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THE NATURAL OPERATORS LIFTING CONNECTIONS TO HIGHER ORDER COTANGENT BUNDLES

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Abstract. We prove that the problem of finding all $\mathcal{M}f_m$-natural operators $C\colon Q \rightsquigarrow QT^{r*}$ lifting classical linear connections ∇ on m-manifolds M into classical linear connections $C_M(\nabla)$ on the r-th order cotangent bundle $T^{r*}M = J^r(M,\mathbb{R})_0$ of M can be reduced to the well known one of describing all $\mathcal{M}f_m$-natural operators $D\colon Q \rightsquigarrow \bigotimes^p T \otimes \bigotimes^q T^*$ sending classical linear connections ∇ on m-manifolds M into tensor fields $D_M(\nabla)$ of type (p,q) on M.

Keywords: classical linear connection; natural operator

MSC 2010: 58A20, 58A32

All manifolds are assumed to be smooth, Hausdorff, finite dimensional and without boundaries. Maps are assumed to be smooth (of class C^∞). The category of m-dimensional manifolds and their embeddings is denoted by $\mathcal{M}f_m$.

A linear connection on a vector bundle E over a manifold M is a bilinear map $D\colon \mathcal{X}(M)\times\Gamma E\to\Gamma E$ such that $D_{fX}\sigma = fD_X\sigma$ and $D_X f\sigma = Xf\sigma + fD_X\sigma$ for any smooth map $f\colon M\to\mathbb{R}$, any vector field $X\in\mathcal{X}(M)$ on M and any smooth section $\sigma\in\Gamma E$ of $E\to M$. In particular, a linear connection ∇ in the tangent space TM of M is called a classical linear connection on M.

In [6], M. Kureš described completely all $\mathcal{M}f_m$-natural operators $B\colon Q_\tau \rightsquigarrow QT^*$ lifting torsion free classical linear connections ∇ on m-manifolds M into classical linear connections $B_M(\nabla)$ on the cotangent bundle T^*M of M.

In [5], the authors studied the similar problem of describing all $\mathcal{M}f_m$-natural operators $B\colon Q \rightsquigarrow Q(\bigotimes^k T^*)$ transforming classical linear connections ∇ on m-manifolds M into classical linear connections $B_M(\nabla)$ on the k-th tensor power $\bigotimes^k T^*M$ of the cotangent bundle T^*M of M. They proved that this problem can be reduced to the well known one of describing all $\mathcal{M}f_m$-natural operators $D\colon Q \rightsquigarrow \bigotimes^p T\otimes\bigotimes^q T^*$

sending classical linear connections ∇ on m-manifolds M into tensor fields $D_M(\nabla)$ of type (p,q) on M.

In the present note we study the similar problem of describing all $\mathcal{M}f_m$-natural operators $C\colon Q \rightsquigarrow QT^{r*}$ lifting classical linear connections ∇ on m-manifolds M into classical linear connections $C_M(\nabla)$ on the r-th order cotangent bundle $T^{r*}M = J^r(M,\mathbb{R})_0$ of M. We prove that this problem can be reduced to the well known one of describing all $D\colon Q \rightsquigarrow \bigotimes^p T \otimes \bigotimes^q T^*$, too.

The r-th order cotangent bundle is a functor $T^{r*}\colon \mathcal{M}f_m \to \mathcal{VB}$ sending any m-manifold M into $T^{r*}M := J^r(M,\mathbb{R})_0$ (the vector bundle of r-jets $M \to \mathbb{R}$ with target 0) and any embedding $\varphi\colon M_1 \to M_2$ of two m-manifolds into $T^{r*}\varphi\colon T^{r*}M_1 \to T^{r*}M_2$ given by $T^{r*}\varphi(j^r_x\gamma) = j^r_{\varphi(x)}(\gamma\circ\varphi^{-1})$, $j^r_x\gamma \in T^{r*}M$. If $r=1$, then $T^{1*}M \cong T^*M$ (the usual cotangent bundle) by $j^1_x\gamma \cong d_x\gamma$.

A general definition of natural operators can be found in [4]. In particular, an $\mathcal{M}f_m$-natural operator $C\colon Q \rightsquigarrow QT^{r*}$ is an $\mathcal{M}f_m$-invariant system $C = \{C_M\}_{M\in \mathrm{obj}(\mathcal{M}f_m)}$ of regular operators (functions)

$$C_M\colon \underline{Q}(M) \to \underline{Q}(T^{r*}M)$$

for any m-manifold M, where $\underline{Q}(M)$ is the set of all classical linear connections on M. More precisely, the $\mathcal{M}f_m$-invariance of C means that if $\nabla_1 \in \underline{Q}(M_1)$ and $\nabla_2 \in \underline{Q}(M_2)$ are φ-related by an embedding $\varphi\colon M_1 \to M_2$ of m-manifolds (i.e. φ is (∇_1,∇_2)-affine), then $C_{M_1}(\nabla_1)$ and $C_{M_2}(\nabla_2)$ are $T^{r*}\varphi$-related. The regularity means that C_M transforms smoothly parametrized families of connections into smoothly parametrized ones.

Similarly, an $\mathcal{M}f_m$-natural operator (natural tensor) $D\colon Q \rightsquigarrow \bigotimes^p T \otimes \bigotimes^q T^*$ is an $\mathcal{M}f_m$-invariant system $D = \{D_M\}_{M\in \mathrm{obj}(\mathcal{M}f_m)}$ of regular operators

$$D_M\colon \underline{Q}(M) \to \mathcal{T}^{p,q}(M)$$

for any $M \in \mathcal{M}f_m$, where $\mathcal{T}^{p,q}(M)$ is the set of tensor fields of type (p,q) on M.

Because of the general result in [7], since $T^{r*}\colon \mathcal{M}f \to \mathcal{VB}$ is a vector natural bundle, there exists an $\mathcal{M}f_m$-natural operator $C\colon Q \rightsquigarrow QT^{r*}$. An explicit example of a natural operator $C\colon Q \rightsquigarrow QT^{r*}$ it will be presented in Example 1, too.

A full description of all $\mathcal{M}f_m$-natural operators $Q_\tau \rightsquigarrow \bigotimes^p T^* \otimes \bigotimes^q T$ transforming torsion free classical connections on m-manifolds into tensor fields of types (p,q) can be found in Lemma in Section 33.4 in [4]. This description is as follows. Each covariant derivative of the curvature $\mathcal{R}(\nabla) \in C^\infty_M(\wedge^2 T^*M \otimes T^*M \otimes TM)$ of a classical linear connection ∇ is an ($\mathcal{M}f_m$-)natural tensor. Further, every tensor multiplication of two natural tensors and every contraction on one covariant and one contravariant entry of a natural tensor give a new natural tensor. Finally, we can tensor any natural tensor with a connection independent natural tensor, we can permute any number of entries in the tensor product and we can repeat these steps and take linear combinations. In this way we can obtain any natural tensor of types (p,q) depending on a torsion free classical linear connection. All natural tensors of a (not necessarily torsion free) classical linear connection ∇ can be obtained provided we also include the torsion tensor $\mathcal{T}(\nabla)$ and their covariant derivatives in the above procedure.

Affine natural liftings of classical linear connections to some other natural bundles (for example to Weil bundles) have been studied by many authors, see e.g. [1].

1. We are going to present an example of an $\mathcal{M}f_m$-natural operator $C^{(r)}\colon Q \rightsquigarrow QT^{r*}$. We start with some preparations.

It is well-known (see [3]) that if ∇ is a classical linear connection on a manifold M and $x \in M$ then there is a ∇-normal coordinate system $\varphi\colon (M,x) \to (\mathbb{R}^m,0)$ with center x. If $\psi\colon (M,x) \to (\mathbb{R}^m,0)$ is another ∇-normal coordinate system with center x then there is $A \in GL(m)$ such that $\psi = A \circ \varphi$ near x.

We have the following important proposition.

Proposition 1. *Let ∇ be a classical linear connection on M. Then there is a (canonical in ∇) vector bundle isomorphism*

$$I_\nabla\colon T^{r*}M \to \bigoplus_{k=1}^{r} S^kT^*M$$

covering the identity map of M.

P r o o f. Let $v \in T^{r*}_xM$, $x \in M$. Let $\varphi\colon (M,x) \to (\mathbb{R}^m,0)$ be a ∇-normal coordinate system with center x. We put

$$I_\nabla(v) = I^\varphi_\nabla(v) := \bigoplus_{k=1}^{r} S^kT^*\varphi^{-1} \circ I \circ T^{r*}\varphi(v),$$

where $I\colon T^{r*}_0\mathbb{R}^m = J^r_0(\mathbb{R}^m,\mathbb{R})_0 \to \bigoplus_{k=1}^{r} S^kT^*_0\mathbb{R}^m = \bigoplus_{k=1}^{r} S^k\mathbb{R}^{m*}$ is the obvious $GL(m)$-invariant vector space isomorphism. If $\psi\colon (M,x) \to (\mathbb{R}^m,0)$ is another ∇-normal coordinate system with center x, then $\psi = A \circ \varphi$ (near x) for some $A \in GL(m)$. Using the $GL(m)$-invariance of I we deduce that $I^\psi_\nabla(v) = I^\varphi_\nabla(v)$. So, the definition of $I_\nabla(v)$ is independent of the choice of φ. □

In [2], J. Gancarzewicz presented a canonical construction of a classical linear connection on the total space of a vector bundle E over M from a linear connection

D in E by means of a classical linear connection ∇ on M. More precisely, if X is a vector field on M and σ is a section of E, then $D_X\sigma$ is a section of E. Further, let X^D denote the horizontal lift of a vector field X with respect to D. Moreover, using the translations in the individual fibres of E, we derive from every section $\sigma\colon M \to E$ a vertical vector field σ^V on E called the vertical lift of σ. In [2], J. Gancarzewicz proved the following fact.

Proposition 2. *For every linear connection D in a vector bundle E over M and every classical linear connection ∇ on M there exists a unique classical linear connection $\Theta = \Theta(D, \nabla)$ on the total space E with the following properties:*

$$\Theta_{X^D} Y^D = (\nabla_X Y)^D, \quad \Theta_{X^D}\sigma^V = (D_X\sigma)^V,$$
$$\Theta_{\sigma^V} X^D = 0, \quad \Theta_{\sigma^V}\sigma_1^V = 0$$

for all vector fields X, Y on M and all sections σ, σ_1 of E.

It is well-known (see [3]) that every classical linear connection ∇ on an m-manifold M can be extended to a linear connection $D^{(r)}_\nabla = \nabla$ in $\bigoplus_{k=1}^{r} S^kT^*M$ by $(\nabla_X A)(X_1, \ldots, X_k) = XA(X_1, \ldots, X_k) - \sum_{i=1}^{k} A(X_1, \ldots, \nabla_X X_i, \ldots, X_k)$, $A \in \Gamma(S^kT^*M)$, $X_1, \ldots, X_k \in \mathcal{X}(M)$, $k = 1, \ldots, r$.

Now, we are in position to present a natural operator $C^{(r)}\colon Q \rightsquigarrow QT^{r*}$.

Example 1. Given a classical linear connection ∇ on M, by Propositions 1 and 2 we have the classical linear connection $\nabla^{(r)}$ on $T^{r*}M$ given by

$$\nabla^{(r)} := (I_\nabla)_*^{-1}\Theta(D^{(r)}_\nabla, \nabla).$$

Clearly, the family $C^{(r)}\colon Q \rightsquigarrow QT^{r*}$ of operators

$$C^{(r)}_M\colon \underline{Q}(M) \to \underline{Q}(T^{r*}M),\ C^{(r)}_M(\nabla) := \nabla^{(r)},$$

where $M \in \mathrm{obj}(\mathcal{M}f_m)$ and $\nabla \in \underline{Q}(M)$, is an $\mathcal{M}f_m$-natural operator.

2. The set of all $\mathcal{M}f_m$-natural operators $C\colon Q \rightsquigarrow QT^{r*}$ is an affine space with the corresponding vector space of all $\mathcal{M}f_m$-natural operators $\Delta\colon Q \rightsquigarrow (\otimes^2 T^* \otimes T)T^{r*}$ lifting classical linear connections ∇ on m-manifolds M into tensor fields $\Delta_M(\nabla)$ of type $(1, 2)$ on $T^{r*}M$ (the definition is quite similar to the one of natural operators $Q \rightsquigarrow QT^{r*}$). Actually, given $\mathcal{M}f_m$-natural operators $C\colon Q \rightsquigarrow QT^{r*}$ and $\Delta\colon Q \rightsquigarrow (\bigotimes^2 T^* \otimes T)T^{r*}$ we have an $\mathcal{M}f_m$-natural operator $C + \Delta\colon Q \rightsquigarrow QT^{r*}$ given by

$$(C + \Delta)_M(\nabla) := C_M(\nabla) + \Delta_M(\nabla),\ \nabla \in \underline{Q}(M),\ M \in \mathcal{M}f_m.$$

So, to describe all $\mathcal{M}f_m$-natural operators $C\colon Q \rightsquigarrow QT^{r*}$ it is sufficient to describe all $\mathcal{M}f_m$-natural operators $\Delta\colon Q \rightsquigarrow (\bigotimes^2 T^* \otimes T)T^{r*}$. Further, because of Proposition 2, we can put $\bigoplus_{k=1}^{r} S^k T^*$ instead of T^{r*}, and our problem of describing all $\mathcal{M}f_m$-natural operators $C\colon Q \rightsquigarrow QT^{r*}$ is reduced to the one of finding all $\mathcal{M}f_m$-natural operators

$$\Delta\colon Q \rightsquigarrow \Big(\bigotimes^2 T^* \otimes T\Big) \bigoplus_{k=1}^{r} S^k T^*$$

lifting classical linear connections ∇ on m-manifolds into tensor fields $\Delta_M(\nabla)$ of type $(1,2)$ on $\bigoplus_{k=1}^{r} S^k T^* M$.

Given a classical linear connection ∇ on M we have

$$T_v\Big(\bigoplus_{k=1}^{r} S^k T^* M\Big) = V_v\Big(\bigoplus_{k=1}^{r} S^k T^* M\Big) \oplus H_v^{\nabla} \cong \bigoplus_{k=1}^{r} S^k T_x^* M \oplus T_x M$$

for any $v \in \bigoplus_{k=1}^{r} S^k T_x^* M$, $x \in M$, where H_v^{∇} is the ∇-horizontal subspace and the identification $\cong$ is the standard one. Then, by linear algebra,

$$\begin{aligned}
&\Big(T_v\Big(\bigoplus_{k=1}^{r} S^k T^* M\Big)\Big)^* \otimes \Big(T_v\Big(\bigoplus_{k=1}^{r} S^k T^* M\Big)\Big)^* \otimes T_v\Big(\bigoplus_{k=1}^{r} S^k T^* M\Big)\\
&= (T_x^* M \otimes T_x^* M \otimes T_x M) \oplus \bigoplus_{l=1}^{r} (T_x^* M \otimes T_x^* M \otimes S^l T_x^* M)\\
&\quad \oplus \bigoplus_{l=1}^{r} (T_x^* M \otimes S^l T_x M \otimes T_x M) \oplus \bigoplus_{l,l_1=1}^{r} (T_x^* M \otimes S^l T_x M \otimes S^{l_1} T_x^* M)\\
&\quad \oplus \bigoplus_{l=1}^{r} (S^l T_x M \otimes T_x^* M \otimes T_x M) \oplus \bigoplus_{l,l_1=1}^{r} (S^l T_x M \otimes T_x^* M \otimes S^{l_1} T_x^* M)\\
&\quad \oplus \bigoplus_{l,l_1=1}^{r} (S^l T_x M \otimes S^{l_1} T_x M \otimes T_x M) \oplus \bigoplus_{l,l_1,l_2=1}^{r} (S^l T_x M \otimes S^{l_1} T_x M \otimes S^{l_2} T_x^* M).
\end{aligned}$$

Consequently, our problem of finding all $\mathcal{M}f_m$-natural operators $C\colon Q \rightsquigarrow QT^{r*}$ is reduced to the one of finding all systems $\Delta^C = ((\Delta^1), \dots, (\Delta^8_{l,l_1,l_2}))$ of systems

$(\Delta^1), \dots, (\Delta^8_{l,l_1,l_2})$ of $\mathcal{M}f_m$-natural operators

$$
\begin{aligned}
\Delta^1\colon Q &\rightsquigarrow \Big(\bigoplus_{k=1}^r S^kT^*, T^*\otimes T^*\otimes T\Big),\\
\Delta^2_l\colon Q &\rightsquigarrow \Big(\bigoplus_{k=1}^r S^kT^*, T^*\otimes T^*\otimes S^lT^*\Big),\\
\Delta^3_l\colon Q &\rightsquigarrow \Big(\bigoplus_{k=1}^r S^kT^*, T^*\otimes S^lT\otimes T\Big),\\
\Delta^4_{l,l_1}\colon Q &\rightsquigarrow \Big(\bigoplus_{k=1}^r S^kT^*, T^*\otimes S^lT\otimes S^{l_1}T^*\Big),\\
\Delta^5_l\colon Q &\rightsquigarrow \Big(\bigoplus_{k=1}^r S^kT^*, S^lT\otimes T^*\otimes T\Big),\\
\Delta^6_{l,l_1}\colon Q &\rightsquigarrow \Big(\bigoplus_{k=1}^r S^kT^*, S^lT\otimes T^*\otimes S^{l_1}T^*\Big),\\
\Delta^7_{l,l_1}\colon Q &\rightsquigarrow \Big(\bigoplus_{k=1}^r S^kT^*, S^lT\otimes S^{l_1}T\otimes T\Big),\\
\Delta^8_{l,l_1,l_2}\colon Q &\rightsquigarrow \Big(\bigoplus_{k=1}^r S^kT^*, S^lT\otimes S^{l_1}T\otimes S^{l_2}T^*\Big)
\end{aligned}
$$

transforming classical linear connections ∇ on m-manifolds M into fibred maps $\Delta^1_M(\nabla)\colon \bigoplus_{k=1}^r S^kT^*M \to T^*M\otimes T^*M\otimes TM, \dots, \Delta^8_{l,l_1,l_2M}(\nabla)\colon \bigoplus_{k=1}^r S^kT^*M \to S^lTM\otimes S^{l_1}TM\otimes S^{l_2}T^*M$ covering the identity map of M, where $l, l_1, l_2 = 1, \dots, r$.

3. To obtain a more extensive reduction than the above one, we need a preparation.

A tensor natural subbundle (of type (p,q)) is a natural vector bundle $F\colon \mathcal{M}f_m \to \mathcal{VB}$ such that (modulo a natural vector bundle isomorphism) $FM \subset \bigotimes^p TM\otimes \bigotimes^q T^*M$ and $F\varphi = \bigotimes^p T\varphi\otimes\bigotimes^q T^*\varphi_{|FM}$ for any m-manifold M and any $\mathcal{M}f_m$-map $\varphi\colon M\to M^1$.

Proposition 3. *Let $F\colon \mathcal{M}f_m\to\mathcal{VB}$ be a tensor natural subbundle of type (p,q). The $\mathcal{M}f_m$-natural operators $B\colon Q\rightsquigarrow \Big(\bigoplus_{k=1}^r S^kT^*, F\Big)$ transforming classical linear connections ∇ on m-manifolds M into fibred maps $B_M(\nabla)\colon \bigoplus_{k=1}^r S^kT^*M\to FM$ covering id_M are in bijection with the systems $E = (E^{(k_1,\dots,k_j)})$ of $\mathcal{M}f_m$-natural* *operators* $E^{(k_1,\dots,k_j)}\colon Q \rightsquigarrow (S^{k_1}T\odot\dots\odot S^{k_j}T)\otimes F$ *for systems* $(k_1,\dots,k_j)$ *of integers* $k_1,\dots,k_j$ *with* $1\leqslant k_1\leqslant\dots\leqslant k_j\leqslant r$, $k_1+\dots+k_j\leqslant q-p$, $j=0,1,2,\dots$ *. If* $j=0$, *then* $(k_1,\dots,k_j)=\emptyset$, *and* $E^{\emptyset}\colon Q\rightsquigarrow F$. *If* $q-p<0$, *any* B *is the zero operator.* (*For* $\odot$, *see Remark* 1.)

More precisely, the natural operator $B^E\colon Q\rightsquigarrow \Big(\bigoplus_{k=1}^{r} S^kT^*, F\Big)$ corresponding to a system $E=(E^{(k_1,\dots,k_j)})$ (as above) is defined by

$$B^E_M(\nabla)_x(v)=\sum\langle E^{(k_1,\dots,k_j)}_M(\nabla)_x, v_{k_1}\otimes\dots\otimes v_{k_j}\rangle,$$

$\nabla\in \underline{Q}(M)$, $M\in\mathrm{obj}(\mathcal{M}f_m)$, $x\in M$, $v=(v_1,\dots,v_r)\in\bigoplus_{k=1}^{r}S^kT^*_xM$, where the (finite) sum $\sum$ is over all systems $(k_1,\dots,k_j)$ of integers with $1\leqslant k_1\leqslant\dots\leqslant k_j\leqslant r$, $k_1+\dots+k_j\leqslant q-p$, $j=0,1,2,\dots$.

Conversely, the system $E^B=(E^{B;(k_1,\dots,k_j)})$ corresponding to a natural operator $B\rightsquigarrow\Big(\bigoplus_{k=1}^{r}S^kT^*,F\Big)$ is well-defined by

$$\langle E^{B;(k_1,\dots,k_j)}_M(\nabla)_x, v_{k_1}\otimes\dots\otimes v_{k_j}\rangle=\frac{1}{\alpha!}\frac{\partial}{\partial t^{k_1}}\dots\frac{\partial}{\partial t^{k_j}}B_M(\nabla)_x(t^1v_1,\dots,t^rv_r)_{|_{t^1,\dots,t^r=0}}$$

where $v=(v_1,\dots,v_r)=\bigoplus_{k=1}^{r}S^kT^*_xM$, $x\in M$, $\alpha=1_{k_1}+\dots+1_{k_j}\in\mathbb{N}^r$.

Remark 1. In Proposition 3, we used the following notation. Given a sequence $V_1,\dots,V_r$ of different vector spaces and a system $(k_1,\dots,k_j)$ of integers with $1\leqslant k_1\leqslant\dots\leqslant k_j\leqslant r$, $V_{k_1}\odot\dots\odot V_{k_j}$ denotes the factor space $V_{k_1}\otimes\dots\otimes V_{k_j}/\sim$, where for any $u,w\in V_{k_1}\otimes\dots\otimes V_{k_j}$, $u\sim w$ iff $\langle u,\varphi_{k_1}\otimes\dots\otimes\varphi_{k_j}\rangle=\langle w,\varphi_{k_1}\otimes\dots\varphi_{k_j}\rangle$ (the usual pairing (contraction)) for any $(\varphi_1,\dots,\varphi_r)\in\bigoplus_{k=1}^{r}V^*_k$.

Proof. By the nonlinear Petree theorem (see [4]) B is of finite order. Further, by the invariance with respect to manifold charts, B is determined by the values

$$(B_{\mathbb{R}^m}(\nabla))_0(v)\in F_0\mathbb{R}^m$$

for all classical linear connections ∇ on $\mathbb{R}^m$ and all points $v=(v_1,\dots,v_r)\in\bigoplus_{k=1}^{r}S^kT^*_0\mathbb{R}^m$. We can assume that the coordinates (symbols) of ∇ are polynomials of degree being the finite order of B. Next, by the invariance of B with respect to the homotheties we have

$$B_{\mathbb{R}^m}((t\mathrm{id}_{\mathbb{R}^m})_*\nabla)_0\bigg(\bigoplus_{k=1}^{r}S^kT^*(t\mathrm{id}_{\mathbb{R}^m})(v)\bigg)=t^{p-q}B_{\mathbb{R}^m}(\nabla)_0(v)$$

for $t>0$. So, the homogeneous function theorem and the Taylor's theorem complete the proof. □

4. Clearly, Proposition 3 is applicable to natural operators $\Delta^1, \dots, \Delta^8_{l,l_1,l_2}$ from item 2. In particular, we have the following corollaries.

Corollary 1. *Given $l = 1, \dots, r$, any $\mathcal{M}f_m$-natural operator $\Delta^3_l \colon Q \rightsquigarrow \Big(\bigoplus_{k=1}^{r} S^k T^* M,\, T^* \oplus S^l T \otimes T\Big)$ is the zero one.*

Corollary 2. *Given $l = 1, \dots, r$, any $\mathcal{M}f_m$-natural operator $\Delta^5_l \colon Q \rightsquigarrow \Big(\bigoplus_{k=1}^{r} S^k T^*,\, S^l T \otimes T^* \otimes T\Big)$ is the zero one.*

Corollary 3. *Given $l, l_1 = 1, \dots, r$, any $\mathcal{M}f_m$-natural operator $\Delta^7_{l,l_1} \colon Q \rightsquigarrow \Big(\bigoplus_{k=1}^{r} S^k T^*,\, S^l T \otimes S^{l_1} T \otimes T\Big)$ is the zero one.*

Corollary 4. *The $\mathcal{M}f_m$-natural operators $\Delta^1 \colon Q \rightsquigarrow \Big(\bigoplus_{k=1}^{r} S^k T^*, T^* \otimes T^* \otimes T\Big)$ are in (the) bijection with the systems $E^{\Delta^1} = (E^{\Delta^1;\emptyset}, E^{\Delta^1;(1)})$ of $\mathcal{M}f_m$-natural operators $E^{\Delta^1;\emptyset} \colon Q \rightsquigarrow T^* \otimes T^* \otimes T$ and $E^{\Delta^1;(1)} \colon Q \rightsquigarrow T \otimes T^* \otimes T^* \otimes T$.*

Corollary 5. *Given a natural number $l = 1, \dots, r$, the $\mathcal{M}f_m$-natural operators $\Delta^2_l \colon Q \rightsquigarrow \Big(\bigoplus_{k=1}^{r} S^k T^*, T^* \otimes T^* \otimes S^l T^*\Big)$ are in (the) bijection with the systems $E^{\Delta^2_l} = (E^{\Delta^2_l;(k_1,\dots,k_j)})$ of $\mathcal{M}f_m$-natural operators $E^{\Delta^2_l;(k_1,\dots,k_j)} \colon Q \rightsquigarrow (S^{k_1} T \odot \dots \odot S^{k_j} T) \otimes T^* \otimes T^* \otimes S^l T^*$ for systems $(k_1, \dots, k_j)$ of integers with $1 \leqslant k_1 \leqslant \dots \leqslant k_j \leqslant r$, $k_1 + \dots + k_j \leqslant l + 2$, $j = 0, 1, 2, \dots$.*

Corollary 6. *Given natural numbers $l, l_1 = 1, \dots, r$, the $\mathcal{M}f_m$-natural operators $\Delta^4_{l,l_1} \colon Q \rightsquigarrow \Big(\bigoplus_{k=1}^{r} S^k T^*, T^* \otimes S^l T \otimes S^{l_1} T^*\Big)$ are in (the) bijection with the systems $E^{\Delta^4_{l,l_1}} = (E^{\Delta^4_{l,l_1};(k_1,\dots,k_j)})$ of $\mathcal{M}f_m$-natural operators $E^{\Delta^4_{l,l_1};(k_1,\dots,k_j)} \colon Q \rightsquigarrow (S^{k_1} T \odot \dots \odot S^{k_j} T) \otimes T^* \otimes S^l T \otimes S^{l_1} T^*$ for systems $(k_1, \dots, k_j)$ of integers with $1 \leqslant k_1 \leqslant \dots \leqslant k_j \leqslant r$, $k_1 + \dots + k_j \leqslant l_1 + 1 - l$, $j = 0, 1, \dots$.*

Corollary 7. *Given natural numbers $l, l_1 = 1, \dots, r$, the $\mathcal{M}f_m$-natural operators $\Delta^6_{l,l_1} \colon Q \rightsquigarrow \Big(\bigoplus_{k=1}^{r} S^k T^*, S^l T \otimes T^* \otimes S^{l_1} T^*\Big)$ are in (the) bijection with the systems $E^{\Delta^6_{l,l_1}} = (E^{\Delta^6_{l,l_1};(k_1,\dots,k_j)})$ of $\mathcal{M}f_m$-natural operators $E^{\Delta^6_{l,l_1};(k_1,\dots,k_j)} \colon Q \rightsquigarrow (S^{k_1} T \odot \dots \odot S^{k_j} T) \otimes S^l T \otimes T^* \otimes S^{l_1} T^*$ for systems $(k_1, \dots, k_j)$ of integers with $1 \leqslant k_1 \leqslant \dots \leqslant k_j \leqslant r$, $k_1 + \dots + k_j \leqslant l_1 + 1 - l$, $j = 0, 1, \dots$.*

Corollary 8. *Given natural numbers $l, l_1, l_2 = 1, \dots, r$, the $\mathcal{M}f_m$-natural operators $\Delta^8_{l,l_1,l_2}\colon Q \rightsquigarrow \Big(\bigoplus_{k=1}^{r} S^k T^*, S^l T \otimes S^{l_1} T \otimes S^{l_2} T^*\Big)$ are in (the) bijection with the systems $E^{\Delta^8_{l,l_1,l_2}} = (E^{\Delta^8_{l,l_1,l_2};(k_1,\dots,k_j)})$ of $\mathcal{M}f_m$-natural operators $E^{\Delta^8_{l,l_1,l_2};(k_1,\dots,k_j)}\colon Q \rightsquigarrow (S^{k_1}T \odot \dots \odot S^{k_j}T) \otimes S^l T \otimes S^{l_1} T \otimes S^{l_2} T^*$ for systems $(k_1, \dots, k_j)$ of integers with $1 \leqslant k_1 \leqslant \dots \leqslant k_j \leqslant r$, $k_1 + \dots + k_j \leqslant l_2 - l_1 - l$, $j = 0, 1, \dots$.*

5. Summing up, we have proved the following (roughly written) theorem.

Theorem 1. *The $\mathcal{M}f_m$-natural operators $C\colon Q \rightsquigarrow QT^{r*}$ are in (the) bijection with the systems $\Delta^C = ((\Delta^1), (\Delta^2_l), (\Delta^4_{l,l_1}), (\Delta^6_{l,l_1}), (\Delta^8_{l,l_1,l_2}))$ of systems (Δ^1), $(\Delta^2_l), \dots, (\Delta^8_{l,l_1,l_2})$ of $\mathcal{M}f_m$-natural operators corresponding to systems of $\mathcal{M}f_m$-natural operators (of the form (almost) $Q \rightsquigarrow \bigotimes^p T \otimes \bigotimes^q T^*$) as wrote in Corollaries 4–8.*

Let us explain our result for $r = 1$ (i.e., in the case of the cotangent bundle).

Now, $l, l_1, l_2 = 1$, only. Consequently, we have $E^{\Delta^1} = (E^{\Delta^1;\emptyset},\ E^{\Delta^1;(1)})$, $E^{\Delta^2_1} = (E^{\Delta^2_1;\emptyset},\ E^{\Delta^2_1;(1)},\ E^{\Delta^2_1;(1,1)},\ E^{\Delta^2_1;(1,1,1)})$, $E^{\Delta^4_{1,1}} = (E^{\Delta^4_{1,1};\emptyset},\ E^{\Delta^4_{1,1};(1)})$, $E^{\Delta^6_{1,1}} = (E^{\Delta^6_{1,1};\emptyset},\ E^{\Delta^6_{1,1};(1)})$, $E^{\Delta^8_{1,1,1}} = (0)$. Thus Theorem 1 for $r = 1$ can be read as follows.

The $\mathcal{M}f_m$-natural operators $C\colon Q \rightsquigarrow QT^*$ are in the bijection with the systems of $\mathcal{M}f_m$-natural operators

$$\begin{aligned}
E^{\Delta^1;\emptyset}&\colon Q \rightsquigarrow T^* \otimes T^* \otimes T,\\
E^{\Delta^1;(1)}&\colon Q \rightsquigarrow T \otimes T^* \otimes T^* \otimes T,\\
E^{\Delta^2_1;\emptyset}&\colon Q \rightsquigarrow T^* \otimes T^* \otimes T^*,\\
E^{\Delta^2_1;(1)}&\colon Q \rightsquigarrow T \otimes T^* \otimes T^* \otimes T^*,\\
E^{\Delta^2_1;(1,1)}&\colon Q \rightsquigarrow (T \odot T) \otimes T^* \otimes T^* \otimes T^*,\\
E^{\Delta^2_1;(1,1,1)}&\colon Q \rightsquigarrow (T \odot T \odot T) \otimes T^* \otimes T^* \otimes T^*,\\
E^{\Delta^4_{1,1};\emptyset}&\colon Q \rightsquigarrow T^* \otimes T \otimes T^*,\\
E^{\Delta^4_{1,1};(1)}&\colon Q \rightsquigarrow T \otimes T^* \otimes T \otimes T^*,\\
E^{\Delta^6_{1,1};\emptyset}&\colon Q \rightsquigarrow T^* \otimes T \otimes T^*,\\
E^{\Delta^6_{1,1};(1)}&\colon Q \rightsquigarrow T \otimes T^* \otimes T \otimes T^*.
\end{aligned}$$

Further, by the general description of natural tensors (item 0) we could describe explicitly the above 10 types of operators. (For example, any natural operator $E^{\Delta^1;\emptyset}\colon Q \rightsquigarrow T^* \otimes T^* \otimes T \cong T^* \otimes T \otimes T^*$ is the linear combination (with real coefficients) of three natural operators (the connection torsion operator T_∇, the operator $\delta_M \otimes C^1_1 T_\nabla$ (the tensor multiplication of the identity tensor field $\delta_M\colon TM \to TM$ and the contraction of the connection torsion) and the operator $C_1^1 T_\nabla \otimes \delta_M^*$. In the case of torsion free connection any such operator is the zero one. Similarly, any natural operator $E^{\Delta^1;(1)}: Q \rightsquigarrow T \otimes T^* \otimes T^* \otimes T \cong T^* \otimes T^* \otimes T \otimes T$ is a linear combination of two connection independent natural tensors (from the identity tensor $TM \otimes TM \to TM \otimes TM$ by means of permutations of indices). In this way we could reobtain (in another form) the result of M. Kureš [6] (in the case of natural operators $Q_\tau \rightsquigarrow QT^*$) and could obtain a new result in the case of not necessarily torsion free connections.

The explanation of our result in the case $r = 2$ is more complicated but (it seems) possible.

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