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MULTIPLICITY AND UNIQUENESS FOR A CLASS OF DISCRETE FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. The paper deals with a class of discrete fractional boundary value problems. We construct the corresponding Green's function, analyse it in detail and establish several of its key properties. Then, by using the fixed point index theory, the existence of multiple positive solutions is obtained, and the uniqueness of the solution is proved by a new theorem on an ordered metric space established by M. Jleli, et al. (2012).

Keywords: fractional order; discrete fractional boundary value problem; fractional difference equation; positive solution

MSC 2010: 26A33, 39A12, 39A05

1. Introduction

In this paper, we are concerned with the discrete fractional boundary value problems

$$\begin{cases} -\Delta^{\nu}_{\nu-2} y(t) = f(t+\nu-1, y(t+\nu-1)), & t \in \mathbb{N}_{0,b}, \\ \Delta y(\nu-2) = y(\nu+b) = 0, \end{cases} \tag{1.1}$$

where $\Delta^{\nu}_{\nu-2}$ is a discrete fractional operator, $1 < \nu < 2$, $\mathbb{N}_{0,b} := \{0, 1, 2, \ldots, b\}$, $b \in \mathbb{N}$, $b \geqslant 3$ and $f\colon \{\nu-2, \nu-1, \ldots, \nu+b\} \times \mathbb{R} \to \mathbb{R}$.

Fractional calculus is a generalization of the ordinary differentiation and integration. It has played a significant role in science, engineering, economy, and other fields [29], [32], [30]. Today there is a large number of papers dealing with the continuous fractional calculus. However, the discrete fractional calculus has seen slower

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progress, for it is still a relatively new and emerging area of mathematics. We refer the reader to [1], [2], [3], [4], [7], [5], [13], [26], [27], [9] and the references therein for the history and basic theory of the discrete fractional calculus. Of particular interest is that Atici and Şengül have shown the usefulness of fractional difference equations in tumor growth modeling in [8]. We can see that they will provide a new tool to model physical phenomena in the future. Thus, to study the fractional difference equations is meaningful, necessary and significant.

Recent interests in the discrete fractional calculus are shown by Atici and Eloe [3], [4]; in [4], they developed the commutativity properties of the fractional sum and the fractional difference operators, and were the first to study a class of initial value problems. Then a number of papers appeared investigating the discrete fractional boundary value problems, such as [1], [6], [12], [16], [17], [18], [19], [20], [23], [21], [24], [15], [26], [31], [33], [5], [14], [22].

In [6], the authors discussed the two-point boundary value problems for finite fractional difference equations

$$\begin{gathered}-\Delta_{\nu-2}^{\nu} y(t)=f(t+\nu-1, y(t+\nu-1)), \quad 1<\nu \leqslant 2, \\ y(\nu-2)=y(\nu+b+1)=0.\end{gathered}$$

They proved the existence of positive solutions by the Krasnosel'skii fixed-point theorem. In [17], Goodrich considered a class of fractional difference equations with nonlocal conditions, and proved the existence and uniqueness of solution by using a variety of tools including the contraction mapping theorem, the Brouwer theorem, and the Krasnosel'skii theorem.

To our best of knowledge, most of the recent papers are concerning the existence of solutions by the Krasnosel'skii fixed-point theorem, and there are few papers dealing with the existence of multiple solutions. Moreover, most of them dealt with the uniqueness of the solution by the contraction mapping theorem. Motivated by [6], [16], [17], [28], [11], we investigate problem (1.1). We obtain the multiplicity of solutions by the fixed-point index theory, and prove the uniqueness of the solution by a new tool established by M. Jleli et al. in [28]. In Section 3, we will see that the uniqueness theorem is a result different from those obtained by the contraction mapping theorem.

The rest of the paper is organized as follows. In Section 2, we introduce some notation, definitions, and preliminary facts that will be used in the remainder of the paper. We get Green's function $G(t,s)$ and discuss its properties in Section 3. Then in Sections 4 and 5, we obtain the multiplicity and uniqueness of positive solutions for problem (1.1), and present examples to demonstrate the application of our result.

2. Preliminaries

Now we present some fundamental facts on the discrete fractional calculus theory which will be found in the recent literature (cf. [1], [4], [6], [13], [16], [26]). For convenience, we introduce the following notation which will be used in the sequel:

$$\mathbb{N}_a = \{a, a+1, a+2, \ldots, \}, \quad a \in \mathbb{R},$$
$$\mathbb{N}_{c,d} = \{c, c+1, c+2, \ldots, d\}, \quad c, d \in \mathbb{R},\ d-c>0,\ c-d \in \mathbb{Z}.$$

We also assume that the empty sums are zero.

Definition 2.1 ([26]). Let $f\colon \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$ be given. Then the νth-order fractional sum of f is given by

$$(\Delta_a^{-\nu} f)(t) = \Delta_a^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} f(s) \quad \text{for } t \in \mathbb{N}_{a+\nu}.$$

Also, we define the trivial sum by $\Delta_a^0 f(t) := f(t)$ for $t \in \mathbb{N}_a$.

Remark 2.1 ([26]). The σ-function in Definition 2.1 comes from the general theory of time scales. It denotes the next point in the time scale after s. In this case, $\sigma(s) = s+1$ for all $s \in \mathbb{N}_a$. The term $(t-\sigma(s))^{\underline{\nu-1}}$ in Definition 2.1 is the so-called generalized falling function, defined by

$$t^{\underline{\mu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\mu)},$$

for any $t, \mu \in \mathbb{R}$ for which the right-hand side is well-defined. We appeal to the convention that if $t+1-\mu$ is a pole of the Gamma function while $t+1$ is not a pole, then $t^{\underline{\mu}} = 0$.

Definition 2.2 ([26]). Let $f\colon \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$ be given, and let $N \in \mathbb{N}$ be chosen such that $N-1 < \nu \leqslant N$. Then the νth-order fractional difference of f is given by

$$(\Delta_a^{\nu} f)(t) = \Delta_a^{\nu} f(t) := \Delta^N \Delta_a^{-(N-\nu)} f(t) \quad \text{for } t \in \mathbb{N}_{a+N-\nu}.$$

Remark 2.2. In [26], Holm proved that

$$\Delta_a^{\nu} f(t) = \begin{cases} \dfrac{1}{\Gamma(-\nu)} \displaystyle\sum_{s=a}^{t+\nu} (t-\sigma(s))^{\underline{-\nu-1}} f(s), & N-1 < \nu < N, \\ \Delta^N f(t), & \nu = N. \end{cases}$$

Lemma 2.1 ([26]).

(1) *Let $a \in \mathbb{R}$ and $\mu > 0$ be given. Then*

$$\Delta(t-a)^{\underline{\mu}} = \mu(t-a)^{\underline{\mu-1}},$$
$$\Delta(a-t)^{\underline{\mu}} = -\mu(a-t-1)^{\underline{\mu-1}}$$

for any t for which both sides are well defined.

(2) *Let $f\colon \mathbb{N}_a \to \mathbb{R}$ be given. For any $k \in \mathbb{N}_0$ and $\mu > 0$ with $M-1 < \mu \leqslant M$ we have*

$$\Delta^k \Delta_a^{-\mu} f(t) = \Delta_a^{k-\mu} f(t) \quad \text{for } t \in \mathbb{N}_{a+\mu}.$$

(3) *Let $f\colon \mathbb{N}_{\mu-M} \to \mathbb{R}$ and $\mu > 0$ with $M-1 < \mu \leqslant M$ be given. Then for $t \in \mathbb{N}_{\mu-M}$,*

$$\Delta_0^{-\mu}\Delta_{\mu-M}^{\mu} f(t) = f(t) - \sum_{j=0}^{M-1} C_j t^{\underline{\mu-M+j}}, \quad C_j \in \mathbb{R}\ (j = 0, 1, \ldots, M-1).$$

The following well-known theorem is very important in our arguments, see [10], [25] for more details about the fixed-point index.

Theorem 2.1 ([11], [10], [25]). *Let X be a Banach space, $K \subseteq X$ a cone in X. For $q > 0$, define $K_q = \{x \in K;\ |x| \leqslant q\}$. Assume that $Q\colon K_q \to K$ is a compact map such that $Qx \neq x$ for $x \in \partial K_q = \{x \in K;\ |x| = q\}$.*

(i) *If $|x| \leqslant |Qx|$ for $x \in \partial K_q$, then*

$$i(Q, K_q, K) = 0.$$

(ii) *If $|x| \geqslant |Qx|$ for $x \in \partial K_q$, then*

$$i(Q, K_q, K) = 1.$$

Next, let us present the fixed-point theorem on ordered metric spaces established in [28]. First, we collect some definitions and notation that are used in the new theorem. These can be found in [28]. Let $(X, d, \preceq)$ be a partially ordered metric space.

Let Φ denote the set of all functions $\varphi\colon [0,\infty) \to [0,\infty)$ satisfying

(a) φ is continuous nondecreasing;

(b) $\varphi^{-1}(\{0\}) = \{0\}$.

Let Ψ denote the set of all functions $\psi\colon\ [0,\infty)\to[0,\infty)$ satisfying

(c) $\lim\limits_{t\to r^+}\psi(t)>0$ (and finite) for all $r>0$;

(d) $\lim\limits_{t\to 0^+}\psi(t)=0$.

Let Θ denote the set of all functions $\theta\colon\ [0,\infty)^4\to[0,\infty)$ satisfying

(e) θ is continuous;

(f) $\theta(s_1,s_2,s_3,s_4)=0$ if and only if $s_1s_2s_3s_4=0$.

Examples of typical function θ are given in [28].

Definition 2.3 ([28]). We say that a mapping $F\colon\ X\times X\to X$ is mixed monotone if

$$x,y,u,v\in X,\ x\preceq u,\ y\succeq v\Longrightarrow F(x,y)\preceq F(u,v).$$

Definition 2.4 ([28]). Let $(X,\preceq)$ be a partially ordered set endowed with a metric d. We say that $(X,d,\preceq)$ is $\uparrow\downarrow$-regular if X has the property that

(i) if a nondecreasing sequence $\{x_n\}\subset X$ converges to x, then $x_n\preceq x$ for all n;

(ii) if a nonincreasing sequence $\{x_n\}\subset X$ converges to x, then $x_n\succeq x$ for all n.

Let $F\colon\ X\times X\to X$ be a given mapping. We consider the mappings $A\colon\ X\times X\to[0,\infty)$ and $B\colon\ X\times X\times X\times X\to[0,\infty)$ defined by

$$A(x,y)=\frac{d(x,F(x,y))+d(y,F(y,x))}{2},\quad (x,y)\in X\times X,$$
$$B(x,y,u,v)=\frac{d(x,F(u,v))+d(y,F(v,u))}{2},\quad (x,y,u,v)\in X\times X\times X\times X.$$

Theorem 2.2 ([28]). *Let $(X,d,\preceq)$ be a partially ordered complete metric space and $F\colon\ X\times X\to X$ a mixed monotone mapping for which there exist $\varphi\in\Phi$, $\psi\in\Psi$, and $\theta\in\Theta$ such that for all $x,y,u,v\in X$ with $x\succeq u$, $y\preceq v$,*

$$\begin{aligned}(2.1)\qquad \varphi\Big(\frac{d(F(x,y),F(u,v))+d(F(y,x),F(v,u))}{2}\Big)&\leqslant\varphi\Big(\frac{d(x,u)+d(y,v)}{2}\Big)\\ -\psi\Big(\frac{d(x,u)+d(y,v)}{2}\Big)&+\theta(A(x,y),A(u,v),B(x,y,u,v),B(u,v,x,y)).\end{aligned}$$

Suppose also that $(X,d,\preceq)$ is $\uparrow\downarrow$-regular and there exist $x_0,y_0\in X$ such that

$$(2.2)\qquad x_0\preceq F(x_0,y_0),\quad y_0\succeq F(y_0,x_0)$$

or

$$(2.3)\qquad x_0\succeq F(x_0,y_0),\quad y_0\preceq F(y_0,x_0).$$

Then F admits a coupled fixed point; that is, there exists $(a,b)\in X\times X$ such that $a=F(a,b)$, $b=F(b,a)$. Further, suppose that every pair of elements in $X\times X$ has either a lower bound or an upper bound. Then F has a unique coupled fixed point. Moreover, if $x_0\preceq y_0$, then $a=b$, that is, $F(a,a)=a$.

Let us conclude this section by the definition of the solution to problem (1.1).

Define the Banach space

$$X=\{x;\ x\colon \mathbb{N}_{\nu-2,\nu+b}\to\mathbb{R}\}$$

endowed with the norm $\|x\|=\max\limits_{t\in\mathbb{N}_{\nu-2,\nu+b}}|x(t)|$.

Definition 2.5. Any $y\in X$ is called a solution of problem (1.1) if $y\in X$ and y satisfies the boundary conditions.

3. Green's function and its properties

In this section, we derive the definite expression of the corresponding Green's function $G(t,s)$ associated with problem (1.1) and prove some of its important properties.

Lemma 3.1. *Let $1<\nu<2$ and $h\colon \mathbb{N}_{\nu-1,\nu+b-1}\to\mathbb{R}$ be given. Then y is a solution of the discrete fractional boundary value problem*

$$\begin{cases} -\Delta^{\nu}_{\nu-2}y(t)=h(t+\nu-1), & t\in\mathbb{N}_{0,b},\\ \Delta y(\nu-2)=y(\nu+b)=0, \end{cases}$$

if and only if $y(t)$ for $t\in\mathbb{N}_{\nu-2,\nu+b}$, has the form

$$y(t)=\sum_{s=0}^{b}G(t,s)h(s+\nu-1),$$

where

$$G(t,s)=\begin{cases} C(\nu,b)(\nu+b-\sigma(s))^{\underline{\nu-1}}, & t=\nu-2,\ \nu-1,\\ \dfrac{C(\nu,b)}{\Gamma(\nu)}[(2-\nu)t^{\underline{\nu-1}}+(\nu-1)t^{\underline{\nu-2}}](\nu+b-\sigma(s))^{\underline{\nu-1}} & \\ \qquad\qquad -\dfrac{1}{\Gamma(\nu)}(t-\sigma(s))^{\underline{\nu-1}}, & 0\leqslant s\leqslant t-\nu\leqslant b,\\ \dfrac{C(\nu,b)}{\Gamma(\nu)}[(2-\nu)t^{\underline{\nu-1}}+(\nu-1)t^{\underline{\nu-2}}](\nu+b-\sigma(s))^{\underline{\nu-1}}, & \\ & 0\leqslant t-\nu<s\leqslant b, \end{cases} \tag{3.1}$$

and

$$C(\nu,b)=\frac{1}{(2-\nu)(\nu+b)^{\underline{\nu-1}}+(\nu-1)(\nu+b)^{\underline{\nu-2}}}$$
$$=\frac{1}{(\nu+b)^{\underline{\nu-2}}[(\nu-1)+(2-\nu)(b+2)]}.$$

P r o o f. By virtue of Definition 2.1 and Lemma 2.1, we can deduce the result via arguments similar to those in the proof of Lemma 3.1 in [6]. Here we omit it. □

Lemma 3.2. *Green's function $G(t,s)$ has the following properties:*

1°. *For each $t\in\{\nu-2,\nu-1\}$, $G(t,s)$ is nonincreasing on $s\in\mathbb{N}_{0,b}$.*

2°. *For each fixed $t\in\mathbb{N}_{\nu,\nu+b}$, $G(t,s)$ is nondecreasing on $s\in\mathbb{N}_{0,t-\nu+1}$ and nonincreasing on $s\in\mathbb{N}_{t-\nu+1,b}$.*

3°. *$G(\nu,0)<G(\nu-1,0)$, and for $s\neq 0$, $G(\nu-1,s)<G(\nu,s)$.*

4°. *For each fixed $s\in\mathbb{N}_{0,b}$, $G(t,s)$ is nondecreasing on $t\in\mathbb{N}_{\nu,\nu+s-1}$ and nonincreasing on $t\in\mathbb{N}_{\nu+s-1,\nu+b}$.*

5°. *For all $(t,s)\in\mathbb{N}_{\nu-2,\nu+b}\times\mathbb{N}_{0,b}$, $G(t,s)\geqslant 0$.*

6°.

$$\sum_{s=0}^{b}G(t,s)=\begin{cases}\dfrac{C(\nu,b)}{\nu}(\nu+b)^{\underline{\nu}}=\frac{(b+2)(b+1)}{\nu[1+(2-\nu)(b+1)]}, & t=\nu-2,\ \nu-1,\\[2mm] \dfrac{C(\nu,b)(\nu+b)^{\underline{\nu}}}{\Gamma(\nu+1)}[(2-\nu)t^{\underline{\nu-1}}+(\nu-1)t^{\underline{\nu-2}}]-\dfrac{1}{\Gamma(\nu+1)}t^{\underline{\nu}}, & t\in\mathbb{N}_{\nu,\nu+b}.\end{cases}\tag{3.2}$$

P r o o f. 1°: For $t=\nu-2,\ \nu-1$, by Lemma 2.1 we can derive

$$\Delta_sG(t,s)=C(\nu,b)\Delta_s(\nu+b-\sigma(s))^{\underline{\nu-1}}=-(\nu-1)C(\nu,b)(\nu+b-\sigma(s)-1)^{\underline{\nu-2}}<0,$$

which implies that 1° is true.

2°: For $s\in\mathbb{N}_{0,t-\nu-1}$, we have that

$$\begin{aligned}\Delta_sG(t,s)&=\frac{1}{\Gamma(\nu-1)}\Big[-C(\nu,b)((2-\nu)t^{\underline{\nu-1}}+(\nu-1)t^{\underline{\nu-2}})\frac{\Gamma(\nu+b-s-1)}{\Gamma(b-s+1)}\\&\quad+\frac{\Gamma(t-s-1)}{\Gamma(t-s-\nu+1)}\Big]\\&=\frac{1}{\Gamma(\nu-1)}\Big[-\frac{\Gamma(t+1)}{\Gamma(t-\nu+3)}\frac{\Gamma(b+3)}{\Gamma(\nu+b+1)}\frac{\Gamma(\nu+b-s-1)}{\Gamma(b-s+1)}\\&\quad\times\frac{(2-\nu)(t-\nu+2)+(\nu-1)}{(2-\nu)(b+2)+(\nu-1)}+\frac{\Gamma(t-s-1)}{\Gamma(t-s-\nu+1)}\Big]\geqslant 0,\end{aligned}\tag{3.3}$$

which is immediately obtained from the following two facts:

$$(b+2-k)(t-k)-(t-\nu+2-k)(\nu+b-k)=(2-\nu)(t-\nu-b)\leqslant 0,\quad \text{for } k\in\mathbb{N}_{0,s+1},$$

and

$$\begin{aligned}&\frac{\Gamma(t+1)}{\Gamma(t-\nu+3)}\frac{\Gamma(b+3)}{\Gamma(\nu+b+1)}\frac{\Gamma(\nu+b-s-1)}{\Gamma(b-s+1)}\frac{(2-\nu)(t-\nu+2)+(\nu-1)}{(2-\nu)(b+2)+(\nu-1)}\\&\times\frac{\Gamma(t-s-\nu+1)}{\Gamma(t-s-1)}=\frac{(2-\nu)(t-\nu+2)+(\nu-1)}{(2-\nu)(b+2)+(\nu-1)}\prod_{k=0}^{s+1}\frac{(b+2-k)(t-k)}{(t-\nu+2-k)(\nu+b-k)}\leqslant 1.\end{aligned}$$

For $s=t-\nu$, we have

$$\Delta_s G(t,s)\big|_{s=t-\nu}=-\frac{C(\nu,b)}{\Gamma(\nu-1)}[(2-\nu)t^{\underline{\nu-1}}+(\nu-1)t^{\underline{\nu-2}}](\nu+b-t+\nu-2)^{\underline{\nu-2}}+1.$$

Let $h_1(t)=(2-\nu)t^{\underline{\nu-1}}+(\nu-1)t^{\underline{\nu-2}}$ and $k_1(t)=(\nu+b-t+\nu-2)^{\underline{\nu-2}}$. We have

$$\begin{aligned}&\Delta(h_1(t)k_1(t))=k_1(t+1)\Delta h_1(t)+h_1(t)\Delta k_1(t)\\&=(\nu+b-t+\nu-3)^{\underline{\nu-2}}(2-\nu)(\nu-1)[t^{\underline{\nu-2}}-t^{\underline{\nu-3}}]\\&\quad+(2-\nu)[(2-\nu)t^{\underline{\nu-1}}+(\nu-1)t^{\underline{\nu-2}}](\nu+b-t+\nu-3)^{\underline{\nu-3}}>0,\quad \text{for } t\in\mathbb{N}_{\nu,\nu+b-1}.\end{aligned}$$

By the inequalities

$$(\nu-1)b+2(\nu-1)<b+\nu$$

and

$$(\nu-1)+(2-\nu)(b+1)<(\nu-1)+(2-\nu)(b+2),$$

we get $(s\leqslant b-1)$

$$\begin{aligned}\Delta_s G(t,s)\big|_{s=t-\nu}&=-\frac{C(\nu,b)}{\Gamma(\nu-1)}h_1(t)k_1(t)+1\\&\geqslant-\frac{C(\nu,b)}{\Gamma(\nu-1)}h_1(\nu+b-1)k_1(\nu+b-1)+1\\&=-\frac{(\nu-1)b+2(\nu-1)}{b+\nu}\frac{(\nu-1)+(2-\nu)(b+1)}{(\nu-1)+(2-\nu)(b+2)}+1>0.\end{aligned}\tag{3.4}$$

For $s\in\mathbb{N}_{t-\nu+1,b-1}$, we have

$$\Delta_s G(t,s)=-(\nu-1)\frac{C(\nu,b)}{\Gamma(\nu)}[(2-\nu)t^{\underline{\nu-1}}+(\nu-1)t^{\underline{\nu-2}}](\nu+b-\sigma(s)-1)^{\underline{\nu-2}}<0.\tag{3.5}$$

Therefore, by (3.3), (3.4), and (3.5), we see that 2° holds.

3°: We have

$$\begin{aligned}
G(\nu,0)-G(\nu-1,0) &= \frac{C(\nu,b)}{\Gamma(\nu)}[(2-\nu)\nu^{\underline{\nu-1}}+(\nu-1)\nu^{\underline{\nu-2}}](\nu+b-1)^{\underline{\nu-1}} \\
&\quad -\frac{1}{\Gamma(\nu)}(\nu-1)^{\underline{\nu-1}}-C(\nu,b)(\nu+b-1)^{\underline{\nu-1}} \\
&= C(\nu,b)\Big[(2-\nu)\nu+\frac{(\nu-1)\nu}{2}\Big](\nu+b-1)^{\underline{\nu-1}}-1-C(\nu,b)(\nu+b-1)^{\underline{\nu-1}} \\
&= C(\nu,b)(\nu+b-1)^{\underline{\nu-1}}\frac{(\nu-1)(2-\nu)}{2}-1 \\
&= \frac{(b+1)}{(\nu+b)}\frac{(\nu-1)(b+2)(2-\nu)}{2[(\nu-1)+(2-\nu)(b+2)]}-1<0,
\end{aligned}$$

which follows from the facts that

$$(\nu-1)(b+2)(2-\nu)<2[(\nu-1)+(2-\nu)(b+2)] \quad \text{and} \quad (b+1)<(\nu+b).$$

So, $G(\nu,0)<G(\nu-1,0)$. Then, for $s\neq 0$,

$$G(\nu,s)-G(\nu-1,s)=C(\nu,b)(\nu+b-s-1)^{\underline{\nu-1}}\frac{(\nu-1)(2-\nu)}{2}>0.$$

4°: For $t\in\mathbb{N}_{\nu,\nu+s-2}$,

$$\Delta_t G(t,s)=\frac{C(\nu,b)}{\Gamma(\nu)}(2-\nu)(\nu-1)[t^{\underline{\nu-2}}-t^{\underline{\nu-3}}](\nu+b-\sigma(s))^{\underline{\nu-1}}>0. \tag{3.6}$$

For $t=\nu+s-1$, we have

$$\begin{aligned}
&\Delta_t G(t,s)\big|_{t=\nu+s-1} \\
&= \frac{C(\nu,b)}{\Gamma(\nu)}(2-\nu)(\nu-1)[t^{\underline{\nu-2}}-t^{\underline{\nu-3}}]\big|_{t=\nu+s-1}(\nu+b-\sigma(s))^{\underline{\nu-1}}-1 \\
&= \frac{(2-\nu)(b+2)}{\Gamma(\nu-1)[(\nu-1)+(2-\nu)(b+2)]}\frac{s+1}{s+2}\frac{\Gamma(\nu+s)}{\Gamma(s+2)}\prod_{k=0}^{s}\frac{b+1-k}{\nu+b-k}-1<0.
\end{aligned} \tag{3.7}$$

For $t\in\mathbb{N}_{\nu+s,\nu+b-1}$, we obtain

$$\begin{aligned}
\Delta_t G(t,s) &= \frac{1}{\Gamma(\nu-1)}\Big[\frac{(2-\nu)(t-\nu+2)}{[(\nu-1)+(2-\nu)(b+2)]} \\
&\quad \times\frac{(\nu+b-\sigma(s))^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-2}}}\frac{\Gamma(t+1)}{\Gamma(t-\nu+4)}-(t-s-1)^{\underline{\nu-2}}\Big]<0.
\end{aligned} \tag{3.8}$$

In fact, for $k \in \mathbb{N}_{0,s}$,

$$\begin{aligned}(t-k)(b+1-k)-(\nu+b-k)(t-\nu+2-k) &= (\nu-1)(\nu+b-t)-(\nu+b-k)\\ &< (\nu-2)(\nu+b-k) < 0,\end{aligned}$$

so

$$\begin{aligned}&\frac{(2-\nu)(t-\nu+2)}{[(\nu-1)+(2-\nu)(b+2)]}\frac{(\nu+b-\sigma(s))^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-2}}}\frac{\Gamma(t+1)}{\Gamma(t-\nu+4)}\frac{1}{(t-s-1)^{\underline{\nu-2}}}\\ &= \frac{(2-\nu)(b+2)}{[(\nu-1)+(2-\nu)(b+2)]}\frac{t-\nu+2}{t-\nu+3}\prod_{k=0}^{s}\frac{(t-k)(b+1-k)}{(\nu+b-k)(t-\nu+2-k)} < 1.\end{aligned}$$

Combining (3.6), (3.7), and (3.8), we conclude that 4° is satisfied.

5°: In view of $G(t,s)$, in order to prove $G(t,s) \geqslant 0$, we only need to check that $G(t,s) \geqslant 0$ when $0 \leqslant s \leqslant t-\nu \leqslant b$.

By 2°, for $0 \leqslant s \leqslant t-\nu$,

$$\begin{aligned}G(t,s) \geqslant G(t,0) &= \frac{1}{\Gamma(\nu)}\frac{(2-\nu)t^{\underline{\nu-1}}+(\nu-1)t^{\underline{\nu-2}}}{(\nu-1)+(2-\nu)(b+2)}\frac{(\nu+b-1)^{\underline{\nu-1}}}{(\nu+b)^{\underline{\nu-2}}}-\frac{(t-1)^{\underline{\nu-1}}}{\Gamma(\nu)}\\ &= \frac{1}{\Gamma(\nu)}\frac{\Gamma(t+1)}{\Gamma(t-\nu+3)}k_2(t) \geqslant 0,\end{aligned}$$

where

$$k_2(t) = \frac{(\nu-1)+(2-\nu)(t-\nu+2)}{(\nu-1)+(2-\nu)(b+2)}\frac{(b+2)(b+1)}{\nu+b}-\frac{(t-\nu+2)(t-\nu+1)}{t}.$$

In fact, we can treat $k_2(t)$ as a continuous function of t on the interval $[\nu,\nu+b]$, and by simple calculation, we have that

$$k_2'(t) = \frac{-(\nu-1)[(2-\nu)(b+2)+(\nu+b)]}{(\nu+b)[(\nu-1)+(2-\nu)(b+2)]}-\frac{(2-\nu)(\nu-1)}{t^2} < 0,$$

which implies that

$$k_2(t) \geqslant k_2(\nu+b) = 0.$$

Thus, we have proved 5°.

6°: By using Lemma 2.4 in [15], we get the result easily. □

Remark 3.1. From 1° and 2° of Lemma 3.2, we have that for each $t \in \{\nu-2, \nu-1\}$,

$$\max_{s\in\mathbb{N}_{0,b}} G(t,s) = G(t,0), \quad \min_{s\in\mathbb{N}_{0,b}} G(t,s) = G(t,b),$$

and for each $t \in \mathbb{N}_{\nu,\nu+b}$,

$$\max_{s\in\mathbb{N}_{0,b}} G(t,s) = G(t, t-\nu+1), \qquad \min_{s\in\mathbb{N}_{0,b}} G(t,s) = \max\{G(t,0), G(t,b)\}.$$

From 1°, 3° and 4° of Lemma 3.2, we get that for each $s \in \mathbb{N}_{0,b}$,

$$\max_{t\in\mathbb{N}_{\nu-2,\nu+b}} G(t,s) = G(\nu+s-1, s),$$
$$\min_{t\in\mathbb{N}_{\nu-2,\nu+b}} G(t,s) = \min\{G(\nu+b,s), G(\nu,s), G(\nu-1,s)\} = G(\nu+b,s) = 0,$$

and

$$\min_{t\in\mathbb{N}_{\nu,\nu+b-1}} G(t,s) = \min\{G(\nu,s), G(\nu+b-1,s)\}.$$

From 6° of Lemma 3.2, we obtain that

$$M_0 := \max_{t\in\mathbb{N}_{\nu-2,\nu+b}} \sum_{s=0}^{b} G(t,s) > 0, \quad m_0 := \min_{t\in\mathbb{N}_{\nu-2,\nu+b}} \sum_{s=0}^{b} G(t,s) = 0.$$

Lemma 3.3. *For all $t \in \mathbb{N}_{\nu,\nu+b-1}$ we have*

$$G(t,s) \geqslant \eta G(s+\nu-1, s), \tag{3.9}$$

where $\eta = \min\{1, \eta_1, \eta_2, \eta_3\}$,

$$\eta_1 = \frac{(\nu-1)b[(4-\nu)b+6-\nu]}{2(b+2)(b+1)},$$
$$\eta_2 = \frac{(2-\nu)\nu^{\underline{\nu-1}} + (\nu-1)\nu^{\underline{\nu-2}}}{(2-\nu)(\nu+b-1)^{\underline{\nu-1}} + (\nu-1)(\nu+b-1)^{\underline{\nu-2}}},$$

and

$$\eta_3 = \min_{s\in\mathbb{N}_{0,b-1}} \Big[\frac{(2-\nu)(\nu+b-1)^{\underline{\nu-1}} + (\nu-1)(\nu+b-1)^{\underline{\nu-2}}}{(2-\nu)(\nu+s-1)^{\underline{\nu-1}} + (\nu-1)(\nu+s-1)^{\underline{\nu-2}}} - \frac{1}{C(\nu,b)}\frac{b-s}{\nu+b-s-1}\Big] > 0.$$

P r o o f. By the property 4° of Lemma 3.2, we get that

$$\min_{t\in\mathbb{N}_{\nu,\nu+b-1}} G(t,s) = \min\{G(\nu,s), G(\nu+b-1,s)\}.$$

For $s=0$ we have

$$\frac{G(\nu,s)}{G(\nu+s-1,s)}\Big|_{s=0} = \frac{G(\nu,0)}{G(\nu-1,0)} = \frac{(\nu-1)b[(4-\nu)b+6-\nu]}{2(b+2)(b+1)} = \eta_1 > 0.$$

Let $k_3(s) = (2-\nu)(\nu+s-1)^{\underline{\nu-1}} + (\nu-1)(\nu+s-1)^{\underline{\nu-2}}$. Then, for $s \in \mathbb{N}_{0,b-1}$, we have

$$\Delta_s k_3(s) = (\nu-1)(2-\nu)[(\nu+s-1)^{\underline{\nu-2}} - (\nu+s-1)^{\underline{\nu-3}}] > 0.$$

Thus, for $s \neq 0$,

$$\begin{aligned}\frac{G(\nu,s)}{G(\nu+s-1,s)} &= \frac{(2-\nu)\nu^{\underline{\nu-1}} + (\nu-1)\nu^{\underline{\nu-2}}}{(2-\nu)(\nu+s-1)^{\underline{\nu-1}} + (\nu-1)(\nu+s-1)^{\underline{\nu-2}}} \\ &\geqslant \frac{(2-\nu)\nu^{\underline{\nu-1}} + (\nu-1)\nu^{\underline{\nu-2}}}{(2-\nu)(\nu+b-1)^{\underline{\nu-1}} + (\nu-1)(\nu+b-1)^{\underline{\nu-2}}} = \eta_2 > 0.\end{aligned}$$

For $s = b$, we have

$$\frac{G(\nu+b-1,s)}{G(\nu+s-1,s)}\bigg|_{s=b} = 1.$$

By Remark 3.1, we have that

$$\max_{t\in\mathbb{N}_{\nu-2,\nu+b}} G(t,s) = G(\nu+s-1,s).$$

Thus,

$$G(\nu+s-1,s) \geqslant G(\nu+b-1,s).$$

For $s \neq b$, by 2° of Lemma 3.2, we have $G(\nu+b-1,s) \geqslant G(\nu+b-1,0)$. By (3.8) of Lemma 3.2 (the proof of 4°), we have $G(\nu+b-1,0) > G(\nu+b,0) = 0$.

Then, we have that

$$\frac{G(\nu+b-1,s)}{G(\nu+s-1,s)} > 0. \tag{3.10}$$

By direct computation, we obtain

$$\begin{aligned}&\frac{G(\nu+b-1,s)}{G(\nu+s-1,s)} \\ &= \frac{(2-\nu)(\nu+b-1)^{\underline{\nu-1}} + (\nu-1)(\nu+b-1)^{\underline{\nu-2}}}{(2-\nu)(\nu+s-1)^{\underline{\nu-1}} + (\nu-1)(\nu+s-1)^{\underline{\nu-2}}} - \frac{1}{C(\nu,b)}\frac{b-s}{\nu+b-s-1}.\end{aligned}$$

In Lemma 3.3, we have defined η_3 as follows:

$$\begin{aligned}\eta_3 = \min_{s\in\mathbb{N}_{0,b-1}} \bigg[&\frac{(2-\nu)(\nu+b-1)^{\underline{\nu-1}} + (\nu-1)(\nu+b-1)^{\underline{\nu-2}}}{(2-\nu)(\nu+s-1)^{\underline{\nu-1}} + (\nu-1)(\nu+s-1)^{\underline{\nu-2}}} \\ &- \frac{1}{C(\nu,b)}\frac{b-s}{\nu+b-s-1}\bigg].\end{aligned}$$

In view of (3.10), we have that $\eta_3 > 0$. Therefore,

$$\frac{G(\nu+b-1,s)}{G(\nu+s-1,s)} \geqslant \eta_3.$$

Thus, we conclude that

$$\frac{G(t,s)}{G(\nu+s-1,s)} \geqslant \eta \quad \text{for all } t \in \mathbb{N}_{\nu,\nu+b-1}.$$

□

4. Multiplicity

In Section 1, we have defined the Banach space X. Now let us define a cone K in X by

$$K = \{x \in X;\ x(t) \geqslant 0,\ t \in \mathbb{N}_{\nu-2,\nu+b} \quad \text{and} \quad \min_{t\in\mathbb{N}_{\nu,\nu+b-1}} x(t) \geqslant \eta\|x\|\}.$$

Define an operator $T\colon\ X \to X$ as

$$(Tx)(t) = \sum_{s=0}^{b} G(t,s) f(s+\nu-1, x(s+\nu-1)).$$

Then, by Lemma 3.1, problem (1.1) can be written as

$$x = Tx, \quad x \in X.$$

Next, we give multiplicity results for the problem (1.1). To be precise, we introduce the conditions on $f(t,x)$:

(F) $f\colon\ \mathbb{N}_{\nu-2,\nu+b} \times \mathbb{R} \to [0,+\infty)$ is continuous.

(H_1) There exists a $\gamma_1 > 0$ such that $0 \leqslant x \leqslant \gamma_1$ and $t \in \mathbb{N}_{\nu-2,\nu+b}$ implies $f(t,x) \leqslant \gamma_1/\Lambda_1$, where $\Lambda_1 = \sum\limits_{s=0}^{b} G(\nu+s-1,s)$.

(H_2)

$$\lim_{x\to 0^+} \min_{t\in\mathbb{N}_{\nu-2,\nu+b}} \frac{f(t,x)}{x} = \infty, \quad \text{and} \quad \lim_{x\to +\infty} \min_{t\in\mathbb{N}_{\nu-2,\nu+b}} \frac{f(t,x)}{x} = \infty.$$

(H_3) There exists a $\gamma_2 > 0$ such that $\eta\gamma_2 \leqslant x \leqslant \gamma_2$ and $t \in \mathbb{N}_{\nu-2,\nu+b}$ implies $f(t,x) > \gamma_2\Lambda_2$, where $\Lambda_2^{-1} = \eta \sum\limits_{s=1}^{b} G(\nu+1,s)$.

(H_4)

$$\lim_{x\to 0^+} \max_{t\in\mathbb{N}_{\nu-2,\nu+b}} \frac{f(t,x)}{x} = 0 \quad \text{and} \quad \lim_{x\to +\infty} \max_{t\in\mathbb{N}_{\nu-2,\nu+b}} \frac{f(t,x)}{x} = 0.$$

Lemma 4.1. *T is completely continuous and $T(K) \subset K$.*

Proof. It is obvious that T is completely continuous under the condition (F). Now we prove $T(K) \subset K$.

From Lemma 3.3, for $x \in K$ we have

$$\begin{aligned}\min_{t\in\mathbb{N}_{\nu,\nu+b-1}}(Tx)(t) &= \min_{t\in\mathbb{N}_{\nu,\nu+b-1}}\sum_{s=0}^{b}G(t,s)f(s+\nu-1,x(s+\nu-1))\\ &\geqslant \eta\sum_{s=0}^{b}G(\nu+s-1,s)f(s+\nu-1,x(s+\nu-1))\\ &\geqslant \eta\max_{t\in\mathbb{N}_{\nu-2,\nu+b}}\sum_{s=0}^{b}G(t,s)f(s+\nu-1,x(s+\nu-1))\\ &= \eta\|Tx\|,\end{aligned}$$

i.e. $Tx \in K$. Thus we complete the proof. $\square$

Now, we present the main results of the section.

Theorem 4.1. *Assume that $f(t,x)$ satisfies* (F), (H_1), *and* (H_2). *Then problem* (1.1) *has at least two positive solutions y_1 and y_2 such that $0 < \|y_1\| < \gamma_1 < \|y_2\|$.*

Proof. Choose a positive number $\xi > 0$ such that

$$\xi\eta\sum_{s=1}^{b}G(\nu+1,s) > 1.$$

By (H_2) there exists $0 < r_1 < \gamma_1$ such that $f(t,x) \geqslant \xi x$ for all $0 \leqslant x \leqslant r_1$. Then, for $x \in \partial K_{r_1}$, we have that

$$\begin{aligned}(Tx)(\nu+1) &= \sum_{s=0}^{b}G(\nu+1,s)f(s+\nu-1,x(\nu+s-1))\\ &\geqslant \xi\sum_{s=1}^{b}G(\nu+1,s)|x(s+\nu-1)|\\ &\geqslant \xi\eta\sum_{s=1}^{b}G(\nu+1,s)\|x\| > \|x\|,\end{aligned}$$

which implies that $\|Tx\| > \|x\|$ for $x \in \partial K_{r_1}$. Hence, by Theorem 2.1, we obtain that

$$i(T, K_{r_1}, K) = 0. \tag{4.1}$$

(H_2) also implies that there is an $R_3 > 0$ such that $f(t,x) \geqslant \xi x$ for all $x \geqslant R_3$. Choose $R_1 = \max\{\gamma_1, R_3/\eta\}$. Then for $x \in \partial K_{R_1}$ we have

$$\min_{t\in\mathbb{N}_{\nu,\nu+b-1}} x(t) \geqslant \eta\|x\| > R_3,$$

and

$$\begin{aligned}(Tx)(\nu+1) &= \sum_{s=0}^{b} G(\nu+1,s) f(s+\nu-1, x(s+\nu-1)) \\ &\geqslant \xi \sum_{s=1}^{b} G(\nu+1,s)|x(s+\nu-1)| \\ &\geqslant \xi\eta \sum_{s=1}^{b} G(\nu+1,s)\|x\| > \|x\|,\end{aligned}$$

which implies that $\|Tx\| > \|x\|$ for $x \in \partial K_{R_1}$ and

$$i(T, K_{R_1}, K) = 0. \tag{4.2}$$

On the other hand, by (H_1) for $x \in \partial K_{\gamma_1}$ we obtain

$$\begin{aligned}\|Tx\| &= \max_{t\in\mathbb{N}_{\nu-2,\nu+b}} \sum_{s=0}^{b} G(t,s) f(s+\nu-1, x(s+\nu-1)) \\ &< \sum_{s=0}^{b} G(\nu+s-1,s) f(s+\nu-1, x(s+\nu-1)) \\ &\leqslant \frac{\gamma_1}{\Lambda_1} \sum_{s=0}^{b} G(\nu+s-1,s) \\ &\leqslant \gamma_1 = \|x\|.\end{aligned}$$

Hence, $\|Tx\| < \|x\|$ for $x \in \partial K_{\gamma_1}$. Obviously, $Tx \neq x$ for $x \in \partial K_{\gamma_1}$. Then

$$i(T, K_{R_1}, K) = 1. \tag{4.3}$$

Now, combining (4.1), (4.2), and (4.3), we get

$$i(T, K_{R_1} \setminus \mathring{K}_{\gamma_1}, K) = -1 \quad \text{and} \quad i(T, K_{\gamma_1} \setminus \mathring{K}_{r_1}, K) = 1.$$

Consequently, T has two fixed points y_1 and y_2 in $K_{R_1} \setminus \mathring{K}_{\gamma_1}$ and $K_{\gamma_1} \setminus \mathring{K}_{r_1}$, respectively. Both of them are positive solutions of problem (1.1). Thus we complete the proof. □

E x a m p l e 4.1. Assume that

$$f(t,x)=\frac{x^2+1}{t^2+c_1},\quad t\in\mathbb{N}_{\nu-2,\nu+b},\ x\in\mathbb{R},$$

where c_1 is a positive constant satisfying

$$\Big(\frac{c_1}{\Lambda_1}\Big)^2-4>0.$$

If we take $\gamma_1>0$ satisfying the inequality

$$\gamma_1^2-\frac{c_1}{\Lambda_1}\gamma_1+1\leqslant 0,$$

then the conditions (F), (H_1) and (H_2) hold.

Theorem 4.2. *Assume that $f(t,x)$ satisfies (F), (H_3), and (H_4). Then problem (1.1) has at least two positive solutions y_3 and y_4 such that $0<\|y_3\|<\gamma_2<\|y_4\|$.*

P r o o f. By (H_4), for all $\varepsilon>0$, there exists $\zeta>0$ such that

$$f(t,x)\leqslant\zeta+\varepsilon x\quad\text{for } x\geqslant 0,\ t\in\mathbb{N}_{\nu-2,\nu+b}.$$

Choose $0<\varepsilon<\Lambda_1^{-1}$ and $R_2>\max\{\zeta\Lambda_1/(1-\varepsilon\Lambda_1),\gamma_2\}$. Then we have that for $x\in\partial K_{R_2}$

$$\begin{aligned}(Tx)(t)&=\sum_{s=0}^{b}G(t,s)f(s+\nu-1,x(s+\nu-1))\\&\leqslant\sum_{s=0}^{b}G(s+\nu-1,s)[\zeta+\varepsilon x(s+\nu-1)]\\&\leqslant\Lambda_1[\zeta+\varepsilon\|x\|]<R_2=\|x\|.\end{aligned}$$

Therefore, $\|Tx\|<\|x\|$ for $x\in\partial K_{R_2}$ and

$$i(T,K_{R_2},K)=1. \tag{4.4}$$

Similarly, for some small $0<r_2<\gamma_2$,

$$i(T,K_{r_2},K)=1. \tag{4.5}$$

On the other hand, for $x\in\partial K_{\gamma_2}$ we have

$$\min_{t\in\mathbb{N}_{\nu,\nu+b-1}}x(t)\geqslant\eta\|x\|=\eta\gamma_2,$$

and

$$
\begin{aligned}
(Tx)(\nu+1) &= \sum_{s=0}^{b} G(\nu+1,s) f(s+\nu-1, x(s+\nu-1)) \\
&\geqslant \sum_{s=1}^{b} G(\nu+1,s) f(s+\nu-1, x(s+\nu-1)) \\
&> \gamma_2 \Lambda_2 \sum_{s=1}^{b} G(\nu+1,s) = \gamma_2 = \|x\|.
\end{aligned}
$$

Hence, $\|Tx\| > \|x\|$ for $x \in \partial K_{\gamma_2}$. It is clear that $Tx \neq x$ for $x \in \partial K_{\gamma_2}$. Then

$$
i(T, K_{\gamma_2}, K) = 0. \tag{4.6}
$$

Then proceeding as in the proof in Theorem 4.1, we prove the theorem. □

Example 4.2. Assume that

$$
f(t,x) = (t^2+1)\frac{x^2}{x^2+c_2}, \quad t \in \mathbb{N}_{\nu-2,\nu+b},\ x \in \mathbb{R},
$$

where c_2 is a positive constant satisfying

$$
\eta^2 - 4\Lambda_2^2 c_2 > 0.
$$

If we take $\gamma_2 > 0$ satisfying the inequality

$$
(\Lambda_2 \eta^2)\gamma_2^2 - \eta^2 \gamma_2 + c_2 \Lambda_2 \leqslant 0,
$$

then the conditions (F), (H_3), and (H_4) hold.

5. Uniqueness

In this section, we state and prove the uniqueness theorem of our paper.

Let U denote the set of functions $\omega\colon [0,+\infty) \to [0,+\infty)$ satisfying

(i) ω in nondecreasing;

(ii) there exists $\psi \in \Psi$ such that for all $\tau \in [0,+\infty)$ we have

$$
\omega(\tau) = \frac{1}{2\Lambda_1}\Big(\frac{\tau}{2} - \psi\Big(\frac{\tau}{2}\Big)\Big).
$$

Theorem 5.1. *Assume that* $f(t,x) = g(t,x,x)$, *where* $g\colon \mathbb{N}_{\nu-2,\nu+b} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a function. Under the assumptions*

(P1) *there exists* $\omega \in U$ *such that for all* $x, y, u, v \in \mathbb{R}$, *with* $x \geqslant u$ *and* $y \leqslant v$, *for* $t \in \mathbb{N}_{\nu-2,\nu+b}$,

$$0 \leqslant g(t,x,y) - g(t,u,v) \leqslant \omega(x-u) + \omega(v-y); \tag{5.1}$$

(P2) *there exist* α, $\beta \in X$ *satisfying*

$$\alpha(t) \leqslant \sum_{s=0}^{b} G(t,s) g(s+\nu-1, \alpha(s+\nu-1), \beta(s+\nu-1)), \quad t \in \mathbb{N}_{\nu-2,\nu+b}, \tag{5.2}$$

and

$$\beta(t) \geqslant \sum_{s=0}^{b} G(t,s) g(s+\nu-1, \beta(s+\nu-1), \alpha(s+\nu-1)), \quad t \in \mathbb{N}_{\nu-2,\nu+b}; \tag{5.3}$$

(P3) $\alpha(t) \leqslant \beta(t)$ *for all* $t \in \mathbb{N}_{\nu-2,\nu+b}$,

problem (1.1) *has a unique solution.*

Proof. It is clear that $(X, d, \preceq)$ is an ordered complete metric space with the partial order $\preceq$ and metric d:

$$u, v \in X, \quad u \preceq v \iff u(t) \leqslant v(t) \quad \text{for all } t \in \mathbb{N}_{\nu-2,\nu+b}$$

and

$$d(u,v) = \|u - v\|.$$

Also, it is easy to show that $(X, d, \preceq)$ is $\uparrow\downarrow$-regular and that every pair of elements in $X \times X$ has either a lower bound or an upper bound.

Introduce the mapping $L\colon X \times X \to X$ by

$$L(x,y)(t) = \sum_{s=0}^{b} G(t,s) g(s+\nu-1, x(s+\nu-1), y(s+\nu-1)), \quad t \in \mathbb{N}_{\nu-2,\nu+b}, \ x, y \in X.$$

(P1) implies that L is a mixed monotone mapping. For all $t \in \mathbb{N}_{\nu-2,\nu+b}$ and $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$, we have

$$
\begin{aligned}
|L(x,y)(t) - L(u,v)(t)| &\leqslant \sum_{s=0}^{b} G(t,s)\omega(x(s+\nu-1) - u(s+\nu-1)) \\
&\quad + \sum_{s=0}^{b} G(t,s)\omega(v(s+\nu-1) - y(s+\nu-1)) \\
&\leqslant \sum_{s=0}^{b} G(s+\nu-1,s)[\omega(\|x-u\|) + \omega(\|v-y\|)] \\
&\leqslant \Lambda_1[\omega(\|x-u\|) + \omega(\|v-y\|)],
\end{aligned}
$$

and

$$
\begin{aligned}
|L(y,x)(t) - L(v,u)(t)| &\leqslant \sum_{s=0}^{b} G(t,s)\omega(v(s+\nu-1) - y(s+\nu-1)) \\
&\quad + \sum_{s=0}^{b} G(t,s)\omega(x(s+\nu-1) - u(s+\nu-1)) \\
&\leqslant \sum_{s=0}^{b} G(s+\nu-1,s)[\omega(\|v-y\|) + \omega(\|x-u\|)] \\
&\leqslant \Lambda_1[\omega(\|x-u\|) + \omega(\|v-y\|)].
\end{aligned}
$$

Then we get that

$$
\frac{d(L(x,y), L(u,v)) + d(L(y,x), L(v,u))}{2} \leqslant \Lambda_1[\omega(\|x-u\|) + \omega(\|v-y\|)].
$$

Since ω is nondecreasing, we have

$$
\begin{aligned}
\Lambda_1[\omega(\|x-u\|) + \omega(\|v-y\|)] &\leqslant 2\Lambda_1\omega(\|x-u\| + \|v-y\|) \\
&= \frac{d(x,u) + d(y,v)}{2} - \psi\Big(\frac{d(x,u) + d(y,v)}{2}\Big).
\end{aligned}
$$

Thus,

$$
\frac{d(L(x,y), L(u,v)) + d(L(y,x), L(v,u))}{2} \leqslant \frac{d(x,u) + d(y,v)}{2} - \psi\Big(\frac{d(x,u) + d(y,v)}{2}\Big).
$$

If we take $\varphi(t) = t$, then we can see that (2.1) of Theorem 2.2 is satisfied. By the assumptions (P2) and (P3), we have that $\alpha \preceq L(\alpha, \beta)$, $\beta \succeq L(\beta, \alpha)$ and $\alpha \preceq \beta$. An application of Theorem 2.2 shows that there exists a unique solution y with $L(y, y) = y$, which implies that problem (1.1) has a unique solution. The proof is complete. □

R e m a r k 5.1. Assume that the function g in Theorem 5.1 has a positive lower and upper bound g_m and g_M, respectively. If we take

$$\alpha(t) = g_m \sum_{s=0}^{b} G(t,s), \quad \beta(t) = g_M \sum_{s=0}^{b} G(t,s),$$

then the conditions (P2) and (P3) hold.

E x a m p l e 5.1. Assume that

$$g(t,x,y) = \frac{1}{2\Lambda_1}\frac{1}{t^2+2}\Big(1 + \frac{1}{1+\mathrm{e}^{-x}} + \frac{1}{1+\mathrm{e}^{y}}\Big), \quad t \in \mathbb{N}_{\nu-2,\nu+b}, \ (x,y) \in \mathbb{R}^2,$$

and let

$$\psi(\tau) = \frac{\tau}{2}, \quad \omega(\tau) = \frac{1}{2\Lambda_1}\Big(\frac{\tau}{2} - \psi\Big(\frac{\tau}{2}\Big)\Big) = \frac{1}{2\Lambda_1}\frac{\tau}{4}.$$

Then (P1) is satisfied. Since g is bounded, we can take

$$\alpha(t) = \sum_{s=0}^{b} G(t,s), \quad \beta(t) = 3\sum_{s=0}^{b} G(t,s).$$

Obviously, conditions (P2) and (P3) hold.

E x a m p l e 5.2. Assume that

$$g(t,x,y) = g_1(t)(\arctan x - y), \quad t \in \mathbb{N}_{\nu-2,\nu+b}, \ (x,y) \in \mathbb{R}^2,$$

where

$$g_1(t) = \frac{1}{G_0}\Big(\frac{1}{8} + \frac{1}{t^2+8}\Big)$$

and

$$G_0 > \sqrt{\max_{s\in\mathbb{N}_{0,b}} \sum_{\tau_1=0}^{b}\sum_{\tau_2=0}^{b} G(s+\nu-1,\tau_1)G(\tau_1+\nu-1,\tau_2)}.$$

Let

$$\psi(\tau) = \frac{\tau}{2}, \quad \omega(\tau) = \frac{1}{G_0}\Big(\frac{\tau}{2} - \psi\Big(\frac{\tau}{2}\Big)\Big) = \frac{\tau}{4G_0}.$$

Then (P1) is satisfied. Define M^* as

$$M^* := \max_{s\in\mathbb{N}_{0,b}} \sum_{\tau_1=0}^{b} G(s+\nu-1,\tau_1) > 0.$$

Set

$$\alpha(t) = -C_1 \sum_{s=0}^{b} G(t,s), \quad 0 < C_1 \leqslant \frac{16}{15}\Big(\frac{\pi}{8G_0} + \frac{\pi}{32G_0^2}M^*\Big)$$

and

$$\beta(t) = \frac{1}{4G_0} \sum_{s=0}^{b} \bigg[G(t,s)\bigg(\frac{\pi}{2} + C_1 \sum_{\tau_2=0}^{b} G(s+\nu-1,\tau_2)\bigg)\bigg].$$

Then conditions (P2) and (P3) hold. Condition (P2) is obvious. For (P3) we have that

$$\begin{aligned}
\alpha(t) = & -C_1 \sum_{s=0}^{b} G(t,s) \leqslant -\Big(\frac{\pi}{8G_0} + \frac{\pi}{32G_0^2}M^* + \frac{C_1}{16}\Big) \sum_{s=0}^{b} G(t,s) \\
= & -\frac{1}{4G_0}\bigg\{\sum_{s=0}^{b} G(t,s)\Big(\frac{\pi}{2} + \frac{\pi}{8G_0}M^* + \frac{C_1}{4G_0}G_0^2\Big)\bigg\} \\
\leqslant & -\frac{1}{4G_0}\bigg\{\sum_{s=0}^{b} G(t,s)\bigg(\frac{\pi}{2} + \frac{\pi}{8G_0}\sum_{\tau_1=0}^{b} G(s+\nu-1,\tau_1) \\
& + \frac{C_1}{4G_0}\sum_{\tau_1=0}^{b}\sum_{\tau_2=0}^{b} G(s+\nu-1,\tau_1)G(\tau_1+\nu-1,\tau_2)\bigg)\bigg\} \\
\leqslant & \sum_{s=0}^{b} G(t,s)g_1(s+\nu-1) \\
& \times \bigg(-\frac{\pi}{2} - \frac{1}{4G_0}\sum_{\tau_1=0}^{b}\bigg[G(s+\nu-1,\tau_1)\bigg(\frac{\pi}{2} + C_1\sum_{\tau_2=0}^{b} G(\tau_1+\nu-1,\tau_2)\bigg)\bigg]\bigg) \\
= & \sum_{s=0}^{b} G(t,s)g_1(s+\nu-1)\Big(-\frac{\pi}{2} - \beta(s+\nu-1)\Big) \\
\leqslant & \sum_{s=0}^{b} G(t,s)g_1(s+\nu-1)(\arctan\alpha(s+\nu-1) - \beta(s+\nu-1)) \\
= & \sum_{s=0}^{b} G(t,s)g(s+\nu-1,\alpha(s+\nu-1),\beta(s+\nu-1)), \quad t \in \mathbb{N}_{\nu-2,\nu+b},
\end{aligned}$$

and

$$\begin{aligned}
\beta(t) = & \frac{1}{4G_0} \sum_{s=0}^{b} \bigg[G(t,s)\bigg(\frac{\pi}{2} + C_1 \sum_{\tau_2=0}^{b} G(s+\nu-1,\tau_2)\bigg)\bigg] \\
& \geqslant \sum_{s=0}^{b} G(t,s)g_1(s+\nu-1)\Big(\frac{\pi}{2} - \alpha(s+\nu-1)\Big)
\end{aligned}$$

$$\geqslant \sum_{s=0}^{b} G(t,s) g_1(s+\nu-1)(\arctan \beta(s+\nu-1) - \alpha(s+\nu-1))$$

$$= \sum_{s=0}^{b} G(t,s) g(s+\nu-1, \beta(s+\nu-1), \alpha(s+\nu-1)), \ t \in \mathbb{N}_{\nu-2,\nu+b}.$$

References

[1] *K. Alyousef*: Boundary Value Problems for Discrete Fractional Equations. Ph.D. dissertation, The University of Nebraska-Lincoln, 2012.

[2] *G. A. Anastassiou*: Nabla discrete fractional calculus and nabla inequalities. Math. Comput. Modelling *51* (2010), 562–571.

[3] *F. M. Atici, P. W. Eloe*: A transform method in discrete fractional calculus. Int. J. Difference Equ. *2* (2007), 165–176.

[4] *F. M. Atici, P. W. Eloe*: Initial value problems in discrete fractional calculus. Proc. Am. Math. Soc. *137* (2009), 981–989.

[5] *F. M. Atici, P. W. Eloe*: Linear systems of fractional nabla difference equations. Rocky Mt. J. Math. *41* (2011), 353–370.

[6] *F. M. Atıcı, P. W. Eloe*: Two-point boundary value problems for finite fractional difference equations. J. Difference Equ. Appl. *17* (2011), 445–456.

[7] *F. M. Atıcı, P. W. Eloe*: Gronwall's inequality on discrete fractional calculus. Comput. Math. Appl. *64* (2012), 3193–3200.

[8] *F. M. Atıcı, S. Şengül*: Modeling with fractional difference equations. J. Math. Anal. Appl. *369* (2010), 1–9.

[9] *N. R. O. Bastos, R. A. C. Ferreira, D. F. M. Torres*: Discrete-time fractional variational problems. Signal Process. *91* (2011), 513–524.

[10] *K. Deimling*: Nonlinear Functional Analysis. Springer, New York, 1985.

[11] *L. H. Erbe, S. C. Hu, H. Y. Wang*: Multiple positive solutions of some boundary value problems. J. Math. Anal. Appl. *184* (1994), 640–648.

[12] *R. A. C. Ferreira*: Positive solutions for a class of boundary value problems with fractional q-differences. Comput. Math. Appl. *61* (2011), 367–373.

[13] *R. A. C. Ferreira*: A discrete fractional Gronwall inequality. Proc. Am. Math. Soc. *140* (2012), 1605–1612.

[14] *R. A. C. Ferreira, C. S. Goodrich*: Positive solution for a discrete fractional periodic boundary value problem. Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal. *19* (2012), 545–557.

[15] *C. S. Goodrich*: Continuity of solutions to discrete fractional initial value problems. Comput. Math. Appl. *59* (2010), 3489–3499.

[16] *C. S. Goodrich*: Solutions to a discrete right-focal fractional boundary value problem. Int. J. Difference Equ. *5* (2010), 195–216.

[17] *C. S. Goodrich*: Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions. Comput. Math. Appl. *61* (2011), 191–202.

[18] *C. S. Goodrich*: Existence of a positive solution to a system of discrete fractional boundary value problems. Appl. Math. Comput. *217* (2011), 4740–4753.

[19] *C. S. Goodrich*: Existence of a positive solution to systems of differential equations of fractional order. Comput. Math. Appl. *62* (2011), 1251–1268.

[20] *C. S. Goodrich*: On positive solutions to nonlocal fractional and integer-order difference equations. Appl. Anal. Discrete Math. *5* (2011), 122–132.

[21] *C. S. Goodrich*: On a discrete fractional three-point boundary value problem. J. Difference Equ. Appl. *18* (2012), 397–415.
[22] *C. S. Goodrich*: On a first-order semipositone discrete fractional boundary value problem. Arch. Math. *99* (2012), 509–518.
[23] *C. S. Goodrich*: On a fractional boundary value problem with fractional boundary conditions. Appl. Math. Lett. *25* (2012), 1101–1105.
[24] *C. S. Goodrich*: On discrete sequential fractional boundary value problems. J. Math. Anal. Appl. *385* (2012), 111–124.
[25] *D. Guo, V. Lakshmikantham*: Nonlinear Problems in Abstract Cones. Notes and reports in mathematics in science and engineering 5, Academic Press, Boston, 1988.
[26] *M. Holm*: The Theory of Discrete Fractional Calculus: Development and Application. Ph.D. dissertation, The University of Nebraska-Lincoln, 2011.
[27] *F. Jarad, T. Abdeljawad, D. Baleanu, K. Biçen*: On the stability of some discrete fractional nonautonomous systems. Abstr. Appl. Anal. *2012* (2012), Article ID 476581, 9 pages, doi:10.1155/2012/476581.
[28] *M. Jleli, V. Č. Rajić, B. Samet, C. Vetro*: Fixed point theorems on ordered metric spaces and applications to nonlinear elastic beam equations. J. Fixed Point Theory Appl. *12* (2012), 175–192.
[29] *A. A. Kilbas, H. M. Srivastava, J. J. Trujillo*: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies 204, Elsevier, Amsterdam, 2006.
[30] *V. Lakshmikantham, S. Leela, J. Vasundhara Devi*: Theory of Fractional Dynamic Systems. Cambridge Academic Publishers, Cambridge, 2009.
[31] *W. D. Lv*: Existence of solutions for discrete fractional boundary value problems with a p-Laplacian operator. Adv. Difference Equ. *2012* (2012), 10 pages, doi:10.1186/1687-1847-2012-163.
[32] *K. S. Miller, B. Ross*: An Introduction to the Fractional Calculus and Fractional Differential Equations. John Wiley & Sons, New York, 1993.
[33] *Y. Y. Pan, Z. L. Han, S. Sun, Y. Zhao*: The existence of solutions to a system of discrete fractional boundary value problems. Abstr. Appl. Anal. *2012* (2012), Article ID 707631, 15 pages, doi:10.1155/2012/707631.

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