Saieed Akbari; Maryam Aliakbarpour; Naryam Ghanbari; Emisa Nategh; Hossein Shahmohamad

Join of two graphs admits a nowhere-zero 3-flow

Czechoslovak Mathematical Journal, Vol. 64 (2014), No. 2, 433–446

Persistent URL: `http://dml.cz/dmlcz/144007`

Czechoslovak Mathematical Journal, 64 (139) (2014), 433–446

JOIN OF TWO GRAPHS ADMITS A NOWHERE-ZERO 3-FLOW

Saieed Akbari, Maryam Aliakbarpour, Naryam Ghanbari,
Emisa Nategh, Tehran, Hossein Shahmohamad, Rochester

(Received January 22, 2013)

Abstract. Let G be a graph, and λ the smallest integer for which G has a nowhere-zero λ-flow, i.e., an integer λ for which G admits a nowhere-zero λ-flow, but it does not admit a $(\lambda-1)$-flow. We denote the minimum flow number of G by $\Lambda(G)$. In this paper we show that if G and H are two arbitrary graphs and G has no isolated vertex, then $\Lambda(G \vee H) \leqslant 3$ except two cases: (i) One of the graphs G and H is K_2 and the other is 1-regular. (ii) $H = K_1$ and G is a graph with at least one isolated vertex or a component whose every block is an odd cycle. Among other results, we prove that for every two graphs G and H with at least 4 vertices, $\Lambda(G \vee H) \leqslant 3$.

Keywords: nowhere-zero λ-flow; minimum nowhere-zero flow number; join of two graphs

MSC 2010: 05C20, 05C21, 05C78

1. Introduction

Throughout this paper all graphs are simple with no multiple edges. Let G be a graph. We denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. Let $v \in V(G)$. We denote the neighbors of v in G by $N_G(v)$. Let $S \subseteq V(G)$. For every $v \in V(G)$, define $N_S(v) = N_G(v) \cap S$. The complement of a graph G is denoted by $\overline{G}$. The degree of the vertex v in G is denoted by $d_G(v)$ (for abbreviation $d(v)$). For every positive integer k, a k-regular graph is a graph in which each vertex has degree k. For every integer r, rG denotes the disjoint union of r copies of G. An *even graph* is a graph in which all degrees are even (an *odd graph* is similarly defined). A *block* of G is a maximal connected subgraph having no cut vertex. A *leaf block* of a connected graph G is a block of G containing exactly one cut vertex of G. A *bracelet graph* is a connected graph whose each block is an odd cycle. A *broken bracelet graph* is a graph one of whose the components is a bracelet graph or an isolated vertex. The complete graph and the cycle of order n is denoted by K_n

and C_n, respectively. Let D denote the graph which is the union of K_1 and K_2. For positive integers $m_1, \ldots, m_k$ ($k \geqslant 2$), let $K_{m_1,\ldots,m_k}$ denote the complete k-partite graph with part sizes $m_1, \ldots, m_k$. The *join* of two graphs G and H, $G \vee H$, is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H. Let (D, f) be an ordered pair, where D is an orientation of $E(G)$ and let $f\colon E(G) \to \mathbb{Z}$ be an integer-valued function called a *flow*. For a vertex $v \in V(G)$, let $E_G^+(v)$ and $E_G^-(v)$ denote the sets of all edges of G with tails at v and heads at v, respectively. Let λ be a positive integer. A λ-*flow* of a graph G is a flow f such that $|f(e)| < \lambda$ for every $e \in E(G)$ and for every $v \in V(G)$,

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e).$$

A *nowhere-zero* λ-*flow* (abbreviated as a λ-NZF) of a graph G is an ordered pair (D, f) such that for every edge $e \in E(G)$, $f(e) \in \{1, \ldots, \lambda - 1\}$. Let G be a graph, and λ the smallest integer for which G has a λ-NZF, i.e., an integer λ for which G admits a λ-NZF, but it does not admit a $(\lambda - 1)$-NZF. We denote the minimum flow number of G by $\Lambda(G)$. Let A be an abelian additive group. An A-NZF is a flow with values in $A \setminus \{0\}$. The *boundary* of f is a function $\partial f\colon V(G) \to A$ defined by

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e).$$

The concept of a nowhere-zero λ-flow was introduced by Tutte [8] as a generalization for face coloring problems in planar graphs. The Four-Color Theorem says that every planar graph is 4-colorable. The Four-Color Theorem is equivalent to saying that every bridgeless planar graph has a 4-NZF. However, in [7], Tutte formulated his famous 5-flow conjecture which is still open:

Conjecture. Every bridgeless graph admits a 5-NZF.

Jaeger [1] proved that every bridgeless graph has an 8-NZF, and Seymour [4] improved Jaeger's result by showing that every bridgeless graph has a 6-NZF. Tutte conjectured that every 4-edge connected graph admits a 3-NZF. Jaeger [2] proved that every 4-edge connected graph has a 4-NZF. In [3], the authors showed that if every edge of a graph G is contained in a cycle of length at most 4, then G admits a 4-NZF. In this paper, we determine the exact minimum flow number of $G \vee H$ for all graphs G and H.

Tutte [7] proved the following interesting result.

Theorem 1. *A multigraph admits a* λ-NZF if and only if it admits a $\mathbb{Z}_\lambda$-NZF.

The following theorem characterizes all graphs which admit a 2-NZF, see [9], page 308.

Theorem 2. *A graph has a* 2-NZF if and only if it is an even graph.

In this paper, we show that if G is a connected graph of order $n \geqslant 2$ which is not a bracelet graph, then $\Lambda(K_1 \vee G) \leqslant 3$. Also, if G is a graph of order at least 2 and it is not a union of one isolated vertex and a 1-regular graph, then $\Lambda(\overline{K_2} \vee G) \leqslant 3$. If $r \geqslant 3$ a positive integer, then we show that $\Lambda(\overline{K_r} \vee G) \leqslant 3$, unless $G \in \{K_1, D\}$. Let $r \geqslant 4$ be an integer. We prove that for every graph G of order at least 4, $\Lambda(\overline{K_r} \vee G) \leqslant 3$. Also it is shown that for every two arbitrary graphs G and H of order at least 4, $\Lambda(G \vee H) \leqslant 3$. Moreover, we prove that if G and H are two graphs such that G has no isolated vertex, then $\Lambda(G \vee H) \leqslant 3$, with the following two exceptions:

(i) One of the graphs G and H is K_2 and the other is 1-regular.

(ii) $H = K_1$ and G is a broken bracelet graph.

2. Minimum NZF for $\overline{K_r} \vee G$

Let G be a graph. In this section, we will determine the minimum flow number of $\overline{K_r} \vee G$ for every positive integer r. The next interesting result was proved by Thomassen and Toft [6].

Theorem 3. *Let G be a* 2*-connected graph of minimum degree at least* 4*. Then G contains an induced cycle C such that $G \setminus V(C)$ is connected and $G \setminus E(C)$ is* 2*-connected.*

The following theorem shows that if every edge of a graph is contained in a small cycle, then the minimum flow number does not exceed 4.

Theorem 4 ([3])**.** *If every edge of a graph is contained in a cycle of length at most* 4*, then the graph admits a* 4-NZF.

Now, we have the following corollary.

Corollary 1. *Let G and H be two graphs. Then $\Lambda(G \vee H) \leqslant 4$, if one of the following holds:*

(i) *$|V(G)| = 1$ and H has no isolated vertices.*

(ii) *$|V(G)| \geqslant 2$ and $|V(H)| \geqslant 2$.*

Corollary 2. *Let G be a bracelet graph. Then $\Lambda(K_1 \vee G) = 4$.*

P r o o f. By Corollary 1, Part (i), $\Lambda(K_1 \vee G) \leqslant 4$. By Theorem 1, it suffices to prove that G has no $\mathbb{Z}_3$-NZF. We prove the corollary by induction on $n = |V(G)|$. Consider a leaf block of G and suppose that this block is C_{2k+1}. Let $V(C_{2k+1}) = \{v_0, \ldots, v_{2k}\}$, where v_0 is a cut vertex of G. By contradiction assume that $K_1 \vee G$ admits a $\mathbb{Z}_3$-NZF. With no loss of generality, assume that 3 edges incident with v_1 are outgoing with value 1 (note that in every $\mathbb{Z}_3$-NZF of a graph one can reverse the orientation of all edges with value 2 and change them to 1). Thus 3 edges incident with v_2 are incoming edges with value 1. By repeating this method, we conclude that 3 edges incident with v_{2k} are incoming edges with value 1. Let $H = K_1 \vee (G \setminus \{v_1, \ldots, v_{2k}\})$. Then H admits a $\mathbb{Z}_3$-NZF, which contradicts the induction hypothesis. Note that by a similar method one can see that, $\Lambda(K_1 \vee C_{2r+1}) > 3$ for every positive integer r. So by induction the proof is complete. □

The next lemma plays a key role in the proofs.

Lemma 1. *Assume that G is a connected even graph of order n and size m and $S \subseteq V(G)$, where $|S|$ is even. Then G admits an orientation in which every edge has value 1 and $\partial f(v) = 2$ for every $v \in S_1$ and $\partial f(v) = -2$ for every $v \in S_2$, where $S_1, S_2 \subset S$ and $|S_i| = |S|/2$ for $i = 1, 2$. Moreover, for any $v \in V(G) \setminus S$, $\partial f(v) = 0$.*

P r o o f. Let $V(G) = \{v_1, \ldots, v_n\}$. With no loss of generality suppose that $V(S) = \{v_1, \ldots, v_{2k}\}$. If $S = \emptyset$, then by Theorem 2, we are done. Thus assume that $S \neq \emptyset$. Since G is even, G has an Eulerian circuit with value 1, say $\mathcal{C}\colon v_1, v_{i_1}, v_{i_2}, \ldots, v_{i_{m-1}}, v_1$. Let j_1 be the smallest index for which $1 < i_{j_1} \leqslant 2k$. Orient the trail $v_1, v_{i_1}, v_{i_2}, \ldots, v_{i_{j_1}}$ is such a way that we obtain a directed trail from v_1 to $v_{i_{j_1}}$. Let $j_1 < j_2$ be the smallest index for which $i_{j_2} \leqslant 2k$ and $i_{j_2} \notin \{1, i_{j_1}\}$. Now, orient the trail $v_{i_{j_1}}, v_{i_{1+j_1}} \ldots, v_{i_{j_2}}$ in such a way that we obtain a directed trail from $v_{i_{j_2}}$ to $v_{i_{j_1}}$. Continue this procedure $2k-3$ times. Finally, orient the edges of the trail $v_{i_{j_{2k-1}}}, v_{i_{1+j_{2k-1}}}, \ldots, v_1$ in such a way that we obtain a directed trail from v_1 to $v_{i_{j_{2k-1}}}$. Now, let

$$S_1 = \{v_1, v_{i_{j_2}}, v_{i_{j_4}}, \ldots, v_{i_{j_{2k-2}}}\}, \quad S_2 = \{v_{i_{j_1}}, v_{i_{j_3}}, \ldots, v_{i_{j_{2k-1}}}\}.$$

This completes the proof. □

Lemma 2. *Let G and H be two graphs such that $\Lambda(G), \Lambda(H) \leqslant 3$. Then $\Lambda(L) \leqslant 3$, where L is the graph shown in Figure 1.*

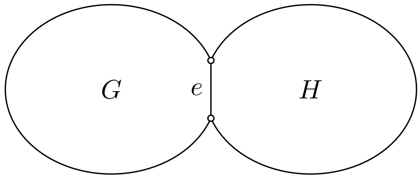

Figure 1. The graph L.

P r o o f. By Theorem 1, both G and H have a $\mathbb{Z}_3$-NZF, say f_1 and f_2, respectively, in which the orientation of e is the same. Now, define the following $\mathbb{Z}_3$-NZF f for L as follows:

$$f(x) = \begin{cases} f_1(x), & x \notin E(H), \\ if_2(x), & x \notin E(G), \\ f_1(e) + if_2(e), & x = e, \end{cases}$$

where $i = 1$ if $f_1(e) = f_2(e)$, and $i = 2$, otherwise. Now, by Theorem 1, we obtain the result. □

Before we determine the minimum flow number of $K_1 \vee G$ for a graph G, we need the following lemma.

Lemma 3. *If G has one of the following properties, then $\Lambda(K_1 \vee G) \leqslant 3$.*
(i) *Every component of G has an even number of vertices.*
(ii) *Every component of G has at least one vertex of odd degree.*

P r o o f. Obviously, it is enough to prove the lemma for a connected graph.

(i) Let $S = \{v \in V(G);\ d_G(v)$ is even$\}$. Since G has an even number of odd vertices, $|S|$ is even. Add a new vertex x and join this vertex to all vertices of odd degree in G and call the new graph H. If G has no odd vertex, then let $H = G$. By Lemma 1, we can find an orientation for H such that $S_1, S_2 \subset S$, $|S_i| = |S|/2$ for $i = 1, 2$, and S_1 and S_2 have the desired property. Now, join x to all vertices of S to form $K_1 \vee G$. Orient all edges incident with x and one endpoint in S_1 from x to S_1 and label them by 2. Do the same for S_2 and orient the edges from S_2 to x. This achieves a 3-NZF for $K_1 \vee G$.

(ii) If $|V(G)|$ is even, then by Part (i), we are done. So suppose that $|V(G)|$ is odd. Now, add a new vertex x to G and join it to all vertices of odd degree in G. Name this new graph H. Clearly, H is even. Define $S = \{v \in V(G);\ d_G(v)$is even$\} \cup \{x\}$. By Lemma 1, we can find an orientation for H such that $S_1, S_2 \subset S$, $|S_i| = |S|/2$ for $i = 1, 2$, and S_1 and S_2 have the desired property. Now, join x to v for every $v \in S_1 \setminus \{x\}$ and orient the edge xv from x to v with value 2. Then join x to every $u \in S_2 \setminus \{x\}$ and orient the edge ux from u to x with value 2. Now, the proof is complete. □

Theorem 5. *If G is a connected graph of order $n \geqslant 2$ which is not a bracelet graph, then $\Lambda(K_1 \vee G) \leqslant 3$.*

Proof. We apply induction on $|V(G)| + |E(G)|$. By Lemma 3, we can assume that G is an even graph of odd order. We consider two cases:

Case 1. G is 2-connected. We divide the proof of this case into two subcases:

Case 1.1. Assume that for every $v \in V(G)$, $d(v) \geqslant 4$. In this case, by Theorem 3 there exists an induced cycle C such that $H = G \setminus E(C)$ is 2-connected. Since C is an induced cycle, it is not hard to see that H is not a bracelet graph. Therefore by induction hypothesis $\Lambda(K_1 \vee G) \leqslant 3$.

Case 1.2. G has a vertex of degree 2. Since G is an even graph which is not an odd cycle, G has a vertex of degree at least 4. Let $V(G) = \{v_1, \ldots, v_n\}$. Since G is connected, there are two adjacent vertices v_1 and v_2 such that $d(v_1) = 2$ and $d(v_2) \geqslant 4$. Consider the following Eulerian circuit of G:

$$\mathcal{C}\colon\ v_1 v_2 v_{m_1} v_{m_2} \ldots v_{m_{|E(G)|-2}} v_1.$$

Since $d(v_2) \geqslant 4$, v_2 appears at least twice in $\mathcal{C}$. Suppose that t is the smallest index such that $v_2 = v_{m_t}$. We claim that there exists a sequence $j_1 < j_2 < \ldots < j_{n-2}$ such that $\{v_3, v_4, \ldots, v_n\} = \{v_{m_{j_1}}, v_{m_{j_2}}, \ldots, v_{m_{j_{n-2}}}\}$ and $S = \{v_{m_{j_k}}\,;\ j_k > t,\ 1 \leqslant k \leqslant n-2\}$ has even cardinality. Since every vertex of G appears at least once in $\mathcal{C}$, we conclude that there exists a sequence $s_1 < s_2 < \ldots < s_{n-2}$ such that $\{v_3, v_4, \ldots, v_n\} = \{v_{m_{s_1}}, v_{m_{s_2}}, \ldots, v_{m_{s_{n-2}}}\}$. Let $S' = \{v_{m_{s_i}}\,;\ s_i > t\}$. If $|S'|$ is even, then we are done. So, suppose that $|S'|$ is odd. Two closed trails $v_2 v_{m_1} v_{m_2} \ldots v_{m_t}$ and $v_{m_t} \ldots v_{m_{|E(G)|-2}} v_1 v_2$ partition all edges of G. If these two trails have only v_2 as a common vertex, then v_2 is a cut vertex and this contradicts the 2-connectedness of G. Thus there is a vertex $u \neq v_2$ in both trails. Since $d(v_1) = 2$, we have $u \neq v_1$. Therefore there exist positive integers p and q such that $p < t < q$ and $u = v_{m_p} = v_{m_q}$. Assume that $u = v_{m_{s_l}}$. If $s_l < t$, then we replace s_l by q. Otherwise, we replace s_l by p. This relabeling of indices makes $|S'|$ even and the claim is proved.

Let $r = n - 2 - |S|$. Since n is odd and $|S|$ is even, $r \neq 0$. Thus $j_r < t < j_{r+1}$. Add a new vertex x and join it to v_1 and v_2, then remove the edge $v_1 v_2$. Clearly, the resultant graph is even. Consider the directed Eulerian circuit $v_1 x v_2 v_{m_1} \ldots v_{m_{|E(G)|-2}} v_1$ in which the values of all edges are 1. Now, if $r \neq n-2$, define the following $n+1$ trails:

$$T_1 = v_1 x,$$
$$T_2 = x v_2 v_{m_1} \ldots v_{m_{j_1}},$$
$$T_3 = v_{m_{j_1}} \ldots v_{m_{j_2}},$$
$$\vdots$$

$$
\begin{aligned}
T_{r+1} &= v_{m_{j_{r-1}}} \dots v_{m_{j_r}},\\
T_{r+2} &= v_{m_{j_r}} \dots v_{m_t},\\
T_{r+3} &= v_{m_t} \dots v_{m_{j_{r+1}}},\\
&\vdots\\
T_n &= v_{m_{j_{n-3}}} \dots v_{m_{j_{n-2}}},\\
T_{n+1} &= v_{m_{j_{n-2}}} \dots v_{m_{|E(G)|-2}} v_1,
\end{aligned}
$$

and if $r = n-2$, define the trails T_i, $1 \leqslant i \leqslant r+2$, as before, define $T_{r+3} = v_{m_t} \dots v_{m_{|E(G)|-2}} v_1$. For $i = 1, \dots, (n+1)/2$, reverse the orientation of all edges of T_{2i}. Since $|S|$ is even, r is odd. It is not hard to see that $\partial f(v_1) = 2$, $\partial f(v_2) = \partial f(x) = -2$ and there are $(n-1)/2$ other vertices whose boundaries are 2 and the boundaries of other vertices are -2. Join x to all vertices with boundary -2 except v_2 and orient these edges with head at x. Also, join x to all vertices with boundary 2 except v_1 and orient them with tail at x. It is straightforward to see that we obtain a 3-NZF for $K_1 \vee G$.

Case 2: G is not 2-connected. First, assume that G has a leaf block, say B, which is an odd cycle. Let $V(K_1) = \{x\}$. Therefore, $K_1 \vee G$ is in Figure 2:

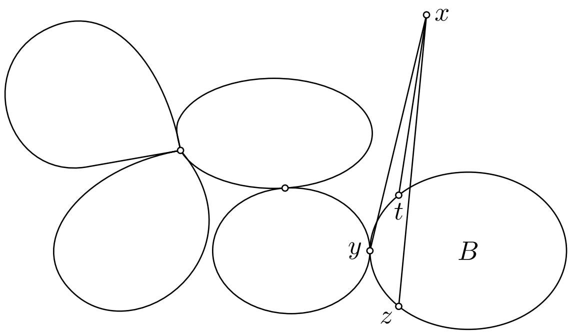

Figure 2. Graph G with an odd cycle as a leaf block.

Let $N_B(y) = \{z, t\}$ and $M = (\{x\} \vee B) \setminus \{xy\}$. Remove y from M and join z to t. By Lemma 3, Part (i), the resultant graph admits a 3-NZF. Since $d_M(y) = 2$, so $\Lambda(M) \leqslant 3$. Since B is an odd cycle and G is not a bracelet graph, $G \setminus (V(B) \setminus \{y\})$ is not a bracelet graph. By induction hypothesis, $\{x\} \vee (G \setminus (V(B) \setminus \{y\}))$ has a 3-NZF. This implies that $\Lambda(K_1 \vee G) \leqslant 3$.

Now, assume that no leaf block is an odd cycle. Consider a leaf block of G, say B. By induction hypothesis, $\Lambda(\{x\} \vee B) \leqslant 3$ and $\Lambda(K_1 \vee (G \setminus (V(B) \setminus \{y\}))) \leqslant 3$. Now, by Lemma 2, $\Lambda(K_1 \vee G) \leqslant 3$ and the proof is complete. □

Now, we have an immediate corollary.

Corollary 3. *Let G be a graph which is not a broken bracelet graph. Then* $\Lambda(K_1 \vee G) \leqslant 3$.

Corollary 4. *If G is not a 1-regular graph, then* $\Lambda(K_2 \vee G) \leqslant 3$.

Proof. We wish to show that if G is not 1-regular, then $K_1 \vee G$ cannot be a bracelet graph. By contradiction, suppose that all blocks of $K_1 \vee G$ are odd cycles. Clearly, for each graph H, $K_1 \vee H$ has at most one cut vertex. Therefore, $K_1 \vee G$ can only be in Figure 3:

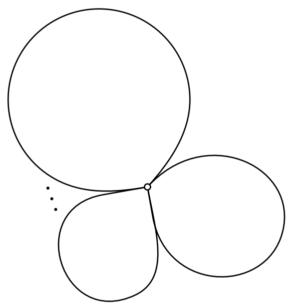

Figure 3. The graph $K_1 \vee G$.

Since each vertex other than the cut vertex has degree 2, G should be 1-regular, which is a contradiction. So by Theorem 5, $\Lambda(K_1 \vee (K_1 \vee G) = K_2 \vee G) \leqslant 3$. □

Remark 1. Let $H = nK_2$ for some positive integer n, and $G = K_2 \vee H$. We show that $\Lambda(G) = 4$. To see this, by Corollary 1, Part (ii), we have $\Lambda(G) \leqslant 4$. By contradiction assume that $\Lambda(G) = 3$. By Theorem 1, G has a $\mathbb{Z}_3$-NZF. We can assume that the value of each edge is 1, because by reversing the orientation of any edge labeled by 2 and changing 2 to 1, we achieve a $\mathbb{Z}_3$-NZF with value 1. If $v \in V(H)$ and $d_G(v) = 3$, then clearly all edges incident with v are outgoing or incoming. This yields that the value of the edge xy should be zero, where $V(K_2) = \{x, y\}$, a contradiction. Therefore $\Lambda(G) = 4$.

Theorem 6. *If G is a graph of order at least 2 and G is not a union of one isolated vertex and a 1-regular graph, then* $\Lambda(\overline{K_2} \vee G) \leqslant 3$.

Proof. If G contains an isolated vertex t, then by Corollary 4, $\Lambda(K_2 \vee (G \setminus \{t\})) \leqslant 3$. Since the degree of t is 2 in $\overline{K_2} \vee G$, it is not hard to see that $\Lambda(\overline{K_2} \vee G) \leqslant 3$. Thus we can assume that G has no isolated vertex. We claim that if H is a graph and $\Lambda(K_1 \vee H) \leqslant 3$, then $\Lambda(\overline{K_2} \vee H) \leqslant 3$. Let $V(\overline{K_2}) = \{x, y\}$. Assume that f is a $\mathbb{Z}_3$-NZF for $\{x\} \vee H$. Now, we define a $\mathbb{Z}_3$-NZF, say g, for $\overline{K_2} \vee H$ as follows:

For every $v \in V(H)$ orient the edge yv, in the same way as the edge xv, and keep the orientation of all edges of H and define $g(xv) = g(yv) = f(xv)$, $g(e) = 2f(e)$, for every $e \in E(H)$. It is straightforward to see that g is a $\mathbb{Z}_3$-NZF for $\overline{K_2} \vee H$ and so $\Lambda(\overline{K_2} \vee H) \leqslant 3$ and the claim is proved.

Let $G_1, \ldots, G_r$ be the connected components of G, for some positive integer r. If G_i is not a bracelet graph, then by Theorem 5, $\Lambda(K_1 \vee G_i) \leqslant 3$. So by the claim, we conclude that $\Lambda(\overline{K_2} \vee G_i) \leqslant 3$. Now, if G_i is a bracelet graph, then $\Lambda(G_i) = 2$. On the other hand by [5], $\Lambda(K_{2,|V(G_i)|}) \leqslant 3$. Therefore $\Lambda(\overline{K_2} \vee G) \leqslant 3$. □

Remark 2. If G is a union of an isolated vertex and a 1-regular graph and $|V(G)| \geqslant 2$, then $\Lambda(\overline{K_2} \vee G) = 4$, since if t is an isolated vertex, the existence of a λ-NZF for $\overline{K_2} \vee G$ is equivalent to the existence of a λ-NZF for $K_2 \vee (G \setminus \{t\})$. Now, by Remark 1, $\Lambda(\overline{K_2} \vee G) = 4$.

In the next result we obtain the minimum flow number of the join of $\overline{K_r}$, $(r \geqslant 3)$ and an arbitrary graph.

Theorem 7. *Let $r \geqslant 3$ be a positive integer. Then $\Lambda(\overline{K_r} \vee G) \leqslant 3$, unless $G \in \{K_1, D\}$.*

Proof. If $n = |V(G)| = 2$, then by Corollary 4 and [5], the assertion holds. Thus assume that $n \geqslant 3$. Let $V(\overline{K_r}) = \{v_1, \ldots, v_r\}$. Suppose there is no positive integer s such that $G = K_1 \cup sK_2$. By Theorem 6, $\Lambda(\overline{K_2} \vee G) \leqslant 3$. First, let $r = 4$. By [5], $\Lambda(K_{2,n}) \leqslant 3$ and so $\Lambda(\overline{K_4} \vee G) \leqslant 3$. Now, assume that $r = 3$. Consider a $\mathbb{Z}_3$-NZF, say f_1, for $\{v_1, v_2\} \vee G$ and a $\mathbb{Z}_3$-NZF, say f_2, for $\{v_2, v_3\} \vee G$ such that the orientation and the value of all edges of G and all edges incident with v_2 in f_1 and f_2 are the same. Clearly, $f_1 + f_2$ is a $\mathbb{Z}_3$-NZF for $\{v_1, v_2, v_3\} \vee G \cong \overline{K_3} \vee G$.

Now, let $r \in \{3, 4\}$ and $G = K_1 \cup sK_2$, for some positive integer $s > 1$. By induction on s, we show that $\Lambda(\overline{K_r} \vee G) \leqslant 3$. For $s = 2$, the result follows from Figure 4 (the value of each edge is 1):

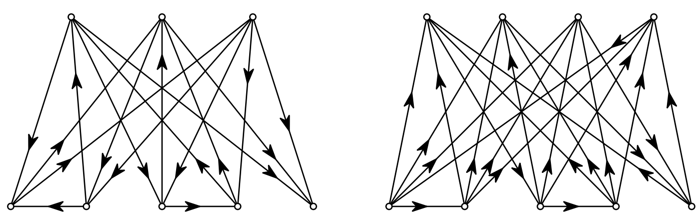

Figure 4. A $\mathbb{Z}_3$-NZF for $\overline{K_r} \vee (K_1 \cup 2K_2)$, $r = 3, 4$.

Now, assume that $s \geqslant 3$. By induction hypothesis, $\Lambda(\overline{K_r} \vee (K_1 \cup (s-1)K_2)) \leqslant 3$. On the other hand, by Corollary 4, $\Lambda(\overline{K_r} \vee K_2) \leqslant 3$. These, yield that $\Lambda(\overline{K_r} \vee G) \leqslant 3$.

Finally, let $r \geqslant 5$. We know that $\Lambda(\overline{K_3} \vee G) \leqslant 3$. On the other hand facts by [5], $\Lambda(K_{r-3,n}) \leqslant 3$ and so $\Lambda(\overline{K_r} \vee G) \leqslant 3$. $\square$

Remark 3. For every positive integer $r \geqslant 2$, $\Lambda(\overline{K_r} \vee D) = 4$. To see this by Theorem 4, it suffices to prove that $\overline{K_r} \vee D$ does not admit a $\mathbb{Z}_3$-NZF. By contradiction assume that $\overline{K_r} \vee D$ admits a $\mathbb{Z}_3$-NZF, f, in which the value of each edge is 1. In such a $\mathbb{Z}_3$-NZF of $\overline{K_r} \vee D$, 3 edges incident with each vertex of $\overline{K_r}$ are incoming or outgoing edges. This yields that $0 = \partial f(z) = \partial f(x) - f(xy)$, where $V(K_1) = \{z\}$ and $V(K_2) = \{x, y\}$. Thus $f(xy) = 0$, a contradiction. So $\Lambda(\overline{K_r} \vee D) = 4$.

3. Minimum NZF for join of two graphs

In this section, we show that except a few cases, the join of two arbitrary graphs has minimum flow number at most 3.

Theorem 8. *Let G and H be two graphs such that G has no isolated vertex. Then $\Lambda(G \vee H) \leqslant 3$, with the following two exceptions:*

(i) *One of the graphs G and H is K_2 and the other is 1-regular.*

(ii) *$H = K_1$ and G is a broken bracelet graph.*

Proof. Let $n_1 = |V(G)|$ and $n_2 = |V(H)|$. We use induction on $|V(G)| + |V(H)| + |E(G)| + |E(H)|$. We divide the proof into three cases:

Case 1. First assume that H has at least two isolated vertices. If $H = \overline{K_2}$, then since G has no isolated vertex by Theorem 6, $\Lambda(G \vee H) \leqslant 3$. Thus assume that $H \neq \overline{K_2}$. Remove two isolated vertices of H and call the resultant graph by H'. If none of the above exceptions holds for two graphs G and H', then by induction hypothesis, $\Lambda(G \vee H') \leqslant 3$. Moreover, by [5], $\Lambda(K_{2,n_1}) \leqslant 3$. So, $\Lambda(G \vee H) \leqslant 3$. Now, suppose that one of the exceptions holds for G or H'. So we have the following subcases:

Case 1.1. Let $G = K_2$ and $H' = rK_2$, for some positive integer r. We have $H = \overline{K_2} \cup H'$. Now, by Corollary 4, $\Lambda(G \vee H) \leqslant 3$.

Case 1.2. Let $H' = K_2$ and $G = rK_2$, for some positive integer r. By Theorem 6, $\Lambda(\overline{K_2} \vee G) \leqslant 3$. Also by Corollary 4, $\Lambda(K_2 \vee \overline{K_{2r}}) \leqslant 3$ and so $\Lambda(G \vee H) \leqslant 3$.

Case 1.3. G is a broken bracelet graph and $H' = K_1$. In this case $H = \overline{K_3}$ and by Theorem 7 we are done.

Case 2. Now, assume that H has exactly one isolated vertex. If $H = K_1$, then by Corollary 3, we are done. Thus let $n_2 \geqslant 2$. If G is not connected and G_1 is a component of G, then by induction hypothesis, $\Lambda((G \setminus V(G_1)) \vee H) \leqslant 3$ and $\Lambda(G_1 \vee \overline{K_{n_2}}) \leqslant 3$. These facts imply that $\Lambda(G \vee H) \leqslant 3$. Now, assume that G is connected. If G is not a bracelet graph, then by Theorem 5, $\Lambda(G \vee K_1) \leqslant 3$. On the other hand, by induction hypothesis $\Lambda(\overline{K_{n_1}} \vee (H \setminus V(K_1))) \leqslant 3$. Hence $\Lambda(G \vee H) \leqslant 3$. Now, let G be a bracelet graph. Call the isolated vertex of H by v. First, assume that $G = C_{2k+1}$ and $V(C_{2k+1}) = \{v_1, \ldots, v_{2k+1}\}$, for some positive integer k. Let P: $v_1v_2v_3$ be a 3-path on the cycle C_{2k+1}. We have $\Lambda(\{v\} \vee P) \leqslant 3$. Also, the union of triangles $vv_4v_5, vv_6v_7, \ldots, vv_{2k}v_{2k+1}$ has a 2-NZF. On the other hand, since $H \setminus \{v\}$ has no isolated vertex, by induction hypothesis $\Lambda((K_1 \cup kK_2) \vee (H \setminus \{v\})) \leqslant 3$. Thus in this case $\Lambda(G \vee H) \leqslant 3$. Now, assume that G is not an odd cycle. Therefore G is in Figure 5:

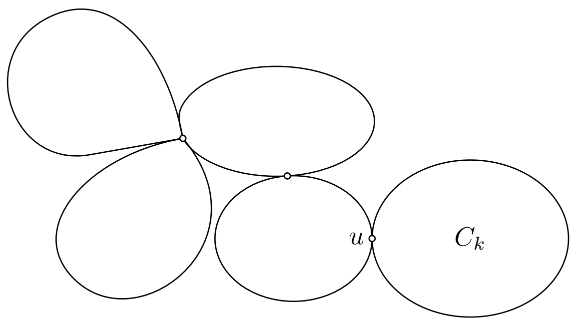

Figure 5. A bracelet graph with a leaf block isomorphic to C_k.

Let $G' = G \setminus (V(C_k) \setminus \{u\})$. By induction hypothesis $\Lambda(G' \vee H) \leqslant 3$. Moreover, $\Lambda(C_k) = 2$. Now, by [5], $\Lambda(K_{k-1,n_2}) \leqslant 3$. Thus $\Lambda(G \vee H) \leqslant 3$.

Case 3. Now, assume that H has no isolated vertex. If one of the graphs G and H is a broken bracelet graph, then by removing the edges of a leaf block of a bracelet component and using induction we obtain a 3-NZF for $G \vee H$. Thus assume that neither of the graphs G and H is a broken bracelet graph. If $n_1 = 2$ or $n_2 = 2$, then by Corollary 4, we are done. So let $n_1, n_2 \geqslant 3$. Now, let $x \in V(G)$ and $y \in V(H)$. By Corollary 3, $\Lambda(\{x\} \vee H) \leqslant 3$ and $\Lambda(\{y\} \vee G) \leqslant 3$. Moreover, by [5], $\Lambda(K_{n_1-1,n_2-1}) \leqslant 3$. Now, using Lemma 2, it is not hard to see that $\Lambda(G \vee H) \leqslant 3$ and the proof is complete. □

The next result proves that the minimum flow number of the join of any two graphs of order at least 4 does not exceed 3.

Theorem 9. *Let G and H be two graphs of order at least 4. Then $\Lambda(G \vee H) \leqslant 3$.*

Proof. Let $n_1 = |V(G)|$ and $n_2 = |V(H)|$. We prove the theorem by induction on $|E(G)| + |E(H)|$. First, let one of the graphs G and H, say G, have at least two isolated vertices. If H has no isolated vertex, then by Theorem 8, $\Lambda(G \vee H) \leqslant 3$.

Thus we can assume that H has at least one isolated vertex. If $G = \overline{K_{n_1}}$, then by Theorem 7, we are done. Thus assume that G has at least one edge. Let $G = G_1 \cup \overline{K_r}$, for some positive integer $r (r \geqslant 2)$, where $\delta(G_1) \geqslant 1$. Now, by Theorem 8, $\Lambda(G_1 \vee H) \leqslant 3$. By [5], $\Lambda(K_{r,n_2}) \leqslant 3$ and so $\Lambda(G \vee H) \leqslant 3$.

Therefore, by Theorem 8, we can assume that both G and H have exactly one isolated vertex. Let $G = G_1 \cup \{u\}$ and $H = H_1 \cup \{v\}$, where u and v are isolated vertices of G and H, respectively. If G contains a cycle, then remove all edges of this cycle and apply the induction to obtain a 3-NZF for $G \vee H$. Thus one can assume that G_1 is not a broken bracelet graph. Similarly, H_1 is not a broken bracelet graph. Let $x \in V(G_1)$ and $y \in V(H_1)$. By Corollary 3, $\Lambda(\{x\} \vee H_1) \leqslant 3$ and $\Lambda(\{y\} \vee G_1) \leqslant 3$. Let $M = \{xp, yq;\ p \in V(H_1), q \in V(G_1)\}$. Now, by Lemma 2, the induced subgraph of $G_1 \vee H_1$ on $E(G_1) \cup E(H_1) \cup M$, L, has a 3-NZF. Let $T = (G \vee H) \setminus (\{x, y\} \cup E(L))$. Clearly, $T \cong K_{n_1-1,n_2-1}$. Consider the 4-cycle with vertex set $\{x, y, u, v\}$ and call it by C. We know that $\Lambda(C) = 2$. We have $E((G \vee H) \setminus (E(L) \setminus \{xy\})) = E(T) \cup E(C)$. Now, by Lemma 2 and [5], $\Lambda((G \vee H) \setminus (E(L) \setminus \{xy\})) \leqslant 3$. Again, using Lemma 2, $\Lambda(G \vee H) \leqslant 3$. The proof is complete. □

We close the paper with the following result.

Theorem 10. *Let G be a graph. Then $\Lambda(D \vee G) = 3$, unless $G = \overline{K_r}$ for some positive integer r.*

Proof. First, notice that since $D \vee G$ has at least one vertex of odd degree, by Theorem 2, $\Lambda(D \vee G) \geqslant 3$. We use induction on $|V(G)| + |E(G)|$. Assume that G has at least one edge and $G_1, \ldots, G_t$ are all components of G of order at least 2 and G has s isolated vertices ($s \geqslant 0$). Suppose that $G \neq K_1 \cup 2K_2$. If $t \geqslant 2$, then there exists a component of G, say G_1, such that $G \setminus V(G_1) \neq D$. Hence by induction hypothesis, $\Lambda(D \vee G_1) = 3$ and by Theorem 7, $\Lambda(\overline{K_3} \vee (G \setminus V(G_1))) \leqslant 3$. Therefore $\Lambda(D \vee G) = 3$. Now, if $G = K_1 \cup 2K_2$, then Figure 6 as well as Theorem 1, show that $\Lambda(D \vee G) = 3$.

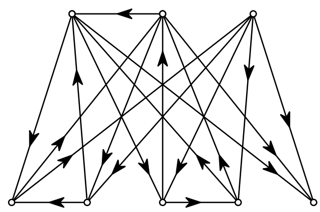

Figure 6. A $\mathbb{Z}_3$-flow for $D \vee (K_1 \cup 2K_2)$.

Now, suppose that $t = 1$. If $s \geqslant 2$, then using induction hypothesis and [5], we have $\Lambda(D \vee G_1) = 3$ and $\Lambda(K_{3,s}) \leqslant 3$, and we are done. Hence, let $s \leqslant 1$. If $s = 0$, then the result follows from Theorem 8. Now, let $s = 1$. If G_1 is a bracelet graph and G_1 is not an odd cycle, then by removing all edges of a block of G_1 and using induction hypothesis the assertion holds. Thus assume that $G_1 = C_{2k+1}$, for some positive integer k. Now, by induction on k, we show that $\Lambda(D \vee (C_{2k+1} \cup K_1)) = 3$. Figure 7 shows that the assertion holds for $k = 1$:

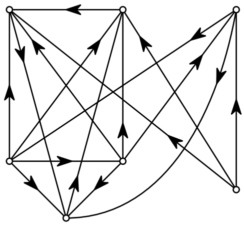

Figure 7. A $\mathbb{Z}_3$-flow for $D \vee (C_3 \cup K_1)$.

Assume that $\Lambda(D \vee (C_{2k-1} \cup K_1)) = 3$. Let $V(C_{2k-1}) = \{v_1, \ldots, v_{2k-1}\}$. Take a 3-NZF for $D \vee (C_{2k-1} \cup K_1)$. Replace the edge v_1v_2 by a path P of order 4, $P\colon\ v_1pqv_2$. Orient and label all edges of P in the same way as v_1v_2. Now, since $\Lambda(K_{2,3}) \leqslant 3$, we find that $\Lambda(D \vee (C_{2k+1} \cup K_1)) = 3$.

Now, suppose that G_1 is not a bracelet graph. If $G_1 = K_2$, since $\Lambda(D \vee D) = 3$, the assertion holds. Hence, let $G_1 \neq K_2$. Thus, $D \vee G$ is shown in Figure 8:

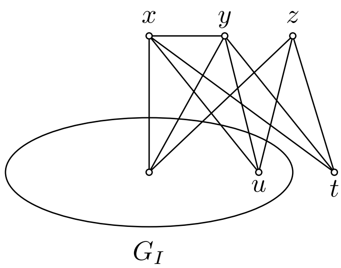

Figure 8. $D \vee (G_1 \cup \{t\})$.

Now, let $L = (D \vee \{u, t\}) \setminus \{xu\}$. Obviously, L admits a 3-NZF. On the other hand, $\{y, z\} \vee (G_1 \setminus (E(G_1) \cup \{u\})) \cong K_{2,n-2}$, where $n = |V(G)|$. By Theorem 5, $\Lambda(\{x\} \vee G_1) \leqslant 3$. Therefore, $\Lambda(D \vee G) = 3$ and the proof is complete. □

References

[1] *F. Jaeger*: Flows and generalized coloring theorems in graphs. J. Comb. Theory, Ser. B *26* (1979), 205–216.
[2] *F. Jaeger*: Nowhere-zero flow problems. Selected Topics in Graph Theory 3. Academic Press, San Diego, 1988, pp. 71–95.
[3] *R. Luo, W. Zang, C. Zhang*: Nowhere-zero 4-flows, simultaneous edge-colorings, and critical partial Latin squares. Combinatorica *24* (2004), 641–657.
[4] *P. D. Seymour*: Nowhere-zero 6-flows. J. Comb. Theory, Ser. B *30* (1981), 130–135.
[5] *H. Shahmohamad*: On minimum flow number of graphs. Bull. Inst. Comb. Appl. *35* (2002), 26–36.
[6] *C. Thomassen, B. Toft*: Non-separating induced cycles in graphs. J. Comb. Theory, Ser. B *31* (1981), 199–224.
[7] *W. T. Tutte*: A contribution to the theory of chromatic polynomials. Can. J. Math. *6* (1954), 80–91.
[8] *W. T. Tutte*: On the imbedding of linear graphs in surfaces. Proc. Lond. Math. Soc., II. Ser. *51* (1949), 474–483.
[9] *D. B. West*: Introduction to Graph Theory. Prentice Hall, Upper Saddle River, 1996.

Authors' addresses: S. Akbari, Department of Mathematical Sciences, Sharif University of Technology; School of Mathematics, Institute for Studies in Theoretical Physics and Mathematics, P. O. Box 19395-5746, Tehran, Iran, e-mail: s_akbari@sharif.edu; M. Aliakbarpour, Department of Computer Engineering, Sharif University of Technology, P. O. Box 19395-5746, Tehran, Iran, e-mail: aliakbarpour@ce.sharif.edu; N. Ghanbari, Department of Electrical Engineering, Sharif University of Technology, P. O. Box 19395-5746, Tehran, Iran, e-mail: niloofar.ghanbari@yahoo.com; E. Nategh, Department of Mathematical Sciences, Sharif University of Technology, P. O. Box 19395-5746, Tehran, Iran, e-mail: e.nategh@gmail.com; H. Shahmohamad, School of Mathematical Sciences, RIT, Rochester, New York 14623, USA, e-mail: hxssma@rit.edu.