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Compatible Idempotent Terms in Universal Algebra

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Abstract

In universal algebra, we oftentimes encounter varieties that are not especially well-behaved from any point of view, but are such that all their members have a "well-behaved core", i.e. subalgebras or quotients with satisfactory properties. Of special interest is the case in which this "core" is a retract determined by an idempotent endomorphism that is uniformly term definable (through a unary term $t(x)$) in every member of the given variety. Here, we try to give a unified account of this phenomenon. In particular, we investigate what happens when various congruence properties—like congruence distributivity, congruence permutability or congruence modularity—are not supposed to hold unrestrictedly in any $\mathbf{A} \in \mathcal{V}$, but only for congruence classes of values of the term operation $t^{\mathbf{A}}$.

Key words: Congruence distributive variety, congruence modular variety, congruence permutable variety, idempotent endomorphism.

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1 Introduction

In universal algebra, we oftentimes encounter varieties that are not especially well-behaved from any point of view—for example, they may not satisfy any nontrivial congruence identity—but are such that all their members have a "well-behaved core", i.e. subalgebras or quotients with satisfactory properties that can be uniformly characterised throughout the variety. Such varieties have been repeatedly investigated by the present authors (and various coworkers) in different contexts ([5], [23], [24], [25], [28], [30]), but a unified account of this phenomenon is still wanted. Of special interest is the case in which the "well-behaved core" is a retract determined by an idempotent endomorphism that is uniformly term definable (through a unary term $t(x)$) in every member of the given variety. This is the topic we will address in the present paper. In particular, we will investigate what happens when various congruence properties—like congruence distributivity, congruence permutability or congruence modularity—are not supposed to hold unrestrictedly in any $\mathbf{A} \in \mathcal{V}$, but only for congruence classes of values of the term operation $t^{\mathbf{A}}$.

The paper is structured as follows. In § 2, for $\mathcal{K}$ a class of algebras we define the notions of $\mathcal{K}$-compatible and $\mathcal{K}$-idempotent term, and provide some examples. Thereafter, we restrict our attention to classes that are varieties. We observe some elementary facts concerning the previously introduced notions and provide a subdirect representation theorem for algebras in varieties having a $\mathcal{V}$-compatible and $\mathcal{V}$-idempotent term t. In § 3, we consider varieties $\mathcal{V}$ of a given type ν, and we let t, s be $\mathcal{V}$-idempotent and $\mathcal{V}$-compatible terms of the same type. We say that $\mathcal{V}$ is (t,s)-*distributive* iff for every $a, b \in \mathbf{A} \in \mathcal{V}$ and for every $\theta, \varphi, \psi \in \mathrm{Con}(\mathbf{A})$, we have

$$(t(a), s(b)) \in \theta \cap (\varphi \vee \psi) \text{ iff } (t(a), s(b)) \in (\theta \cap \varphi) \vee (\theta \cap \psi).$$

The section is devoted to an investigation of this property. In particular, we prove a restricted version of Jónsson's lemma for (t,t)-distributive varieties. In § 4, we relativise in a similar way to $\mathcal{V}$-compatible and $\mathcal{V}$-idempotent terms the concepts of congruence modularity and congruence permutability. Finally, in § 5, we revisit some classical results (see. e.g. [20]) on congruence properties of algebras, all with a marked geometrical flavour, adapting them to the present context.

2 Varieties with a compatible idempotent term

Let $\mathcal{K}$ be a class of algebras of a given type ν. A term $t\,(x, \overrightarrow{y})$ of type ν is said to be $\mathcal{K}$-*idempotent* in case for all $\mathbf{A} \in \mathcal{K}$ and for all $b, \overrightarrow{a} \in A$, it is the case that $t\,(t(b, \overrightarrow{a}), \overrightarrow{a}) = t(b, \overrightarrow{a})$; it is said to be $\mathcal{K}$-*compatible* if the term operation $t^{\mathbf{A}}$ is an endomorphism on any algebra $\mathbf{A} \in \mathcal{K}$.

These concepts are duly exemplified hereafter.

Example 1

(1) Recall that a *locally Boolean band* [29, Chapter II] is a band with zero $\mathbf{A}$ such that, for every $a \in A$, the principal subalgebra $(a] = \{axa \colon x \in A\}$ generated by a is a Boolean lattice with respect to the natural band partial ordering. Define the mapping $t \colon A \to (a]$ by $t(x) = axa$. As noticed in [32, Lemma 1.3.16], every locally Boolean band is normal, i.e. it satisfies the

identity $xyzx \approx xzyx$. Then

$$\begin{aligned} t(xy) &= axya = ayxa = aayxa = a(ay)xa \\ &= axaya = axaaya = t(x)t(y), \end{aligned}$$

whence t is an endomorphism, which is idempotent by definition of band. The same polynomial map is compatible and idempotent also w.r.t. *locally boolean skew lattices* [1, 26, 27].

(2) Consider the variety $\mathcal{V}$ of *pseudocomplemented semilattices* [17]. The Glivenko–Frink Theorem [19, Theorem 1, § 6.4] implies that the term $t(x) = x''$ is $\mathcal{V}$-idempotent and $\mathcal{V}$-compatible, and defines an operation that maps any $\mathbf{A} \in \mathcal{V}$ onto its Boolean skeleton. The same situation holds for *pseudocomplemented MTL algebras* [14], a variety that includes in particular Gödel algebras and Product algebras.

(3) The term $t(x) = \left(\neg\left((\neg x)^2\right)\right)^2$ is $\mathcal{V}$-idempotent and $\mathcal{V}$-compatible, if $\mathcal{V}$ is taken to be the variety of MV algebras [11] generated by Chang's algebra.

(4) Let $\mathcal{V}$ be the variety of *quasi-MV algebras* [25], a generalisation of MV algebras motivated by quantum computation. The term $t(x) = x \oplus 0$ is $\mathcal{V}$-idempotent and $\mathcal{V}$-compatible, and defines an operation that maps any quasi-MV algebra $\mathbf{A}$ onto its MV retract of regular elements.

(5) Let $\mathcal{V}$ be the variety of MV algebras. *State-morphism MV algebras* [13] are among the various formal frameworks that have been developed for many-valued probability. Formally, the language of MV algebras is expanded by a unary term s which is required to be $\mathcal{V}$-idempotent and $\mathcal{V}$-compatible.

(6) Finally, let $\mathcal{V} = \mathcal{LG} \vee \mathcal{IRL}$, the join of the varieties of lattice-ordered groups and of integral residuated lattices in the lattice of subvarieties of residuated lattices [18]. This join coincides with the direct product $\mathcal{LG} \times \mathcal{IRL}$ of the two varieties [21]. The term $t(x) = x(x\backslash 1)$ is $\mathcal{V}$-idempotent and $\mathcal{V}$-compatible, and for any $\mathbf{A} \in \mathcal{V}$ the kernel of the term operation $t^{\mathbf{A}}$ is the projection onto the integral factor of the direct product representation of $\mathbf{A}$.

Henceforth, we will restrict our investigation to cases where the class $\mathcal{K}$ is actually a variety and where the term t is unary, i.e., no parameters are involved.

It will be expedient to introduce from the start some terminological and notational conventions. If t, of type ν, is $\mathcal{V}$-idempotent and $\mathcal{V}$-compatible, then $\mathcal{V}_t$ will denote the subvariety of $\mathcal{V}$ axiomatised w.r.t. $\mathcal{V}$ by the equation $x \approx t(x)$. If $\mathbf{A} \in \mathcal{V}$ and $\theta \in \mathrm{Con}(\mathbf{A})$, by $\overline{\theta}_t$ we mean the smallest congruence on $\mathbf{A}$ containing θ s.t. $\mathbf{A}/\overline{\theta}_t \in \mathcal{V}_t$. The subscripts will be omitted whenever the term t is understood. Clearly, $\overline{\Delta} = \ker(t)$. More generally, we have that:

Lemma 2 $\overline{\theta} = \{(a,b)\colon t(a)\theta t(b)\} = \theta \vee \ker(t)$.

Proof Let $\varphi = \{(a,b)\colon t(a)\theta t(b)\}$. φ is a congruence: it is an equivalence, and preserves the operations since t is $\mathcal{V}$-compatible. We prove the following items in turn.

$\overline{\theta} \subseteq \varphi$. Clearly, $\theta \subseteq \varphi$, and $\mathbf{A}/\varphi \in \mathcal{V}_t$ because t is $\mathcal{V}$-idempotent and thus $t(a)\theta t\,(t(a))$, i.e. $a/\varphi = t(a)/\varphi$ for all $a \in A$. Therefore $\overline{\theta} \subseteq \varphi$.

$\varphi \subseteq \theta \vee \ker(t)$. If $t(a)\theta t(b)$, then $a\,\theta \vee \ker(t)\,t(a)\,\theta \vee \ker(t)\,t(b)\,\theta \vee \ker(t) b$.

$\theta \vee \ker(t) \subseteq \overline{\theta}$. Let $a\theta \vee \ker(t)\,b$. Then there are $c_1, \ldots, c_n \in A$ such that $a\theta c_1 \ker(t) c_2 \ldots c_n \ker(t)\,b$. Consequently,

$$t(a)\theta t(c_1) \ker(t) t\,(c_2) \ldots t(c_n) \ker(t)\,t(b).$$

By definition, for any x, y if $x \ker(t) y$, then $t(x) = t(y)$. Hence the previous chain shortens to $t(a)\theta t(b)$. Since $\theta \subseteq \overline{\theta}$, it follows that $a\overline{\theta}t(a)\overline{\theta}t(b)\overline{\theta}b$. □

Whenever a congruence θ is such that $\overline{\theta}_t = \theta$, we will say that it is *t-closed*, and we denote the set of t-closed congruences by $\mathrm{Clo}_t(\mathbf{A})$.

Let us now state some easy but useful facts concerning these notions (for which cf. also [28, § 3]). For the remainder of this section, unless otherwise specified, we let $\mathcal{V}$ be a variety of type ν and t, of type ν, be a $\mathcal{V}$-idempotent and $\mathcal{V}$-compatible term.

Lemma 3 *Let* $\mathbf{A}$ *be a member of* $\mathcal{V}$. *Then:*

(1) $\mathbf{A}/\ker(t) \simeq t\,(\mathbf{A}) = (t(A), \{f \restriction t(A) \colon f \in \nu\})$ *is a maximal retract of* $\mathbf{A}$ *belonging to* $\mathcal{V}_t$.

(2) $\{\mathbf{A}/\ker(t) \colon \mathbf{A} \in \mathcal{V}\} = \mathcal{V}_t$.

(3) *The retraction from* $\mathbf{A}$ *to* $\mathbf{A}/\ker(t)$ *induces, pointwise, a retraction from* $\mathrm{Con}(\mathbf{A})$ *to*

$$\mathrm{Con}\,(\mathbf{A}/\ker(t)) \simeq \mathrm{Clo}_t(\mathbf{A}) = \{\varphi \in \mathrm{Con}(\mathbf{A}) \colon \varphi \supseteq \ker(t)\}\,.$$

Moreover, this retraction is equivalently specified by the mapping $f(\theta) = \overline{\theta}$.

Proof (1) $f \colon \mathbf{A}/\ker(t) \to t(\mathbf{A})$, defined by $f(a/\ker(t)) = t(a)$, is an isomorphism (use the $\mathcal{V}$-compatibility of t). $\mathbf{A}/\ker(t)$ is a quotient of $\mathbf{A}$, and is isomorphic to $t(\mathbf{A})$, which is a subalgebra of $\mathbf{A}$ (use again the $\mathcal{V}$-compatibility of t). Moreover, $\mathbf{A}/\ker(t)$ is in $\mathcal{V}_t$ by the $\mathcal{V}$-idempotency of t. Finally, if $\mathbf{A}/\theta \in \mathcal{V}_t$, we invoke Lemma 2 to show that

$$\theta = \overline{\theta} = \theta \vee \ker(t) \supseteq \ker(t),$$

whence $\mathbf{A}/\ker(t)$ is maximal among the quotients of $\mathbf{A}$ that belong to $\mathcal{V}_t$.

(2) The left-to-right inclusion follows from (1). If $\mathbf{A} \in \mathcal{V}_t$, then in particular $\mathbf{A} \in \mathcal{V}$, and $\mathbf{A} = \mathbf{A}/\ker(t)$.

(3) The only non-evident part of our claim is the fact that the map $f\,(\theta) = \overline{\theta}$ is a retraction of $\mathrm{Con}(\mathbf{A})$ onto $\mathrm{Clo}_t(\mathbf{A})$. In fact, by Lemma 2,

$$\begin{aligned}(a, b) \in \overline{\theta \cap \psi} &\text{ iff } (t(a), t(b)) \in \theta \cap \psi \\ &\text{ iff } (t(a), t(b)) \in \theta \text{ and } (t(a), t(b)) \in \psi \\ &\text{ iff } (a, b) \in \overline{\theta} \text{ and } (a, b) \in \overline{\psi}.\end{aligned}$$

By the same lemma, $\overline{\theta \vee \psi} = (\theta \vee \psi) \vee \ker(t) = (\theta \vee \ker(t)) \vee (\psi \vee \ker(t)) = \overline{\theta} \vee \overline{\psi}$. Finally, if $\varphi \supseteq \ker(t)$, then $\varphi = \varphi \vee \ker(t) = \overline{\varphi} = f(\varphi)$. □

We observe that the map sending θ to $\overline{\theta}$ also preserves compositions: if $(a, b) \in \overline{\theta \circ \psi}$, then $(t(a), t(b)) \in \theta \circ \psi$. Then, there is a c s.t. $t(a)\theta c \psi t(b)$. So $t(a)\theta t(c)\psi t(b)$, i.e. $a\overline{\theta} c \overline{\psi} b$. For the converse inclusion, if $a\overline{\theta} c \overline{\psi} b$, then $t(a)\theta t(c)\psi t(b)$. Then, there is a y s.t. $t(a)\theta y \psi t(b)$. Hence, $(a, b) \in \overline{\theta \circ \psi}$.

Now, suppose that, for all $\mathbf{A} \in \mathcal{V}$ and for any n-ary term f of type ν (n *strictly* greater than 1), $\mathbf{A}$ satisfies $t(f(x_1, \ldots, x_n)) \approx f(x_1, \ldots, x_n)$. This condition implies rather general results.

Lemma 4 *Let $\mathbf{A} \in \mathcal{V}$ be subdirectly irreducible and let $\Delta \neq \ker(t) \neq \nabla$. Then there is exactly one element $a \in A$ such that $t(a) \neq a$.*

Proof Suppose that there are distinct $a, b \in A$ such that $t(a) \neq a$ and $t(b) \neq b$. Define a relation θ_a as follows: $\{(a, t(a)), (t(a), a)\} \cup \Delta$. It is not hard to verify that $\theta_a \in \mathrm{Con}(\mathbf{A})$. Therefore, upon analogously defining θ_b, one obtains that $\theta_a, \theta_b \in \mathrm{Con}(\mathbf{A})$ and $\theta_a \cap \theta_b = \Delta$. □

The proof of Lemma 4 suggests that for any $a \in A$,

$$\theta(a, t(a)) = \{(a, t(a)), (t(a), a)\} \cup \Delta.$$

Now, define

$$x \tau y \text{ iff } x = y \text{ or } x, y \in t(A).$$

Theorem 5 *Let $\mathbf{A} \in \mathcal{V}$. Then $\mathbf{A} \leq \mathbf{A}/\ker(t) \times \mathbf{A}/\tau$, and this embedding is subdirect.*

Proof For a start, we have to prove that the quotient $\mathbf{A}/_{\tau}$ is well-defined. τ is clearly an equivalence. To prove that $\tau \in \mathrm{Con}(\mathbf{A})$, suppose first that f has arity 1, and let $a\tau b$. If $a = b$, then $f(a) = f(b)$ and we are done. If $a = t(a)$, $b = t(b)$, then $t(f(a)) = f(t(a)) = f(a)$ and similarly $t(f(b)) = f(b)$. If f has arity greater than 1, then f preserves τ by our initial hypothesis.

Moreover, let $(a, b) \in \ker(t) \cap \tau$; then $t(a) = t(b)$, and $a, b \in t(A)$, whence $a = t(a) = t(b) = b$. So, $\ker(t) \cap \tau = \Delta$. Also, $a \ker(t)\, t(a) \tau\, t(b) \ker(t) b$, whereby $\ker(t) \vee \tau = \nabla$. This completes our proof. □

It should be noticed that, for $\mathbf{A} \in \mathcal{V}$, it is not necessarily the case that $\mathbf{A}$ is isomorphic to $\mathbf{A}/\ker(t) \times \mathbf{A}/\tau$, since $\ker(t)$ and τ need not permute (see [25] for counterexamples). Also observe that, whenever it can be proved that τ is a congruence, Theorem 5 can be directly obtained from [23, Theorem 3.13], since $V(\mathbf{A}/\ker(t)), V(\mathbf{A}/\tau)$ are easily shown to be *quasi-independent* and *orthogonal* in the sense of that paper. For related results, see also [24, § 4].

The preceding theorem has as corollaries several known theorems in the structure theories of individual varieties with compatible and idempotent terms: for example, the subdirect embedding theorem for quasi-MV algebras [25], where $t(x) = x \oplus 0$, or the decomposition theorem for a normally presented variety $\mathcal{V}$ [5], where t is the assigned term of $\mathcal{V}$.

3 (t, s)-distributive varieties

Let us now introduce the first relativised congruence property on our list: (t, s)-distributivity.

Definition 6 Let $\mathbf{A}$ be an algebra of type ν, and let t, s be $V(\mathbf{A})$-idempotent and $V(\mathbf{A})$-compatible terms of the same type. $\mathbf{A}$ is (t, s)-*distributive* iff for every $a, b \in \mathbf{A}$ and for every $\theta, \varphi, \psi \in \mathrm{Con}(\mathbf{A})$, we have

$$(t(a), s(b)) \in \theta \cap (\varphi \vee \psi) \text{ iff } (t(a), s(b)) \in (\theta \cap \varphi) \vee (\theta \cap \psi)\,.$$

A variety $\mathcal{V}$ of type ν is (t, s)-*distributive* iff every algebra in $\mathcal{V}$ is (t, s)-distributive with respect to the same terms t, s.

Clearly, when t, s are the identity terms, the notion of (t, s)-distributivity coincides with the notion of congruence distributivity. When t, s are the same term, we call the algebra $\mathbf{A}$ (or the variety $\mathcal{V}$) t-*distributive*, for short.

Example 7 Pseudocomplemented semilattices and quasi-MV algebras satisfy no nontrivial congruence identity in the language of lattices (see, respectively, [31] and [3]), but are t-distributive, where $t(x) = x''$ in the former case and $t(x) = x \oplus 0$ in the latter. The nilpotent shift of any congruence distributive variety [5] is t-distributive, where t is the assigned term of $\mathcal{V}$.

It immediately follows from Lemma 3 that:

Lemma 8 *The following are equivalent:*

(1) $\mathcal{V}$ *is* t*-distributive;*

(2) *for all* $\mathbf{A} \in \mathcal{V}$*, the sublattice* $\mathrm{Clo}_t(\mathbf{A})$ *of all closed congruences of* $\mathbf{A}$ *is distributive;*

(3) *for all* $\mathbf{A} \in \mathcal{V}$*, the retract* $t(\mathbf{A})$ *is congruence distributive.*

Like congruence distributivity, (t, s)-distributivity is a Mal'cev property, as the following theorem shows.

Theorem 9 $\mathcal{V}$ *is* (t, s)*-distributive, iff there exists a set of* $m+1$ *ternary terms* $\{p_j(x, y, z)\}_{j \leq m}$ *s.t.* $\mathcal{V}$ *satisfies the following equations:*

$$\begin{gathered}
p_0(x, y, z) \approx t(x);\\
p_m(x, y, z) \approx s(z);\\
p_j(s(x), s(y), t(x)) \approx s(t(x)) \approx p_j(s(x), t(y), t(x))\\
\textit{for all } j \leq m;\\
p_j(x, t(x), y) \approx p_{j+1}(x, t(x), y) \textit{ for even } j;\\
p_j(x, s(y), y) \approx p_{j+1}(x, s(y), y) \textit{ for odd } j.
\end{gathered}$$

Proof Left to right. We work in the 3-generated free $\mathcal{V}$-algebra $\mathbf{F}(x, y, z)$. Observe that

$$(t(x), s(z)) \in \theta\,(t(x), s(z)) \cap (\theta\,(t(x), y) \vee \theta\,(y, s(z)))$$

and so

$$(t(x), s(z)) \in (\theta(t(x), s(z)t) \cap \theta(t(x), y)) \vee (\theta(t(x), s(z)) \cap \theta(y, s(z))).$$

Hence there are $m+1$ terms $p_0, \ldots, p_m$ s.t.

$$\begin{aligned}
&t(x)\theta\,(t(x), s(z)) \cap \theta\,(t(x), y)\,p_1(x, y, z)\\
&p_1(x, y, z)\theta\,(t(x), s(z)) \cap \theta\,(y, s(z))\,p_2(x, y, z)\\
&\vdots\\
&p_{m-1}(x, y, z)\theta\,(t(x), s(z)) \cap \theta\,(y, s(z))\,s(z)
\end{aligned}$$

and from this the desired equations follow, taking into account the fact that t and s are both $\mathcal{V}$-compatible and therefore $\mathcal{V}$ satisfies $s(t(x)) \approx t(s(x))$.

Right to left. Suppose $a, b \in \mathbf{A} \in \mathcal{V}$, and $(t(a), s(b)) \in \varphi \cap (\psi \vee \chi)$. Then $t(a)\varphi s(b)$ and for some $c_1, \ldots, c_r$ we have

$$t(a)\psi c_1 \chi \ldots c_r \chi s(b)$$

and thus

$$s(t(a))\psi s(c_1)\chi \ldots s(c_r)\chi s(b)$$

whence

$$\begin{aligned}
&p_j(t(a), s(t(a)), s(b))\psi p_j(t(a), s(c_1), s(b))\chi \ldots\\
&\quad p_j(t(a), s(c_r), s(b))\chi p_j(t(a), s(b), s(b)).
\end{aligned}$$

We now argue that

$$\begin{aligned}
&p_j(t(a), s(t(a)), s(b))\varphi p_j(t(a), s(c_1), s(b))\varphi \ldots\\
&\quad p_j(t(a), s(c_r), s(b))\varphi p_j(t(a), s(b), s(b)).
\end{aligned}$$

In fact, for example,

$$\begin{aligned}
p_j(t(a), s(t(a)), s(b))\varphi p_j(s(t(a)), s(t(a)), t(a))&\\
&= s(t(a))\\
&= p_j(s(t(a)), s(c_1), t(a))\\
&\varphi p_j(t(a), s(c_1), s(b)).
\end{aligned}$$

So

$$p_j(t(a), t(a), s(b))(\varphi \cap \psi) \vee (\varphi \cap \chi) p_j(t(a), s(b), s(b)).$$

Then, in view of the given equations, and using freely the idempotence property, if we let $\theta = (\varphi \cap \psi) \vee (\varphi \cap \chi)$, we obtain that

$$\begin{aligned} t(a) &= p_0(t(a), t(a), s(b)) \\ &= p_1(t(a), t(a), s(b)) \\ &\theta p_1(t(a), s(b), s(b)) \\ &= p_2(t(a), s(b), s(b)) \\ &\vdots \\ &p_n(t(a), s(b), s(b)) \\ &= s(b). \end{aligned}$$

□

One of the most desirable properties of congruence distributive varieties is Jónsson's Lemma: if $V(\mathcal{K})$ is congruence distributive, then all subdirectly irreducible members of $V(\mathcal{K})$ are in $HSP_U(\mathcal{K})$. It is well-known that congruence distributivity is a sufficient, but by no means necessary, condition for the lemma to hold—for example, the weaker commutator condition $[\varphi, \psi] = \varphi \cap \psi$ [16] or full bidecomposability [10] are known to suffice. Similarly, versions of Jónsson's Lemma hold for many individual t-distributive varieties, like quasi-MV algebras or nilpotent shifts of congruence distributive varieties ([6], [22]). Abstracting away from these particular examples, we now present a version of Jónsson's Lemma that holds for any t-distributive variety. Since the first part of the proof is identical to Jónsson's original proof, we will be sketchy and refer the reader to [4, Theorem IV-6.8] for the details.

Theorem 10 *Let $\mathcal{V} = V(\mathcal{K})$ be t-distributive. Then all subdirectly irreducible members of $\mathcal{V}_t$ are in $HSP_U(\mathcal{K})$.*

Proof The proof is the same as for Jónsson's Lemma until we get to the point where for congruences θ_J and θ_K we have $\theta_J|_B \cap \theta_K|_B \subseteq \varphi$. We want to show $\theta_J|_B \subseteq \varphi$ or $\theta_K|_B \subseteq \varphi$. Since we assumed $\mathbf{A} = \mathbf{B}/\varphi$ to be a s.i. member of $\mathcal{V}_t$, we have $\varphi \supseteq \ker(t)$. Therefore we have $\overline{\theta_J|_B \cap \theta_K|_B} \subseteq \varphi$. Lemma 3 then yields $\overline{\theta_J|_B} \cap \overline{\theta_K|_B} \subseteq \varphi$. Now we have three closed congruences, and so we can make use of Lemma 8. Therefore, we obtain that $\varphi = \varphi \vee (\overline{\theta_J|_B} \cap \overline{\theta_K|_B}) = (\varphi \vee \overline{\theta_J|_B}) \cap (\varphi \vee \overline{\theta_K|_B})$. Since φ is meet irreducible, we get $\varphi = \varphi \vee \overline{\theta_J|_B}$ or $\varphi = \varphi \vee \overline{\theta_K|_B}$. Hence $\theta_J|_B \subseteq \overline{\theta_J|_B} \subseteq \varphi$ or $\theta_K|_B \subseteq \overline{\theta_K|_B} \subseteq \varphi$ as required. □

4 (t, s)-permutable and (t, s)-modular varieties

Exactly like congruence distributivity, also congruence permutability and congruence modularity can be relativised to a pair of compatible and idempotent terms. These concepts will be the focus of the present section.

Definition 11 Let $\mathbf{A}$ be an algebra of type ν, and let t, s be $V(\mathbf{A})$-idempotent and $V(\mathbf{A})$-compatible terms of the same type. $\mathbf{A}$ is (t, s)*-permutable* iff for every $a, b \in \mathbf{A}$ and for every $\theta, \varphi \in \mathrm{Con}(\mathbf{A})$, we have

$$(t(a), s(b)) \in \theta \circ \varphi \text{ iff } (t(a), s(b)) \in \varphi \circ \theta.$$

$\mathbf{A}$ is (t, s)*-modular* iff for every $a, b \in \mathbf{A}$ and for every $\theta_1, \theta_2, \theta_3 \in \mathrm{Con}(\mathbf{A})$:

$$\text{if } \theta_1 \subseteq \theta_2 \text{ then } (t(a), s(b)) \in \theta_2 \cap (\theta_1 \vee \theta_3) \text{ iff } (t(a), s(b)) \in \theta_1 \vee (\theta_2 \cap \theta_3).$$

A variety $\mathcal{V}$ of type ν is (t, s)*-permutable* ((t, s)*-modular*) iff every algebra in $\mathcal{V}$ is (t, s)-permutable ((t, s)-modular) with respect to the same terms t, s.

The above definition generalises simultaneously the standard notion of congruence permutability and the concept of τ*-permutability* [2]: the former by letting t and s be the identity, and the latter by letting a and b in the above definition be the same element.

As in the case of (t, s)-distributivity, also (t, s)-permutability can be captured by means of Mal'cev conditions.

Theorem 12 *A variety $\mathcal{V}$ is (t, s)-permutable iff there exists a ternary term p s.t.*

$$\mathcal{V} \models p(x, s(y), y) \approx t(x),$$
$$\mathcal{V} \models p(x, t(x), y) \approx s(y).$$

Proof Left to right. We work in the $\mathcal{V}$-free 3-generated algebra $\mathbf{F}(x, y, z)$. Let $\theta = \theta(t(x), y)$ and $\varphi = \theta(y, s(z))$. Therefore, we have $(t(x), s(z)) \in \theta \circ \varphi$ and thus $(t(x), s(z)) \in \varphi \circ \theta$. This means that there is a $p(x, y, z)$ which is φ-congruent to $t(x)$ and θ-congruent to $s(z)$. Now, consider $\mathbf{F}(x, z)$ as naturally embedded in $\mathbf{F}(x, y, z)$. Let $f \colon \mathbf{F}(x, y, z) \to \mathbf{F}(x, z)$ be a homomorphism defined by $f(x) = x$, $f(y) = s(z)$, $f(z) = z$. Its kernel contains φ, because φ is the smallest congruence that identifies y with $s(z)$. Thus, $p(x, s(z), z) = t(x)$ in $\mathbf{F}(x, z)$, and therefore the corresponding equation holds in the whole variety. For the other equation we argue similarly.

Right to left. Let $a, b \in \mathbf{A} \in \mathcal{V}$. Suppose $(t(a), s(b)) \in \varphi \circ \psi$, whence there is c s.t. $t(a) \varphi c \psi s(b)$. Then

$$s(b) = p(a, t(a), b) \varphi p(a, c, b) \psi p(a, s(b), b) = t(a).$$

Our conclusion follows. □

As above, we use the term t*-permutable* in place of the more cumbersome (t, t)-permutable. The varieties in Example 7, with the obvious adjustments, make instances of t-permutable varieties.

A characterization of t-permutability is in terms of joins of closed congruences:

Lemma 13 *Let $\mathbf{A}$ be an algebra, let t be a $V(\mathbf{A})$-compatible and $V(\mathbf{A})$-idempotent term, and let $\theta_1, \theta_2 \in \mathrm{Con}(\mathbf{A})$. Then the following are equivalent:*

(1) *for all* $a, b \in A$, $(t(a), t(b)) \in \theta_1 \circ \theta_2$ *iff* $(t(a), t(b)) \in \theta_2 \circ \theta_1$*;*

(2) $\overline{\theta_1} \circ \overline{\theta_2} = \overline{\theta_1} \vee \overline{\theta_2}$*.*

Proof It follows from Lemma 3. □

It is well known that congruence permutability implies congruence modularity. We now see that a similar result applies in our case.

Theorem 14 *If* **A** *is* t*-permutable, then* **A** *is* t*-modular.*

Proof Suppose $\theta_1 \subseteq \theta_2$, and let $(t(a), s(b)) \in \theta_2 \cap (\theta_1 \vee \theta_3)$. Then, there is $c \in A$ such that $t(a)\theta_1 c\, \theta_3 s(b)$, since **A** is t-permutable. So, since $t(a)\theta_1 c$, then $t(a)\theta_2 c$, because $\theta_1 \subseteq \theta_2$. Thus, $s(b)\theta_2 t(a)\theta_2\, c$, and then $(s(b), c) \in \theta_2 \cap \theta_3$. Therefore, $t(a)\theta_1 c\, (\theta_2 \cap \theta_3) s(b)$, whence $t(a)\theta_1 \vee (\theta_2 \cap \theta_3) s(b)$. □

It makes sense to combine (t, s)-permutability and (t, s)-distributivity into a single notion, which represents the natural counterpart of the notion of arithmetical variety.

Definition 15 An algebra **A**, or a variety $\mathcal{V}$, is (t, s)*-arithmetical* iff it is (t, s)-distributive and (t, s)-permutable.

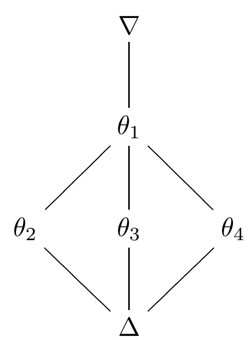
As usual, 't-arithmetical' will stand for '(t, t)-arithmetical'. As the next example shows, Definition 15 leads to a genuine generalisation of the notion of arithmetical variety.

Example 16 Let $\mathbf{4} = \langle \{0, a, b, 1\} \rightarrow, 0 \rangle$ be the pre-BCK algebra (in the sense of W. H. Cornish [12]) whose implication table is as follows:

$\rightarrow$	**0**	**a**	**b**	**1**
0	1	1	1	1
a	0	1	1	1
b	0	1	1	1
1	0	1	1	1

It can be seen that the partitions corresponding to the non-trivial congruences of **4** are:

- $\theta_1 = \{\{a, b, 1\}, \{0\}\}$;
- $\theta_2 = \{\{a, b\}, \{1\}, \{0\}\}$;
- $\theta_3 = \{\{a, 1\}, \{b\}, \{0\}\}$;
- $\theta_4 = \{\{b, 1\}, \{a\}, \{0\}\}$.



$\mathbf{4}$ is not congruence permutable, for $a\theta_3 \circ \theta_4 b$, but it is not the case that $a\theta_4 \circ \theta_3 b$. Moreover, it is evident from the diagram above that $\mathrm{Con}(\mathbf{4})$ is not distributive. Upon defining $t(x) = 1 \to x$, it can be verified that t is $V(\mathbf{4})$-compatible and $V(\mathbf{4})$-idempotent, and $t(\mathbf{4}) = \mathbf{2}$, the two element Tarski algebra. Therefore $V(\mathbf{4})$, though neither congruence distributive nor congruence permutable, is t-arithmetical.

We show that t-arithmeticity is a Mal'cev property as well.

Theorem 17 *For a variety $\mathcal{V}$ the following are equivalent:*

(1) *$\mathcal{V}$ is t-arithmetical.*

(2) *There exists a term p as in Theorem 12, and a term M such that:*

$$\mathcal{V} \models M(x, y, t(x)) \approx M(x, t(x), y) \approx M(y, t(x), x) \approx t(x).$$

(3) *There exists a term m s.t.*

$$\mathcal{V} \models m(x, y, t(x)) \approx m(x, t(y), y) \approx m(y, t(y), x) \approx t(x).$$

Proof (1) implies (2). The term p exists by Theorem 12. Let us show the existence of the term M in the 3-generated $\mathcal{V}$-free algebra $\mathbf{F}(x, y, z)$. As t is idempotent,

$$(t(x), t(z)) \in \theta(t(x), z) \cap (\theta(t(x), y) \vee \theta(y, t(z)))$$

Since $\mathcal{V}$ is t-distributive,

$$(t(x), t(z)) \in (\theta(t(x), z) \cap \theta(t(x), y)) \vee (\theta(t(x), z) \cap \theta(y, t(z)))$$

and since $\mathcal{V}$ is t-permutable,

$$(t(x), t(z)) \in (\theta(t(x), z) \cap \theta(t(x), y)) \circ (\theta(t(x), z) \cap \theta(y, t(z)))$$

whence there exists $M(x, y, z)$ s.t.

$$t(x)\theta(t(x), z) \cap \theta(t(x), y) M(x, y, z) \theta(t(x), z) \cap \theta(y, t(z)) t(z).$$

Now the desired equations follow.

(2) implies (3). Let

$$m(x, y, z) = p(x, M(x, y, z), z).$$

(3) implies (1). Let

$$p(x, y, z) = m(x, y, z),$$

and let

$$M(x, y, z) = m(x, m(x, y, z), z).$$

Then easy but tedious calculations show that these terms yield the required properties. □

5 Some geometrical insights

From the Seventies onwards, several interesting results of geometrical flavour have been discovered in the context of congruence modular varieties (a key reference is [20]; see also [3, 7, 8]). In this section we carry out some investigations along these lines in the context under discussion in this paper.

Definition 18 Let $\mathcal{V}$ be a variety of type ν, and let t, s be $\mathcal{V}$-compatible and $\mathcal{V}$-idempotent terms of the same type. $\mathbf{A} \in \mathcal{V}$ satisfies the (t,s)*-triangular scheme* if for any $x, y, z \in A$ and $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{A})$, with $\alpha \cap \beta \subseteq \gamma$, if $(x, t(y)) \in \alpha$, $(x, s(z)) \in \gamma$ and $(t(y), s(z)) \in \beta$, then $(t(y), s(z)) \in \gamma$. Graphically:

$$\begin{tikzcd} & s(z) \arrow[dl, dash, "\gamma"'] \arrow[d, dash, "\beta"] \\ x \arrow[r, dash, "\alpha"'] & t(y) \end{tikzcd} \qquad \text{and } \alpha \cap \beta \leq \gamma$$

implies

$$\begin{tikzcd} & s(z) \arrow[dl, dash, "\gamma"'] \arrow[d, dash, "\beta"'] \arrow[d, dash, dashed, bend left, "\gamma"] \\ x \arrow[r, dash, "\alpha"'] & t(y) \end{tikzcd}$$

In case $t = s$, we will call the (t,s)-triangular scheme t-triangular.

Lemma 19 *Every* (t,s)*-distributive variety* $\mathcal{V}$ *satisfies the* (t,s)*-triangular scheme.*

Proof Let $\mathcal{V}$ be a (t,s)-distributive variety with $\mathbf{A} \in \mathcal{V}$, $x, y, z \in A$ and $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{A})$, with $\alpha \cap \beta \subseteq \gamma$. Suppose $(x, t(y)) \in \alpha$, $(x, s(z)) \in \gamma$ and $(t(y), s(z)) \in \beta$. Then, $(t(y), s(z)) \in \beta \cap (\alpha \vee \gamma)$. By (t,s)-distributivity, $(t(y), s(z)) \in (\beta \cap \alpha) \vee (\beta \cap \gamma) \subseteq \gamma \vee (\beta \cap \gamma) = \gamma$. Then, the (t,s)-scheme is satisfied. $\square$

Using the previous lemma we can characterise t-distributivity, under the assumption of t-permutability.

Theorem 20 *Let* $\mathbf{A}$ *be a* t*-permutable algebra. Then,* $\mathbf{A}$ *is* t*-distributive if and only if it satisfies the* t*-triangular scheme.*

Proof Let $\mathbf{A}$ be a t-permutable algebra that satisfies the t-triangular scheme. Suppose by way of contradiction that $\mathbf{A}$ is not t-distributive. Then, by Lemma 8, $\mathrm{Clo}_t(\mathbf{A})$ is not distributive. Hence, $\mathrm{Clo}_t(\mathbf{A})$ contains a sublattice isomorphic to M_3, as shown in the figure below:

$$\begin{tikzcd} & \overline{\alpha} \vee \overline{\beta} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \overline{\alpha} \arrow[dr, dash] & \overline{\beta} \arrow[d, dash] & \overline{\gamma} \arrow[dl, dash] \\ & \overline{\alpha} \cap \overline{\beta} & \end{tikzcd}$$

We can see that $\overline{\alpha} \cap \overline{\beta} \subsetneq \overline{\gamma}$. Also, $\overline{\gamma} \subsetneq \overline{\alpha} \vee \overline{\beta} = \overline{\alpha} \circ \overline{\beta}$, by t-permutability, and $\overline{\alpha} \cap \overline{\beta} = \overline{\alpha} \cap \overline{\gamma} = \overline{\beta} \cap \overline{\gamma}$. If $(a, b) \in \overline{\gamma}$, then $(t(a), t(b)) \in \overline{\gamma}$, and then $(t(a), t(b)) \in \overline{\alpha} \circ \overline{\beta}$. Therefore, there is a c such that $t(a)\, \overline{\alpha}\, c\, \overline{\beta}\, t(b)$, and so $t(a) = t(t(a))\, \alpha\, t(c) \beta\, t(t(b)) = t(b)$. That is

$$\begin{tikzcd}
 & t(b) \\
t(a) \arrow[ur, dash, "\gamma"] \arrow[r, dash, "\alpha"'] & t(c) \arrow[u, dash, "\beta"]
\end{tikzcd}$$

which implies by Definition 18

$$\begin{tikzcd}
 & t(b) \\
t(a) \arrow[ur, dash, "\gamma"] \arrow[r, dash, "\alpha"'] & t(c) \arrow[u, dash, "\beta"] \arrow[u, dash, dashed, bend right, "\gamma"']
\end{tikzcd}$$

Then, $(t(a), t(b)) \in \overline{\gamma}$, $(t(a), t(c)) \in \overline{\gamma} \cap \overline{\alpha}$, $(t(c), t(b)) \in \overline{\gamma} \cap \overline{\beta}$, imply that

$$(t(a), t(b)) \in (\overline{\gamma} \cap \overline{\alpha}) \vee (\overline{\gamma} \cap \overline{\beta}) = \overline{\alpha} \cap \overline{\beta},$$

and therefore $(a, b) \in \overline{\alpha} \cap \overline{\beta}$, in contradiction with the fact that $\overline{\alpha} \cap \overline{\beta}$ is strictly below $\overline{\gamma}$. Hence $\mathbf{A}$ is t-distributive. The converse implication follows from Lemma 19. $\square$

Next, we characterise t-distributivity without the assumption of t-permutability. To this aim, we introduce the following concept:

Definition 21 Let $\mathcal{V}$ be a variety of type ν, and let t be a $\mathcal{V}$-compatible and $\mathcal{V}$-idempotent term of the same type. Let $\mathbf{A} \in \mathcal{V}$ and let $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{A})$, $x, y, u, v \in A$. We say that $\mathbf{A}$ satisfies the *strong t-scheme* if

$$\alpha \cap \beta \subseteq \gamma \text{ and } \quad \begin{tikzcd}
t(x) \arrow[r, dash, "\alpha"] \arrow[d, dash, "\beta"'] & u \arrow[dr, dash, "\gamma"] & \\
t(y) \arrow[rr, dash, "\alpha"'] & & v
\end{tikzcd}$$

implies

$$\alpha \cap \beta \subseteq \gamma \text{ and } \quad \begin{tikzcd}
t(x) \arrow[r, dash, "\alpha"] \arrow[d, dash, "\beta"] \arrow[d, dash, dashed, bend right, "\gamma"'] & u \arrow[dr, dash, "\gamma"] & \\
t(y) \arrow[rr, dash, "\alpha"'] & & v
\end{tikzcd}$$

We say that $\mathbf{A}$ satisfies the *weak t-scheme* if the congruence α above is replaced by Φ_n, for any natural number n, where

$$\Phi_0 = \overline{\alpha}, \qquad \Phi_{n+1} = \Phi_n \circ \overline{\gamma} \circ \overline{\alpha}.$$

Lemma 22 *If $\mathbf{A}$ is t-distributive, then it satisfies the strong t-scheme.*

Proof It is well known that a lattice is distributive if and only if it satisfies the condition:

$$\alpha \cap \beta \subseteq \gamma \Longrightarrow \beta \cap (\alpha \vee \gamma) \subseteq \gamma. \tag{I}$$

By Lemma 8, $\mathrm{Con}(\mathbf{A})$ is t-distributive if and only if $\mathrm{Clo}_t(\mathbf{A})$ is such. Suppose that the assumption of the strong t-scheme holds, i.e. for $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{A})$, $x, y, u, v \in A$, $\overline{\alpha} \cap \overline{\beta} \subseteq \overline{\gamma}$ and

$$\begin{tikzcd} t(x) \arrow[r, dash, "\alpha"] \arrow[d, dash, "\beta"'] & u \arrow[d, dash, "\gamma"] \\ t(y) \arrow[r, dash, "\alpha"'] & v \end{tikzcd}$$

We have that $(t(x), t(y)) \in \overline{\beta}$, and we can verify that $t(y)\overline{\alpha}\, v\overline{\gamma}\, u\overline{\alpha}\, t(x)$. Therefore, $(t(x), t(y)) \in \overline{\beta} \cap (\overline{\alpha} \vee \overline{\gamma}\,) \subseteq \overline{\gamma}\,$, by Condition (I). Hence the strong t-scheme is satisfied. □

We can now prove the following:

Theorem 23 *Let $\mathcal{V}$ be a variety of type ν, and let t be a $\mathcal{V}$-compatible and $\mathcal{V}$-idempotent term of the same type. If $\mathbf{A} \in \mathcal{V}$ satisfies the weak t-scheme, then $\mathbf{A}$ is t-distributive.*

Proof Suppose that $\mathbf{A}$ satisfies the weak t-scheme. Let $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{A})$, and $\overline{\alpha} \cap \overline{\beta} \subseteq \overline{\gamma}$. We prove that $\overline{\beta} \cap (\overline{\alpha} \vee \overline{\gamma}) \subseteq \overline{\gamma}$, i.e. Condition (I), which in turn is equivalent to t-distributivity by Lemma 8. By induction on n we show that

$$\overline{\beta} \cap \Phi_n \subseteq \overline{\gamma},$$

where Φ_n is as in Definition 21.

Base: $n = 0$. If $n = 0$, then $\Phi_n = \overline{\alpha}$. Hence, by assumption

$$\overline{\beta} \cap \Phi_n = \overline{\beta} \cap \overline{\alpha} \subseteq \overline{\gamma}.$$

Inductive step: suppose the claim holds for Φ_n. We have to show that $\overline{\beta} \cap \Phi_{n+1} \subseteq \overline{\gamma}$. Let $(t(x), t(y)) \in \overline{\beta} \cap \Phi_{n+1}$. Then, by definition of Φ_{n+1}, $(t(x), t(y)) \in \overline{\beta} \cap \Phi_n \circ \overline{\gamma} \circ \overline{\alpha}$. Notice that $\overline{\beta} \cap \Phi_n \circ \overline{\gamma} \circ \overline{\alpha} \subseteq \overline{\beta} \cap \Phi_n \circ \overline{\gamma} \circ \Phi_n$, since, by definition, $\overline{\alpha} \subseteq \Phi_n$. This implies that there are $u, v \in A$ such that $t(y)\Phi_n\, v\overline{\gamma}\, u\Phi_n\, t(x)$, and $(t(x), t(y)) \in \overline{\beta}$ by inductive hypothesis. Hence, we are in the situation that

$$\begin{tikzcd} t(x) \arrow[r, dash, "\Phi_n"] \arrow[d, dash, "\overline{\beta}"'] & u \arrow[d, dash, "\overline{\gamma}"] \\ t(y) \arrow[r, dash, "\Phi_n"'] & v \end{tikzcd}$$

which is the assumption of the weak t-scheme. Therefore, $(t(x), t(y)) \in \overline{\gamma}$. So $\overline{\beta} \cap \Phi_{n+1} \subseteq \overline{\beta} \cap \Phi_n \circ \overline{\gamma} \circ \Phi_n \subseteq \overline{\gamma}$. □

We now give an appropriate version of a result by Fleischer [15].

Lemma 24 *Let $\mathcal{V}$ be a t-permutable variety and $\mathbf{A}, \mathbf{B} \in \mathcal{V}$. If $\mathbf{C} \leq \mathbf{A} \times \mathbf{B}$, upon denoting by $\mathbf{A}'$ and $\mathbf{B}'$ the images of the projections of $\mathbf{C}$ along the first and the second component, respectively, then*

$$t(C) = \{(a, b) \in t(A' \times B') \colon f(a) = g(b)\},$$

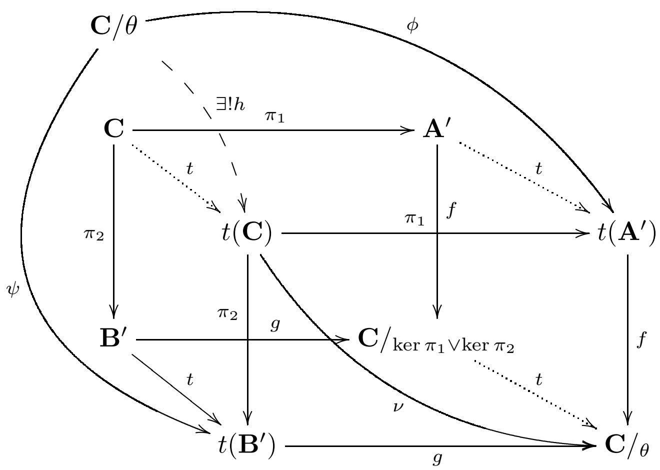
for surjective morphisms $f \colon \mathbf{A}' \to \mathbf{C}/\theta$, $g \colon \mathbf{B}' \to \mathbf{C}/\theta$, where

$$\theta = \overline{\ker(\pi_1) \vee \ker(\pi_2)}.$$

Proof Let $\theta = \overline{\ker(\pi_1) \vee \ker(\pi_2)}$, $\nu \colon \mathbf{C} \to \mathbf{C}/\theta$, and $f \colon \mathbf{A}' \to \mathbf{C}/\theta$, $g \colon \mathbf{B}' \to \mathbf{C}/\theta$ be homomorphisms such that $\nu = f \circ \pi_1$, $\nu = g \circ \pi_2$, respectively. Observe that, since t is term definable in the language of $\mathcal{V}$, t commutes with the projections, i.e. $\pi(t(x)) = t(\pi(x))$, for any x, and so $t(A' \times B') = t(A') \times t(B')$. Now, if $c \in t(C)$, then $c = (\pi_1(c), \pi_2(c))$. Then, $f(t(\pi_1(c))) = \nu(c) = g(t(\pi_2(c)))$. So $c \in \{(a, b) \in t(A' \times B') \colon f(a) = g(b)\}$.

On the other hand, let $(a, b) \in t(A' \times B')$ and $f(a) = g(b)$. Consider $c_1, c_2 \in t(C)$ such that $\pi_1(c_1) = a$ and $\pi_2(c_2) = b$. So, $\nu(c_1) = f(\pi_1(c_1)) = f(a) = g(b) = g(\pi_2(c_2)) = \nu(c_2)$. Therefore, $c_1 \theta c_2$. Then, $c_1 \overline{\ker(\pi_1) \vee \ker(\pi_2)}\, c_2$. It follows that $c_1 \overline{\ker(\pi_1)} \vee \overline{\ker(\pi_2)}\, c_2$, by Lemma 3. Then, there is a $c \in C$ such that $c_1 \overline{\ker(\pi_1)}\, c\, \overline{\ker(\pi_2)}\, c_2$, since $\overline{\ker(\pi_1)}$ and $\overline{\ker(\pi_2)}$ are closed congruences. Therefore, $t(c_1) = c_1 \ker(\pi_1)\, t(c) \ker(\pi_2)\, c_2 = t(c_2)$. We conclude that $t(c) = (a, b) \in t(C)$. □

In other words, for a t-permutable variety $\mathcal{V}$ and $\mathbf{A}, \mathbf{B} \in \mathcal{V}$ such that $\mathbf{C} \leq \mathbf{A} \times \mathbf{B}$, the fibred product of $t(\pi_1(\mathbf{C}))$ and $t(\pi_2(\mathbf{C}))$ over $\mathbf{C}/\overline{\ker \pi_1 \vee \ker \pi_2}$ is given by $t(C)$. A graphical synopsis of the previous Lemma is the following:



Theorem 25 *Let $\mathcal{V}$ be a t-modular variety, $\mathbf{A} \in \mathcal{V}$, and $\theta_0, \theta_1, \psi \in \mathrm{Con}(\mathbf{A})$. Let $a, b, c, d \in A$, $(a, b), (c, d) \in \theta_0$, $(a, c), (b, d) \in \theta_1$, and $\theta_0 \cap \theta_1 \subseteq \psi$. Then, $(b, d) \in \psi$ implies $(a, c) \in \overline{\psi}$. Graphically,*

$$\theta_0 \cap \theta_1 \leq \psi \ \textit{and} \quad \begin{tikzcd} a \arrow[r, dash, "\theta_0"] \arrow[d, dash, "\theta_1"] & b \arrow[d, dash, "\theta_1"'] \arrow[d, bend left, "\psi"] \\ c \arrow[r, dash, "\theta_0"'] & d \end{tikzcd}$$

implies

$$\begin{tikzcd} a \arrow[r, dash, "\theta_0"] \arrow[d, dash, "\theta_1"] \arrow[d, bend right, "\overline{\psi}"'] & b \arrow[d, dash, "\theta_1"'] \arrow[d, bend left, "\psi"] \\ c \arrow[r, dash, "\theta_0"'] & d \end{tikzcd}$$

Proof Suppose the assumptions hold. Hence, $(a, c) \in \theta_1$. Also, $a\theta_0\, b(\theta_1 \cap \psi)d\, \theta_0 c$. Thus, $(a, b) \in \theta_1 \cap (\theta_0 \vee (\theta_1 \cap \psi))$. So, $(t(a), t(b)) \in \overline{\theta_1 \cap (\theta_0 \vee (\theta_1 \cap \psi))}$, i.e. $(a, b) \in \overline{\theta_1} \cap (\overline{\theta_0} \vee (\overline{\theta_1} \cap \overline{\psi}))$, by Lemma 3. By t-modularity, $(a, b) \in \overline{\theta_1} \cap (\overline{\theta_0} \vee (\overline{\theta_1} \cap \overline{\psi})) = (\overline{\theta_0} \cap \overline{\theta_1}) \vee (\overline{\theta_0} \cap \overline{\psi}) \subseteq \overline{\psi}$. □

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References

[1] Bignall, R. J., Leech, J.: *Skew Boolean algebras and discriminator varieties.* Algebra Universalis **33** (1995), 387–398.

[2] Blok, W. J., Raftery, J. G.: *Assertionally equivalent quasivarieties.* International Journal of Algebra and Computation **18**, 4 (2008), 589–681.

[3] Bou, F., Paoli, F., Ledda, A., Freytes, H.: *On some properties of quasi-MV algebras and $\sqrt{'}$quasi-MV algebras, II.* Soft Computing **12**, 4 (2008), 341–352.

[4] Burris, S., Sankappanavar, H. P.: A Course in Universal Algebra. *Springer-Verlag*, Berlin, 1981.

[5] Chajda, I.: *Normally presented varieties.* Algebra Universalis **34** (1995), 327–335.

[6] Chajda, I.: *Jónsson's lemma for normally presented varieties.* Mathematica Bohemica **122**, 4 (1997), 381–382.

[7] Chajda, I., Czédli, G., Horváth, E. K.: *Trapezoid Lemma and congruence distributivity.* Mathematica Slovaca **53**, 3 (2003), 247–253.

[8] Chajda, I., Czédli, G., Horváth, E. K.: *Shifting Lemma and shifting lattice identities.* Algebra Universalis **50** (2003), 51–60.

[9] Chajda, I., Horváth, E. K.: *A triangular scheme for congruence distributivity.* Acta Sci. Math. (Szeged) **68** (2002), 29–35.

[10] Chajda, I., Rosenberg, I.: *Remarks on Jónsson's lemma.* Houston Journal of Mathematics **22**, 2 (1996), 249–262.

[11] Cignoli, R., D'Ottaviano, I. M. L., Mundici, D.: Algebraic Foundations of Many-Valued Reasoning. *Kluwer*, Dordrecht, 1999.

[12] Cornish, W. H.: *Constructions for BCK-algebras*. Math. Sem. Notes Kobe Univ. **11** (1983), 1–7.

[13] Di Nola, A., Dvurečenskij, A.: *State-morphism MV algebras*. Annals of Pure and Applied Logic **161**, 2 (2009), 161–173.

[14] Esteva, F., Godo, L.: *Monoidal t-norm based logic: Towards a logic of left-continuous t-norms*. Fuzzy Sets and Systems **124** (2001), 271–288.

[15] Fleischer, I.: *A note on subdirect products*. Acta Math. Acad. Sci. Hungar. **6** (1955), 463–465.

[16] Freese, R., McKenzie, R.: Commutator Theory for Congruence Modular Varieties. London Mathematical Society Lecture Notes, **125**, *Cambridge University Press*, Cambridge, 1987.

[17] Frink, O.: *Pseudo-complements in semi-lattices*. Duke Math. J. **29** (1962), 505–514.

[18] Galatos, N., Jipsen, P., Kowalski, T., Ono, H.: Residuated Lattices: An Algebraic Glimpse on Substructural Logics. *Elsevier*, Amsterdam, 2007.

[19] Grätzer, G.: Lattice Theory: First Concepts and Distributive Lattices. *W. H. Freeman and Co.*, San Francisco, 1971.

[20] Gumm, H. P.: Geometrical Methods in Congruence Modular Algebras. Memoirs Amer. Math. Soc., *Amer. Math. Soc.*, 1983.

[21] Jónsson, B., Tsinakis, C.: *Products of classes of residuated structures*. Studia Logica **77** (2004), 267–292.

[22] Kowalski, T., Paoli, F.: *On some properties of quasi-MV algebras and square root quasi-MV algebras, III*. Reports on Mathematical Logic **45** (2010), 161–199.

[23] Kowalski, T., Paoli, F.: *Joins and subdirect products of varieties*. Algebra Universalis **65**, 4 (2011), 371–391.

[24] Kowalski, T., Paoli, F., Spinks, M.: *Quasi-subtractive varieties*. Journal of Symbolic Logic **76**, 4 (2011), 1261–1286.

[25] Ledda, A., Konig, M., Paoli, F., Giuntini, R.: *MV algebras and quantum computation*. Studia Logica **82**, 2 (2006), 245–270.

[26] Leech, J.: *Skew lattices in rings*. Algebra Universalis **26** (1989), 48–72.

[27] Leech, J.: *Recent developments in the theory of skew lattices*. Semigroup Forum **52** (1996), 7–24.

[28] Paoli, F., Ledda, A., Kowalski, T., Spinks, M.: *Quasi-discriminator varieties*. International Journal of Algebra and Computation **24**, 3 (2014), 375–411.

[29] Petrich, I.: Lectures on semigroups. *Wiley and Sons*, New York, 1977.

[30] Salibra, A., Ledda, A., Paoli, F., Kowalski, T.: *Boolean-like algebras*. Algebra Universalis **69**, 2 (2013), 113–138.

[31] Sankappanavar, H. P.: *Congruence lattices of pseudocomplemented semilattices*. Algebra Universalis **9** (1979), 304–316.

[32] Spinks, M.: On the Theory of Pre-BCK Algebras. Ph.D. Thesis, *Monash University*, 2003.