t^\alpha + \lambda)x(t) = h(t), \quad t \in J,\ 1 < \alpha < 2,\ 0 < \beta < 1, \\ ax(0) + bx(1) = c, \quad x'(0) = x'(1) = 0,\ a + b \ne 0, \end{cases} \tag{4}$$

where $h \in C(J, R)$.

Lemma 2.4 *A unique solution of the equation* (4) *satisfies the following integral equation*

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} h(s)\,ds - \lambda x(u) \right) du \\ & - \frac{b}{a+b} \int_0^1 \frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} h(s)\,ds - \lambda x(u) \right) du \\ & + \frac{b - t^\alpha(a+b)}{\alpha(a+b)} \int_0^1 \frac{(1-u)^{\alpha-2}}{\Gamma(\alpha-1)} \left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} h(s)\,ds - \lambda x(u) \right) du \\ & + \frac{c}{a+b}. \end{aligned} \tag{5}$$

Proof Applying Definitions 2.1 and 2.2, one can find the general solution of

$$^cD_t^{\beta}(^cD_t^{\alpha}+\lambda)x(t)=h(t)$$

can be written as

$$\begin{aligned} x(t) =& \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}\left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)}h(s)\,ds-\lambda x(u)\right)du \\ & -\frac{t^{\alpha}}{\Gamma(\alpha+1)}r_0-r_1-r_2t. \end{aligned} \tag{6}$$

Using the conditions for the equation (4), we find that

$$\begin{aligned} r_2 =& \ 0, \\ r_1 =& \ \frac{b}{a+b}\int_0^1 \frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}\left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)}h(s)ds-\lambda x(u)\right)du \\ & -\frac{b}{\alpha(a+b)}\int_0^1 \frac{(1-u)^{\alpha-2}}{\Gamma(\alpha-1)}\left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)}h(s)ds-\lambda x(u)\right)du-\frac{c}{a+b}, \\ r_0 =& \ \Gamma(\alpha)\int_0^1 \frac{(1-u)^{\alpha-2}}{\Gamma(\alpha-1)}\left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)}h(s)ds-\lambda x(u)\right)du. \end{aligned}$$

Substituting the values of r_0, r_1, r_2 in (6), we obtain the solution (5). This completes the proof. □

Now, we state a known result due to Krasnoselskii (see [38]) which is needed to prove the existence of at least one solution of the problem (1).

Theorem 2.5 *Let $\mathcal{M}$ be a closed convex and nonempty subset of a Banach space X. Let $\mathcal{A}, \mathcal{B}$ be two operators such that*
(i) $\mathcal{A}x+\mathcal{B}y\in\mathcal{M}$ whenever $x, y\in\mathcal{M}$,
(ii) $\mathcal{A}$ is a compact and continuous,
(iii) $\mathcal{B}$ is a contraction mapping.
Then there exists a $z\in\mathcal{M}$ such that $z=\mathcal{A}z+\mathcal{B}z$.

3 Main results

In this section, we firstly apply Banach fixed point theorem to show the existence and uniqueness of solutions of the problem (1).

For brevity, set $\sup_{t\in J}|f(t,0)|=M$.

Theorem 3.1 *Assume that $f\colon J\times R\to R$ be a jointly continuous function and satisfies nonlinear D-contraction on the second variable. Then the problem* (1) *has a unique solution in $B_r=\{x\in C(J,R):\|x\|_C\le r\}$, where*

$$r\ge\frac{M\dfrac{(|a+b|+2m)\,\alpha+m\beta}{\alpha|a+b|\Gamma(\alpha+\beta+1)}+\left|\dfrac{c}{a+b}\right|}{1-\Lambda},$$

provided that

$$\Lambda = \frac{(|a+b|+2m)\,\alpha + m\beta}{\alpha|a+b|\Gamma(\alpha+\beta+1)} + |\lambda|\frac{|a+b|+2m}{|a+b|\Gamma(\alpha+1)} < 1, \tag{7}$$

and $m = \max\{|a|, |b|\}$.

Proof Letting $\Omega = J \times B_r$, then Ω is a compact set. Since f a jointly continuous function, we can define $\sup_{(t,x)\in\Omega} |f(t,x)| = f_{\max}$.

Define an operator $F\colon B_r \to C(J,R)$ by

$$\begin{aligned}(Fx)(t) &= \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} f(s,x(s))ds - \lambda x(u)\right) du \\ &\quad - \frac{b}{a+b}\int_0^1 \frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} f(s,x(s))ds - \lambda x(u)\right) du \\ &\quad + \frac{b - t^\alpha(a+b)}{\alpha(a+b)} \int_0^1 \frac{(1-u)^{\alpha-2}}{\Gamma(\alpha-1)} \left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} f(s,x(s))ds - \lambda x(u)\right) du + \frac{c}{a+b}.\end{aligned}$$

It is easy to verify that F is well defined. In fact for every $x \in B_r$ and any $\delta > 0$, $0 < t < t+\delta$, we get

$$\begin{aligned} &|(Fx)(t+\delta) - (Fx)(t)| \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t [(t+\delta-u)^{\alpha-1} - (t-u)^{\alpha-1}] \left|\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} f(s,x(s))ds - \lambda x(u)\right| du \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_t^{t+\delta} (t+\delta-u)^{\alpha-1} \left|\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} f(s,x(s))ds - \lambda x(u)\right| du \\ &\quad + \frac{1}{\alpha}|t^\alpha - (t+\delta)^\alpha| \int_0^1 \frac{(1-u)^{\alpha-2}}{\Gamma(\alpha-1)} \left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} f(s,x(s))ds - \lambda x(u)\right) du \\ &\le \frac{1}{\Gamma(\alpha)} \left(\frac{f_{\max}}{\Gamma(\beta+1)} + \lambda r\right) \\ &\quad \times \left(\int_0^t [(t+\delta-u)^{\alpha-1} - (t-u)^{\alpha-1}]du + \int_t^{t+\delta} (t+\delta-u)^{\alpha-1} du\right) \\ &\quad + \frac{1}{\Gamma(\alpha+1)} \left(\frac{f_{\max}}{\Gamma(\beta+1)} + \lambda r\right) [(t+\delta)^\alpha - t^\alpha] \\ &= \frac{2}{\Gamma(\alpha+1)} \left(\frac{f_{\max}}{\Gamma(\beta+1)} + \lambda r\right) [(t+\delta)^\alpha - t^\alpha] \to 0, \quad \text{as } \delta \to 0.\end{aligned}$$

Step 1. We show that $FB_r \subset B_r$. For $x \in B_r$, $t \in J$, we have

$$
\begin{aligned}
\|Fx\|_C \le \sup_{t\in J}\Big\{ &\int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \\
&\times \Big(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)}(|f(s,x(s))-f(s,0)|+|f(s,0)|)ds + |\lambda x(u)|\Big)du \\
&+ \Big|\frac{b}{a+b}\Big| \Big|\int_0^1 \frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)} \\
&\times \Big(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)}(|f(s,x(s))-f(s,0)|+|f(s,0)|)ds + |\lambda x(u)|\Big)du\Big| \\
&+ \Big|\frac{b-t^\alpha(a+b)}{\alpha(a+b)}\Big| \Big|\int_0^1 \frac{(1-u)^{\alpha-2}}{\Gamma(\alpha-1)} \\
&\times \Big(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)}(|f(s,x(s))-f(s,0)|+|f(s,0)|)ds + |\lambda x(u)|\Big)du\Big| \\
&+ \Big|\frac{c}{a+b}\Big|\Big\} \\
\le \sup_{t\in J}\Big\{ &\int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)}ds\Big)du\Big\}(\Phi(\|x\|_C)+M) \\
&+ \sup_{t\in J}\int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}du|\lambda|\|x\|_C \\
&+ \Big|\frac{b}{a+b}\Big|\int_0^1 \frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)}ds\Big)du(\Phi(\|x\|_C)+M) \\
&+ \Big|\frac{b}{a+b}\Big|\int_0^1 \frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}du|\lambda|\|x\|_C \\
&+ \sup_{t\in J}\Big|\frac{b-t^\alpha(a+b)}{\alpha(a+b)}\Big|\int_0^1 \frac{(1-u)^{\alpha-2}}{\Gamma(\alpha-1)}\Big(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)}ds\Big)du(\Phi(\|x\|_C)+M) \\
&+ \sup_{t\in J}\Big|\frac{b-t^\alpha(a+b)}{\alpha(a+b)}\Big|\int_0^1 \frac{(1-u)^{\alpha-2}}{\Gamma(\alpha)}du|\lambda|\|x\|_C + \Big|\frac{c}{a+b}\Big| \\
\le (r+M)\Big\{ &\frac{1}{\Gamma(\alpha)\Gamma(\beta+1)}\sup_{t\in J}\int_0^t (t-u)^{\alpha-1}u^\beta du \\
&+ \frac{m}{|a+b|\Gamma(\alpha)\Gamma(\beta+1)}\int_0^1 (1-u)^{\alpha-1}u^\beta du \\
&+ \frac{m}{\alpha|a+b|\Gamma(\alpha-1)\Gamma(\beta+1)}\int_0^1 (1-u)^{\alpha-2}u^\beta du\Big\} \\
&+ |\lambda| r\Big\{\sup_{t\in J}\int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}du + \frac{m}{|a+b|}\int_0^1 \frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}du \\
&+ \frac{m}{\alpha|a+b|}\int_0^1 \frac{(1-u)^{\alpha-2}}{\Gamma(\alpha-1)}du + \Big|\frac{c}{a+b}\Big|\Big\}.
\end{aligned}
$$

Using the relation for Beta function $B(\cdot,\cdot)$:

$$B(\beta+1,\alpha)=\int_0^1(1-u)^{\alpha-1}u^\beta du=\frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)},$$
$$\int_0^\eta(\eta-u)^{\alpha-1}u^\beta du=\frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\eta^{\alpha+\beta},$$

we find that

$$\|Fx\|_C\le(r+M)\frac{(|a+b|+2m)\,\alpha+m\beta}{\alpha|a+b|\Gamma(\alpha+\beta+1)}+|\lambda|r\frac{|a+b|+2m}{|a+b|\Gamma(\alpha+1)}+\left|\frac{c}{a+b}\right|\le r.$$

Step 2. We show that F is a contraction mapping.
For $x,y\in B_r$ and for each $t\in J$, we get

$$\|Fx-Fy\|_C\le\sup_{t\in J}\Bigg\{\int_0^t\frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}$$
$$\times\left(\int_0^u\frac{(u-s)^{\beta-1}}{\Gamma(\beta)}(|f(s,x(s))-f(s,y(s))|)ds+|\lambda||x(u)-y(u)|\right)du$$
$$+\left|\frac{b}{a+b}\right|\left|\int_0^1\frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}\right.$$
$$\left.\times\left(\int_0^u\frac{(u-s)^{\beta-1}}{\Gamma(\beta)}(|f(s,x(s))-f(s,y(s))|)ds+|\lambda||x(u)-y(u)|\right)du\right|$$
$$+\left|\frac{b-t^\alpha(a+b)}{\alpha(a+b)}\right|\left|\int_0^1\frac{(1-u)^{\alpha-2}}{\Gamma(\alpha-1)}\right.$$
$$\left.\times\left(\int_0^u\frac{(u-s)^{\beta-1}}{\Gamma(\beta)}(|f(s,x(s))-f(s,y(s))|)ds+|\lambda||x(u)-y(u)|\right)du\right|\Bigg\}$$
$$\le\Phi(\|x-y\|)\Bigg\{\int_0^1\frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}\left(\int_0^u\frac{(u-s)^{\beta-1}}{\Gamma(\beta)}ds\right)du$$
$$+\frac{m}{|a+b|}\int_0^1\frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}\left(\int_0^u\frac{(u-s)^{\beta-1}}{\Gamma(\beta)}ds\right)du$$
$$+\frac{m}{\alpha|a+b|}\int_0^1\frac{(1-u)^{\alpha-2}}{\Gamma(\alpha-1)}\left(\int_0^u\frac{(u-s)^{\beta-1}}{\Gamma(\beta)}ds\right)du\Bigg\}+|\lambda|\|x-y\|$$
$$\times\left\{\int_0^1\frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}du+\frac{m}{|a+b|}\int_0^1\frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}du+\frac{m}{\alpha|a+b|}\int_0^1\frac{(1-u)^{\alpha-2}}{\Gamma(\alpha-1)}du\right\}$$
$$\le\left(\frac{\Big(|a+b|+2m\Big)\alpha+m\beta}{\alpha|a+b|\Gamma(\alpha+\beta+1)}+|\lambda|\frac{|a+b|+2m}{|a+b|\Gamma(\alpha+1)}\right)\|x-y\|$$
$$=\Lambda\|x-y\|.$$

Thus, F is a contraction mapping on B_r due to the condition (7). By applying the well-known Banach's contraction mapping principle we know that the operator F has a unique fixed point on B_r. Therefore, the problem (1) has a unique solution. □

Secondly, we will apply Krasnoselskii fixed point theorem to show another existence result in another ball where $f\colon J\times R\to R$ will be weaken a Carathéodory function and the condition (7) is weakened in some sense.

Theorem 3.2 *Assume that $f\colon J\times R\to R$ be a Carathéodory function and satisfies nonlinear D-contraction on the second variable. In addition, there exists a constant $L>0$ such that $|f(t,x)|\le L(1+|x|)$ for each $t\in J$ and all $x\in R$. Then the problem (1) has at least one solution in $B_{r'}=\{x\in C(J,R):\|x\|_C\le r'\}$ where*

$$r'\ge\frac{\dfrac{(|a+b|+2m)\,\alpha+m\beta}{\alpha|a+b|\Gamma(\alpha+\beta+1)}L+\left|\dfrac{c}{a+b}\right|}{1-\dfrac{(|a+b|+2m)\,\alpha+m\beta}{\alpha|a+b|\Gamma(\alpha+\beta+1)}L-|\lambda|\dfrac{|a+b|+2m}{|a+b|\Gamma(\alpha+1)}},\tag{8}$$

provided that

$$\max\left\{\frac{2m\alpha+m\beta}{\alpha|a+b|\Gamma(\alpha+\beta+1)}+\frac{2m|\lambda|}{|a+b|\Gamma(\alpha+1)},\Lambda_L\right\}<1,\tag{9}$$

and

$$\Lambda_L=L\frac{(|a+b|+2m)\,\alpha+m\beta}{\alpha|a+b|\Gamma(\alpha+\beta+1)}+|\lambda|\frac{|a+b|+2m}{|a+b|\Gamma(\alpha+1)}.$$

Proof We define two operators P and Q on $B_{r'}$ as

$$(Px)(t)=\int_0^t\frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}\left(\int_0^u\frac{(u-s)^{\beta-1}}{\Gamma(\beta)}f(s,x(s))ds-\lambda x(u)\right)du,$$

$$\begin{aligned}(Qx)(t)={}&-\frac{b}{a+b}\int_0^1\frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}\left(\int_0^u\frac{(u-s)^{\beta-1}}{\Gamma(\beta)}f(s,x(s))ds-\lambda x(u)\right)du\\&+\frac{b-t^\alpha(a+b)}{\alpha(a+b)}\int_0^1\frac{(1-u)^{\alpha-2}}{\Gamma(\alpha-1)}\left(\int_0^u\frac{(u-s)^{\beta-1}}{\Gamma(\beta)}f(s,x(s))ds-\lambda x(u)\right)du\\&+\frac{c}{a+b}.\end{aligned}$$

It is not difficult to verify that $Px+Qy\in C(J,R)$ for any $x,y\in B_{r'}$ according to the linear growth condition on f. Moreover, for any $x,y\in B_{r'}$ we find that

$$
\begin{aligned}
&|(Px)(t)+(Qy)(t)| \\
&\leq \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}\left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)}|f(s,x(s))|ds+|\lambda x(u)|\right)du \\
&+\frac{m}{|a+b|}\int_0^1 \frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}\left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)}|f(s,x(s))|ds+|\lambda x(u)|\right)du \\
&+\frac{m}{\alpha|a+b|}\int_0^1 \frac{(1-u)^{\alpha-2}}{\Gamma(\alpha-1)}\left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)}|f(s,x(s))|ds+|\lambda x(u)|\right)du+\left|\frac{c}{a+b}\right| \\
&\leq \frac{L(1+\|x\|_C)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t (t-u)^{\alpha-1}\left(\int_0^u (u-s)^{\beta-1}ds\right)du \\
&+|\lambda|\|x\|_C\int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}du \\
&+\frac{mL(1+\|x\|_C)}{|a+b|\Gamma(\alpha)\Gamma(\beta)}\int_0^1 (1-u)^{\alpha-1}\left(\int_0^u (u-s)^{\beta-1}ds\right)du \\
&+\frac{m|\lambda|\|x\|_C}{|a+b|}\int_0^1 \frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}du \\
&+\frac{mL(1+\|x\|_C)}{\alpha|a+b|\Gamma(\alpha-1)\Gamma(\beta)}\int_0^1 (1-u)^{\alpha-2}\left(\int_0^u (u-s)^{\beta-1}ds\right)du \\
&+\frac{m|\lambda|\|x\|_C}{\alpha|a+b|}\int_0^1 \frac{(1-u)^{\alpha-2}}{\Gamma(\alpha-1)}du+\left|\frac{c}{a+b}\right| \\
&\leq \frac{L(1+r')}{\Gamma(\alpha)\Gamma(\beta+1)}\int_0^t (t-u)^{\alpha-1}u^\beta du+\frac{|\lambda|r'}{\Gamma(\alpha+1)} \\
&+\frac{mL(1+r')}{|a+b|\Gamma(\alpha)\Gamma(\beta+1)}\int_0^1 (1-u)^{\alpha-1}u^\beta du+\frac{m|\lambda|r'}{|a+b|\Gamma(\alpha+1)} \\
&+\frac{mL(1+r')}{\alpha|a+b|\Gamma(\alpha-1)\Gamma(\beta+1)}\int_0^1 (1-u)^{\alpha-2}u^\beta du+\frac{m|\lambda|r'}{\alpha|a+b|\Gamma(\alpha)}+\left|\frac{c}{a+b}\right| \\
&\leq \frac{(|a+b|+2m)\,\alpha+m\beta}{\alpha|a+b|\Gamma(\alpha+\beta+1)}L+\left|\frac{c}{a+b}\right| \\
&+\left(\frac{(|a+b|+2m)\,\alpha+m\beta}{\alpha|a+b|\Gamma(\alpha+\beta+1)}L+\frac{|a+b|+2m}{|a+b|\Gamma(\alpha+1)}|\lambda|\right)r' \\
&\leq r',
\end{aligned}
$$

which implies that $Px+Qy \in B_{r'}$.

For $x, y \in \Omega$ and for each $t \in J$, by the analogous argument to the proof of Theorem 3.1, we obtain

$$
\begin{aligned}
&|(Qx)(t) - (Qy)(t)| \\
&\leq \left|\frac{b}{a+b}\right| \int_0^1 \frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} (|f(s,x(s)) - f(s,y(s))|) ds \right) du \\
&\quad + \left|\frac{b|\lambda|}{a+b}\right| \int_0^1 \frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)} |x(u) - y(u)| du + \left|\frac{b - t^\alpha(a+b)}{\alpha(a+b)}\right| \\
&\quad \times \int_0^1 \frac{(1-u)^{\alpha-2}}{\Gamma(\alpha-1)} \left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} (|f(s,x(s)) - f(s,y(s))|) ds \right) du \Big\} \\
&\quad + \left|\frac{(b - t^\alpha(a+b))|\lambda|}{\alpha(a+b)}\right| \int_0^1 \frac{(1-u)^{\alpha-2}}{\Gamma(\alpha-1)} |x(u) - y(u)| du \\
&\leq \Phi(\|x-y\|_C) \Big\{ \frac{m}{|a+b|} \int_0^1 \frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} ds \right) du \\
&\quad + \frac{m}{\alpha|a+b|} \int_0^1 \frac{(1-u)^{\alpha-2}}{\Gamma(\alpha-1)} \left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} ds \right) du \Big\} \\
&\quad + |\lambda| \|x-y\|_C \Big\{ \frac{m}{|a+b|} \int_0^1 \frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)} du + \frac{m}{\alpha|a+b|} \int_0^1 \frac{(1-u)^{\alpha-2}}{\Gamma(\alpha-1)} du \Big\} \\
&\leq \left(\frac{2m\alpha + m\beta}{\alpha|a+b|\Gamma(\alpha+\beta+1)} + \frac{2m|\lambda|}{|a+b|\Gamma(\alpha+1)} \right) \|x-y\|_C.
\end{aligned}
$$

From the condition (9), it follows that Q is a contraction mapping.

The continuity and linear grown condition of f implies that the operator P is continuous by means of Lebesgue Dominated Convergence Theorem. Also, P is uniformly bounded on $B_{r'}$ since

$$
\|Px\|_C \leq \frac{L(1+r')}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda| r'}{\Gamma(\alpha+1)}.
$$

Now we need to prove the compactness of the operator P. In fact, it is easy to obtain

$$
\begin{aligned}
&|(Px)(t_2) - (Px)(t_1)| \\
&= \Big| \int_0^{t_2} \frac{(t_2-u)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} f(s,x(s)) ds - \lambda x(u) \right) du \\
&\quad - \int_0^{t_1} \frac{(t_1-u)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} f(s,x(s)) ds - \lambda x(u) \right) du \Big| \\
&\leq \frac{L(1+r')}{\Gamma(\alpha+\beta+1)} |t_2^{\alpha+\beta} - t_1^{\alpha+\beta}| + \frac{|\lambda| r'}{\Gamma(\alpha+1)} |t_2^\alpha - t_1^\alpha| \\
&\leq \left(\frac{L(1+r')}{\Gamma(\alpha+\beta)} + \frac{|\lambda| r'}{\Gamma(\alpha+1)} \right) |t_2 - t_1|,
\end{aligned}
$$

which is independent of x and tends to zero as $t_1 \to t_2$. Thus P is relatively compact on $B_{r'}$. Hence, by the Arzela–Ascoli Theorem, P is compact on $B_{r'}$. Thus all the assumption of Lemma 2.5 are satisfied and the conclusion of Lemma 2.5 implies that the problem (1) has at least one solution on J. The proof is completed. □

Corollary 3.3 *Suppose that the following conditions hold:*
(i) there exists a function $\mu(t) \in L^{\frac{1}{\sigma}}(J, R^+)$ *for some* $\sigma \in (0, \beta)$ *such that*

$$|f(t,x) - f(t,y)| \leq \mu(t)|x-y|, \text{ for each } t \in J \text{ and all } x, y \in R.$$

(ii) there exists a function $\mu(t) \in L^{\frac{1}{\sigma}}(J, R^+)$ *for some* $\sigma \in (0, \beta)$ *such that*

$$|f(t,x)| \leq \mu(t)(1+|x|) \text{ for each } t \in J \text{ and all } x \in R.$$

Then the problem (1) *has at least one solution on* J *provided that*

$$\frac{((|a+b|+2m)\alpha + m\beta - m\sigma)\,\Gamma(\beta-\sigma+1)\mu^*}{\alpha|a+b|\Gamma(\beta)\Gamma(\alpha+\beta-\sigma+1)}\left(\frac{1-\sigma}{\beta-\sigma}\right)^{1-\sigma} + \frac{|a+b|+2m}{|a+b|\Gamma(\alpha+1)}|\lambda| < 1, \tag{10}$$

where $m = \max\{|a|, |b|\}$ *and* $\mu^* = \left(\int_0^1 (\mu(s))^{\frac{1}{\sigma}} ds\right)^{\sigma}$.

To end this section, we extend to study the following more general problem

$$\begin{cases} {}^cD_t^{\beta}({}^cD_t^{\alpha} + \lambda)x(t) = t^{\gamma} f(t, x(t)), \ t \in J, \ 1 < \alpha < 2, \ 0 < \beta < 1, \ \gamma > 0, \\ ax(0) + bx(1) = c, \quad x'(0) = x'(1) = 0, \ a + b \neq 0. \end{cases} \tag{11}$$

We remark that the term t^{γ} in the first equation in the problem (11) will weaken the impact from the singular kernel $(t-\cdot)^{\beta-1}$ in the possible singular integrals. Some related work on a weakly singular integral equation has been reported by Wang et al. [39].

Theorem 3.4 *Assume that* $f : J \times R \to R$ *be a jointly continuous function and satisfies nonlinear D-contraction on the second variable. In addition, there exists a constant* $L > 0$ *such that* $|f(t,x)| \leq L(1+|x|)$ *for each* $t \in J$ *and all* $x \in R$. *Then the problem* (11) *has at least one solution provided that*

$$\max\left\{\frac{(2m\alpha + m\beta + m\gamma)\Gamma(\gamma+1)}{\alpha|a+b|\Gamma(\alpha+\beta+\gamma+1)} + \frac{2m|\lambda|}{|a+b|\Gamma(\alpha+1)}, \frac{((|a+b|+2m)\alpha + m\beta + m\gamma)\,\Gamma(\gamma+1)L}{\alpha|a+b|\Gamma(\alpha+\beta+\gamma+1)} + \frac{(|a+b|+2m)|\lambda|}{|a+b|\Gamma(\alpha+1)}\right\} < 1, \tag{12}$$

where $m = \max\{|a|, |b|\}$.

Proof Denote

$$B_{r''} = \{x \in C(J,R) : \|x\|_C \le r'',\}$$

where

$$r'' > \frac{\dfrac{((|a+b|+2m)\alpha + m\beta + m\gamma)\,\Gamma(\gamma+1)L}{\alpha|a+b|\Gamma(\alpha+\beta+\gamma+1)} + |\dfrac{c}{a+b}|}{1 - \dfrac{((|a+b|+2m)\alpha + m\beta + m\gamma)\,\Gamma(\gamma+1)L}{\alpha|a+b|\Gamma(\alpha+\beta+\gamma+1)} - \dfrac{(|a+b|+2m)|\lambda|}{|a+b|\Gamma(\alpha+1)}}.$$

We define two operators P and Q on $B_{r''}$ as follows

$$(Px)(t) = \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} s^\gamma f(s,x(s))ds - \lambda x(u) \right) du,$$

$$(Qx)(t) = -\frac{b}{a+b} \int_0^1 \frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} s^\gamma f(s,x(s))ds - \lambda x(u) \right) du$$

$$+ \frac{b - t^\alpha(a+b)}{\alpha(a+b)} \int_0^1 \frac{(1-u)^{\alpha-2}}{\Gamma(\alpha-1)} \left(\int_0^u \frac{(u-s)^{\beta-1}}{\Gamma(\beta)} s^\gamma f(s,x(s))ds - \lambda x(u) \right) du$$

$$+ \frac{c}{a+b}.$$

The rest proof is very similar to the steps of Theorem 3.2, so we omit it here. □

4 Examples

In this section, we make two examples to illustrate our the above theory results from the subject of the mathematics. Other interesting application the reader can refer to Sandev at al. [34, 36]

Example 4.1 Consider the following boundary value problem

$$\begin{cases} {}^cD_t^{\frac{2}{3}} \left({}^cD_t^{\frac{11}{6}} + \frac{1}{100} x(t) \right) = \frac{|x(t)|}{25(1+|x(t)|)}, & t \in [0,1], \\ 2x(0) + x(1) = 0, \quad x'(0) = x'(1) = 0. \end{cases} \tag{13}$$

Set $\alpha = \frac{11}{6}$, $\beta = \frac{2}{3}$, $\lambda = \frac{1}{100}$, $a = 2, b = 1, c = 0$ and

$$f(t,x) = \frac{x}{25(1+x)}, \quad (t,x) \in [0,1] \times R^+.$$

Obviously f is a special case of (3), then f is nonlinear D-contraction on x and $|f(t,x)| \le \frac{1}{25}(1+|x|)$ with $L = \frac{1}{25}$. Further, the condition (7) is equivalent to

$$\frac{\frac{77}{6} + \frac{4}{3}}{\frac{11}{2}\Gamma(\frac{21}{6})} + \frac{7}{300\Gamma(\frac{17}{6})} \cong 0.7886 < 1.$$

Then the problem (13) has a unique solution on J according to Theorem 3.1.

On the one hand, the condition (9) is equivalent to

$$\max\left\{\frac{\frac{77}{6}+\frac{4}{3}}{\frac{11}{2}\Gamma(\frac{21}{6})}\frac{1}{25}+\frac{7}{300\Gamma(\frac{17}{6})},\frac{\frac{44}{6}+\frac{4}{3}}{\frac{11}{2}\Gamma(\frac{21}{6})}+\frac{4}{300\Gamma(\frac{17}{6})}\right\}$$
$$=\max\{0.0445, 0.4819\}<1.$$

Then the problem (13) has at least one solution on J according to Theorem 3.2.

Example 4.2 Consider the following boundary value problem

$$\begin{cases} {}^cD_t^{\frac{3}{4}}\left({}^cD_t^{\frac{3}{2}}+\frac{1}{10}x(t)\right)=\frac{|x(t)|}{(t+5)^2(1+|x(t)|)}, & t\in[0,1],\\ 2x(0)-x(1)=0, \quad x'(0)=x'(1)=0. \end{cases} \tag{14}$$

Set $\alpha=\frac{3}{2}$, $\beta=\frac{3}{4}$, $\lambda=\frac{1}{10}$ and

$$f(t,x)=\frac{x}{(t+5)^2(1+x)}, \quad (t,x)\in[0,1]\times R^+.$$

Obviously, $|f(t,x)|\le\frac{1}{(t+5)^2}(1+|x|)$ and $|f(t,x)-f(t,y)|\le\frac{1}{(t+5)^2}|x-y|$. So we can put $\mu(t)=\frac{1}{(t+5)^2}$. Set $\sigma=\frac{1}{2}$ and $\mu^*=\left(\frac{1}{3\times5^3}-\frac{1}{3\times6^3}\right)^{\frac{1}{2}}=0.0346$. Further, the condition (10) is equivalent to

$$\frac{8\Gamma(\frac{5}{4})\times0.0346\times\sqrt{2}}{(\frac{3}{2})\Gamma(\frac{3}{4})\Gamma(\frac{11}{4})}+\frac{1}{2\Gamma(\frac{5}{2})}\cong0.4961<1.$$

Then the problem (14) has at least one solution according to Corollary 3.3.

5 Final remarks

Sometimes we need to consider the following model

$$\begin{cases} {}^LD_t^{\beta}({}^LD_t^{\alpha}+\lambda)x(t)=f(t,x(t)),\ t\in J:=[0,1],\ 1<\alpha<2,\ 0<\beta<1,\\ x(0)=0,\ \lim_{t\to0^+}(t^{-\alpha}x(t))=x_1, \end{cases} \tag{15}$$

where ${}^LD_t^{\alpha}$ (or ${}^LD_t^{\beta}$) is the Riemann–Liouville fractional derivative of order α (or β) with the lower limit 0, f is a nonlinear D-contraction function, or $f(t,x)=F(t,x)+G(t,x)$ where F is continuous and G is a nonlinear D-contraction function and $F(t,x)\le L(1+|x|)$ with $L>0$.

By proceeding as in Section 2 and Section 3, one can derive the formula of the solution and obtain the existence theorems for the problem (15).

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