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ACCELERATION OF TWO-GRID STABILIZED MIXED FINITE ELEMENT METHOD FOR THE STOKES EIGENVALUE PROBLEM

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Abstract. This paper provides an accelerated two-grid stabilized mixed finite element scheme for the Stokes eigenvalue problem based on the pressure projection. With the scheme, the solution of the Stokes eigenvalue problem on a fine grid is reduced to the solution of the Stokes eigenvalue problem on a much coarser grid and the solution of a linear algebraic system on the fine grid. By solving a slightly different linear problem on the fine grid, the new algorithm significantly improves the theoretical error estimate which allows a much coarser mesh to achieve the same asymptotic convergence rate. Finally, numerical experiments are shown to verify the high efficiency and the theoretical results of the new method.

Keywords: accelerated two grid method; Stokes eigenvalue problem; stabilized method; equal-order pair; error estimate

MSC 2010: 65N25, 65N30, 65N12, 76D07

1. Introduction

Numerical methods of eigenvalue problems have received increasing attention in physical and mathematical fields (see [2]). Thus, in practical applications, it is a very important issue to adopt efficient methods to reduce the computational costs for investigating these problems. At the present time, numerous works are devoted to these problems (see [1], [6]–[8], and the references cited therein).

The two-grid discretization method is one of these efficient methods and has been well developed in recent years. It was first introduced by Xu [25], [26] for the nonsymmetric and nonlinear elliptic problems. To the best of our knowledge, the technique

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has been successfully applied and further investigated for Poisson eigenvalue equations and integral equations in [27], semilinear elliptic eigenvalue problems in [9] and nonselfadjoint elliptic problems in [17] and [29]. The applications of the two-grid method in Stokes eigenvalue problem can be found in [7], [16]. In particular, Hu and Cheng [14] proposed an accelerated two-grid discretization scheme for solving elliptic eigenvalue problems. Yang et al. [28] presented a two-grid discretization scheme based on shifted-inverse power method for elliptic eigenvalue problems and then discussed the adaptive finite element method based on multi-scale discretization for the eigenvalue problems in [19]. The two-grid method for the second order elliptic problems by mixed finite element method has been established in [8], [24]. Influenced by the work mentioned above, we establish a new stabilized finite element two-grid discretization scheme for the Stokes eigenvalue problem in this paper. Compared with the scheme in [16], our accelerated scheme is more efficient: the resulting solution obtained by our scheme can maintain an asymptotically optimal accuracy by taking $h = H^4$ when solving the Stokes eigenvalue problem.

The mixed finite element method is frequently used to obtain approximate solutions to more than one unknown. For example, the Stokes equations are often solved to obtain both pressure and velocity simultaneously. Accordingly, we need a finite element space for each unknown. These two spaces must be chosen carefully so that they satisfy an inf-sup stability condition for the mixed method to be stable. This condition does not allow the use of simple finite element pairs like equal-order ones, which offer some computational advances, as they are simple and have practical uniform data structure and adequate accuracy. Thus, much attention has been paid to the study of the stabilized methods for the Stokes problem.

Recent studies have focused on stabilization of the lower equal-order finite element pair using the projection of the pressure onto the piecewise constant space or the continuous space [4] and [23]. This stabilization technique does not require any calculation of high-order derivatives or edge-based data structures and is free of stabilization parameters and can be implemented at the element level. Therefore, this stabilized method is gaining more and more attention in computational fluid dynamics [18], [11], [15], [16], [3].

The paper focuses on the method which combines accelerated two-grid discretization scheme with a stabilized finite element method based on the pressure projection for the Stokes eigenvalue problem. The rest of this paper is organized as follows. In the next section, we introduce the studied problem, the notation and some well-known results used throughout this paper. We propose a stabilized finite element strategy for solving the Stokes eigenvalue problem in Section 3. Then, in Section 4, the accelerated two-grid algorithm and its error estimates are discussed. In Section 5, numerical experiments are given to illustrate the theoretical results and the high efficiency of the proposed method. Finally, we conclude our presentation in Section 6 with a few comments and also some possible future research topics.

2. Preliminaries

In this paper, we consider the following Stokes eigenvalue problem appearing in many engineering applications:

$$-\Delta \mathbf{u} + \nabla p = \lambda \mathbf{u} \quad \text{in } \Omega, \tag{2.1}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma, \tag{2.3}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded and convex domain with a Lipschitz-continuous boundary Γ, $p(\mathbf{x})$ represents the pressure, $\mathbf{u}(\mathbf{x})$ the velocity vector and $\lambda \in \mathbb{R}$ the eigenvalue.

We shall introduce the following Hilbert spaces

$$\mathbf{V} = [H_0^1(\Omega)]^2, \quad Y = [L^2(\Omega)]^2, \quad W = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \colon \int_\Omega q \, \mathrm{d}x = 0 \right\}.$$

The spaces $[L^2(\Omega)]^m$, $m = 1, 2$, are equipped with the L^2-scalar product $(\cdot,\cdot)$ and L^2-norm $\|\cdot\|_0$. The norm and seminorm in $[H^k(\Omega)]^2$ are denoted by $\|\cdot\|_k$ and $|\cdot|_k$, respectively. The space $\mathbf{V}$ is equipped with the norm $\|\nabla \cdot\|_0$ or its equivalent norm $\|\cdot\|_1$ due to the Poincaré inequality. Spaces consisting of vector-valued functions are denoted in boldface. Furthermore, the norm in the space dual to V is given by

$$\|\mathbf{u}\|_{-1} = \sup_{\mathbf{v}\in\mathbf{V}, \|\mathbf{v}\|_1 = 1} (\mathbf{u}, \mathbf{v}). \tag{2.4}$$

Therefore, we define the following bilinear forms $a(\cdot,\cdot)$, $d(\cdot,\cdot)$ and $b(\cdot,\cdot)$ on $\mathbf{V}\times\mathbf{V}$, $\mathbf{V}\times W$ and $\mathbf{V}\times\mathbf{V}$, respectively, by

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= (\nabla \mathbf{u}, \nabla \mathbf{v}) \quad \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{V}, \\ d(\mathbf{v}, q) &= (\operatorname{div} \mathbf{v}, q) \quad \forall\, \mathbf{v} \in \mathbf{V}, \quad \forall\, q \in W, \\ b(\mathbf{u}, \mathbf{v}) &= (\mathbf{u}, \mathbf{v}) \quad \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{V}, \end{aligned}$$

and a generalized bilinear form $B((\cdot,\cdot),(\cdot,\cdot))$ on $(\mathbf{V}\times W)\times(\mathbf{V}\times W)$ by

$$B((\mathbf{u}, p), (\mathbf{v}, q)) = a(\mathbf{u}, \mathbf{v}) - d(\mathbf{v}, p) + d(\mathbf{u}, q) \quad \forall (\mathbf{u}, p), (\mathbf{v}, q) \in \mathbf{V}\times W. \tag{2.5}$$

With the above notation, the variational formulation of problem (2.1)–(2.3) reads as follows: Find $(\mathbf{u}, p; \lambda) \in (\mathbf{V} \times W) \times \mathbb{R}$ with $\|\mathbf{u}\|_0 = 1$ such that

$$B((\mathbf{u}, p), (\mathbf{v}, q)) = \lambda b(\mathbf{u}, \mathbf{v}) \quad \forall (\mathbf{v}, q) \in \mathbf{V} \times W. \tag{2.6}$$

From [2] we know that the eigenvalue problem (2.5) has an eigenvalue sequence $\{\lambda_j\}$:

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots,$$

and corresponding eigenvectors

$$\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3, \ldots,$$

with the orthogonal property $b(\mathbf{u}_i, \mathbf{u}_j) = \delta_{ij}$.

Let

$$M(\lambda_i) = \{\mathbf{u} \in \mathbf{V} \colon\ \mathbf{u} \text{ is an eigenvector of (2.5) corresponding to } \lambda_i\}.$$

Moreover, the bilinear form $d(\cdot, \cdot)$ satisfies the inf-sup condition for all $q \in W$

$$\sup_{\mathbf{v} \in \mathbf{V}} \frac{|d(\mathbf{v}, q)|}{\|\mathbf{v}\|_1} \geqslant \beta \|q\|_0, \tag{2.7}$$

where $\beta > 0$ is a constant depending only on Ω. Therefore, the generalized bilinear form B satisfies the continuity property and coercive condition

$$|B((\mathbf{u}, p), (\mathbf{v}, q))| \leqslant C(\|\mathbf{u}\|_1 + \|p\|_0) \times (\|\mathbf{v}\|_1 + \|q\|_0), \tag{2.8}$$

$$\sup_{(\mathbf{v}, q) \in (\mathbf{V}, W)} \frac{|B((\mathbf{u}, p), (\mathbf{v}, q))|}{\|\mathbf{v}\|_1 + \|q\|_0} \geqslant \beta_1 (\|\mathbf{u}\|_1 + \|p\|_0), \tag{2.9}$$

where C and β_1 are positive constants depending only on Ω. Throughout the paper we use c or C to denote a generic positive constant whose value may change from place to place but remains independent of the mesh parameter.

3. A stabilized mixed finite element method

Let $\mathcal{F} = \{T_h\}$ be a regular family of partitions of Ω into triangles in the sense of Ciarlet [10]. For $h > 0$, we introduce finite-dimensional subspaces $(\mathbf{V}_h, W_h) \subset (\mathbf{V}, W)$, which are associated with $T_h \in \mathcal{F}$. Now we choose the unstable velocity-pressure pair of finite element spaces with the same order as follows:

$$\mathbf{V}_h = \{v_h = (v_1, v_2) \in [C^0(\Omega)]^2 \cap \mathbf{V}\colon\ v_i \in P_1(T)\ \forall\, T \in T_h,\ i = 1, 2\}, \tag{3.1}$$

$$W_h = \{w \in C^0 \cap W\colon\ w \in P_1(T)\ \forall\, T \in T_h\}, \tag{3.2}$$

where $P_1(T)$ represents the space of linear functions on the element T.

As noted earlier, this choice of the approximate spaces $\mathbf{V}_h$ and W_h does not satisfy the inf-sup condition:

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{|d(\mathbf{v}_h, w_h)|}{\|\mathbf{v}_h\|_1} \geqslant \beta_2 \|w_h\|_0 \quad \forall\, w_h \in W_h, \tag{3.3}$$

where the constant $\beta_2 > 0$ is independent of h.

Now, we give a stabilized finite-element approximation based on the pressure projection stabilization method which was based on the idea of [4] and used a similar technique as in [18], [11], [16], [15].

Let $\Pi\colon\ L^2(\Omega) \to R_0$ be the standard L^2-projection with the following properties:

$$(p, q) = (\Pi p, q) \quad \forall\, p \in W, q \in R_0, \tag{3.4}$$

$$\|\Pi p\|_0 \leqslant c\|p\|_0 \quad \forall\, p \in W, \tag{3.5}$$

$$\|p - \Pi p\|_0 \leqslant ch\|p\|_1 \quad \forall\, p \in H^1(\Omega), \tag{3.6}$$

where $R_0 = \{q \in W\colon\ q|_T \in P_0(T)\ \forall\, T \in T_h\}$. We introduce the pressure projection stabilization term

$$G(p, q) = \nu(p - \Pi p, q) = \nu(p - \Pi p, q - \Pi q) \quad \forall\, p, q \in W_h, \tag{3.7}$$

where $\nu > 0$ is a relaxation parameter independent of h and adjusts the stabilization term to relax the continuity equation so as to allow the application of inf-sup incompatible spaces. For more information on the particular choice of a relaxation parameter we refer to [11]. It is clear that the stabilized form $G(p, q)$ in (3.7) is symmetric and semi-definite. In numerical experiments, we will present the choice of the stabilized operator Π and the specific definition of $G(\cdot, \cdot)$.

The stabilized mixed finite element method is based on the following bilinear form:

$$B_h((\mathbf{u}_h, p_h), (\mathbf{v}, q)) = B((\mathbf{u}_h, p_h), (\mathbf{v}, q)) + G(p_h, q), \quad (\mathbf{v}, q) \in \mathbf{V}_h \times W_h. \tag{3.8}$$

Now, the corresponding discrete variational formulation of (2.6) for the discrete Stokes eigenvalue problem is recast: Find $(\bar{\mathbf{u}}_h, \bar{p}_h; \bar{\lambda}_h) \in (\mathbf{V}_h \setminus \{0\} \times W_h) \times \mathbb{R}$ with $\|\bar{\mathbf{u}}_h\|_0 = 1$, such that

$$B_h((\bar{\mathbf{u}}_h, \bar{p}_h), (\mathbf{v}, q)) = \bar{\lambda}_h b(\bar{\mathbf{u}}_h, \mathbf{v}) \quad \forall\, (\mathbf{v}, q) \in \mathbf{V}_h \times W_h. \tag{3.9}$$

We know from [2] that the discrete Stokes eigenvalue problem (3.9) has eigenvalues

$$0 < \bar{\lambda}_{1,h} \leqslant \bar{\lambda}_{2,h} \leqslant \bar{\lambda}_{3,h} \leqslant \ldots \leqslant \bar{\lambda}_{N_h,h}$$

and the corresponding eigenvectors

$$\bar{\mathbf{u}}_{1,h},\ \bar{\mathbf{u}}_{2,h},\ \bar{\mathbf{u}}_{3,h}, \ldots, \bar{\mathbf{u}}_{N_h,h},$$

with the property $b(\bar{\mathbf{u}}_{i,h}, \bar{\mathbf{u}}_{j,h}) = \delta_{ij}$, $1 \leqslant i,\ j \leqslant N_h$ (N_h is the dimension of $\mathbf{V}_h$).

Let

$$M_h(\lambda_i) = \{\mathbf{u}_h \in \mathbf{V}_h \colon\ \mathbf{u}_h \text{ is an eigenvector of (3.9) corresponding to } \lambda_{ih}\}.$$

The next theorem, which can be found in [4], [18], [11], shows the continuity property and the weak coercivity property of the bilinear form $B_h((\mathbf{u}_h, p_h), (\mathbf{v}, q))$ for the finite element pair $\mathbf{V}_h \times W_h$.

Theorem 3.1. *For all $(\mathbf{u}_h, p_h), (\mathbf{v}, q) \in \mathbf{V}_h \times W_h$ there exist positive constants C and β, independent of h, such that*

$$|B_h((\mathbf{u}_h, p_h), (\mathbf{v}, q))| \leqslant C(\|\mathbf{u}_h\|_1 + \|p_h\|_0) \times (\|\mathbf{v}\|_1 + \|q\|_0), \tag{3.10}$$

$$\sup_{(\mathbf{v},q)\in(\mathbf{V}_h,W_h)} \frac{|B_h((\mathbf{u}_h, p_h), (\mathbf{v}, q))|}{\|\mathbf{v}\|_1 + \|q\|_0} \geqslant \beta(\|\mathbf{u}_h\|_1 + \|p_h\|_0). \tag{3.11}$$

By well-established techniques for the eigenvalue approximation [2], [5], [21], [30], [20] and for the stabilized mixed finite element method [4], [18], one has the following results.

Theorem 3.2. *Let $(\bar{\mathbf{u}}_h, \bar{p}_h; \bar{\lambda}_h)$ be an eigenpair solution of (3.9). Then there exists an exact eigenpair $(\mathbf{u}, p; \lambda)$ of (2.6) satisfying the following error estimates:*

$$\|\mathbf{u} - \bar{\mathbf{u}}_h\|_0 + h(\|\mathbf{u} - \bar{\mathbf{u}}_h\|_1 + \|p - \bar{p}_h\|_0) \leqslant ch^2 \tag{3.12}$$

and

$$|\lambda - \bar{\lambda}_h| \leqslant ch^2. \tag{3.13}$$

4. An accelerated two-grid stabilized scheme and error estimates

In this section, we shall present the main algorithm of the paper and derive some optimal bounds of the errors.

First, we define a new bilinear form as follows: $G_\mu((\mathbf{u},p),(\mathbf{v},q))\colon (\mathbf{V}\times W)\times(\mathbf{V}\times W)\to\mathbb{R}$,

$$G_\mu((\mathbf{u},p),(\mathbf{v},q)) = B((\mathbf{u},p),(\mathbf{v},q)) - \mu b(\mathbf{u},\mathbf{v}) \tag{4.1}$$

and a discrete bilinear form as follows: $G_{\mu_h}((\mathbf{u}_h,p_h),(\mathbf{v},q))\colon (\mathbf{V}_h\times W_h)\times(\mathbf{V}_h\times W_h)\to\mathbb{R}$,

$$G_{\mu_h}((\mathbf{u}_h,p_h),(\mathbf{v},q)) = B_h((\mathbf{u}_h,p_h),(\mathbf{v},q)) - \mu_h b(\mathbf{u}_h,\mathbf{v}). \tag{4.2}$$

For simplicity, we only consider the first eigenvalue. Using a similar technique as in [26], [27], [13], [6], we have the following lemma for the newly introduced bilinear form:

Lemma 4.1. *For all $(\mathbf{u},p)\in(\mathbf{V}\cap M(\lambda)^\perp)\times W$ and $(\mathbf{u}_h,p_h)\in(\mathbf{V}_h\cap M_h(\lambda)^\perp)\times W_h$, if μ and μ_h are not eigenvalues of* (2.6) *and* (3.9)*, respectively, there exists two positive constants $C(\mu)$ and $C(\mu_h)$ independent of the mesh size h such that*

$$\sup_{(\mathbf{v},q)\in(\mathbf{V},W)}\frac{|G_\mu((\mathbf{u},p),(\mathbf{v},q))|}{\|\mathbf{v}\|_1+\|q\|_0}\geqslant C(\mu)(\|\mathbf{u}\|_1+\|p\|_0) \tag{4.3}$$

and

$$\sup_{(\mathbf{v},q)\in(\mathbf{V}_h,W_h)}\frac{|G_{\mu_h}((\mathbf{u}_h,p_h),(\mathbf{v},q))|}{\|\mathbf{v}\|_1+\|q\|_0}\geqslant C(\mu_h)(\|\mathbf{u}_h\|_1+\|p_h\|_0). \tag{4.4}$$

For simplicity, we omit the proof. According to (4.3) and (4.4), if μ is not an eigenvalue, then $G_\mu((\mathbf{u},p),(\mathbf{v},q))=(\mathbf{f},\mathbf{v})$ is uniquely solvable for all $(\mathbf{v},q)\in(\mathbf{V}\times W)$ or $(\mathbf{v},q)\in(\mathbf{V}_h\times W_h)$. If μ is an eigenvalue, then $G_\mu((\mathbf{u},p),(\mathbf{v},q))=(\mathbf{f},\mathbf{v})$ may have no solution (In fact, it has at least one solution if and only if $\mathbf{f}\in M(\mu)^\perp$, see [14]).

Now, let H and $h\ll H<1$ be two real positive parameters tending to zero. Also, a coarse triangulation of T_H of Ω is constructed as in Section 3. A fine triangulation T_h is generated by a mesh refinement process to T_H, such that T_h is nested to T_H. The conforming finite element space pairs $(\mathbf{V}_h,W_h)$ and $(\mathbf{V}_H,W_H)\subset(\mathbf{V}_h,W_h)$ based on the triangulations T_h and T_H, respectively, are constructed as in Section 3.

Accelerated two-grid stabilized finite element approximations are defined as follows. The algorithm has three steps:

Step 1. On the coarse grid T_H, solve the following Stokes eigenvalue problem for $(p_H, \mathbf{u}_H; \lambda_H) \in (W_H \times \mathbf{V}_H) \times R$ with $\|\mathbf{u}_H\|_0 = 1$:

$$B_H((\mathbf{u}_H, p_H), (\mathbf{v}, q)) = \lambda_H b(\mathbf{u}_H, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{V}_H,\ q \in W_H. \tag{4.5}$$

Step 2. On the fine grid T_h, compute $(p_h, \mathbf{u}_h) \in W_h \times \mathbf{V}_h$ to satisfy the following Stokes problem:

$$G_{\lambda_H}((\mathbf{u}_h, p_h), (\mathbf{v}, q)) = b(\mathbf{u}_H, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{V}_h,\ q \in W_h. \tag{4.6}$$

Step 3. Compute the Rayleigh quotient for $(\mathbf{u}_h, p_h)$,

$$\lambda_h = \frac{B_h((\mathbf{u}_h, p_h), (\mathbf{u}_h, p_h))}{b(\mathbf{u}_h, \mathbf{u}_h)}. \tag{4.7}$$

Remark 4.1. Our algorithm is different from [16] in Step 2. In [16], Step 2 reads as follows:

$$B_h((\mathbf{u}_h, p_h), (\mathbf{v}, q)) = \lambda_H b(\mathbf{u}_H, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{V}_h,\ q \in W_h. \tag{4.8}$$

It can be found that the linear system (4.6) is nearly singular, which has been much discussed in the literature [13], [12], [22]. The improved two-grid method is a technique of accelerating convergence based on shifted inverse power method [13]. Moreover, it implies that λ_H is already a good approximation of λ_h when this system actually becomes singular or very close to being singular.

As in [27], we give an important but straightforward identity that relates the errors in the eigenvalue and eigenvector approximation.

Lemma 4.2. *Let* $(\mathbf{u}, p; \lambda)$ *be an eigenvalue pair of* (2.6) *for any* $\mathbf{s} \in \mathbf{V} \setminus \{0\}$ *and* $w \in W$,

$$\frac{B((\mathbf{s}, w), (\mathbf{s}, w))}{b(\mathbf{s}, \mathbf{s})} - \lambda = \frac{B((\mathbf{s} - \mathbf{u}, w - p), (\mathbf{s} - \mathbf{u}, w - p))}{b(\mathbf{s}, \mathbf{s})} - \lambda \frac{b(\mathbf{s} - \mathbf{u}, \mathbf{s} - \mathbf{u})}{b(\mathbf{s}, \mathbf{s})}. \tag{4.9}$$

The following theorem gives the error estimates for our accelerated two-grid scheme.

Theorem 4.1. *Let* $(\mathbf{u}_h, p_h; \lambda_h)$ *be an eigenpair solution of* (4.5)–(4.7). *Then there exists an exact eigenpair* $(\mathbf{u}, p; \lambda)$ *of* (2.6) *satisfying the following error estimates:*

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \leqslant c(h + H^4) \tag{4.10}$$

and

$$|\lambda - \lambda_h| \leqslant c(h^2 + H^8). \tag{4.11}$$

P r o o f. The proof follows the ideas from [6]. Consider an equivalent linear system on the fine grid as follows:

$$G_{\lambda_H}((\tilde{\mathbf{u}}_h, \tilde{p}_h), (\mathbf{v}, q)) = (\bar{\lambda} - \lambda_H) b(\mathbf{u}_H, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{V}_h,\ q \in W_h. \tag{4.12}$$

Note that

$$\lambda_h = \frac{B_h((\mathbf{u}_h, p_h), (\mathbf{u}_h, p_h))}{b(\mathbf{u}_h, \mathbf{u}_h)} = \frac{B_h((\tilde{\mathbf{u}}_h, \tilde{p}_h), (\tilde{\mathbf{u}}_h, \tilde{p}_h))}{b(\tilde{\mathbf{u}}_h, \tilde{\mathbf{u}}_h)}. \tag{4.13}$$

Setting $(e, \eta) = (\mathbf{u} - \bar{\mathbf{u}}_h, p - \bar{p}_h)$ and $(e_h, \eta_h) = (\bar{\mathbf{u}} - \tilde{\mathbf{u}}_h, \bar{p}_h - \tilde{p}_h)$, from (2.6), (3.9) and (4.12) for any $\mathbf{v} \in \mathbf{V}_h$, $q \in W_h$ we have

$$G_{\lambda_H}((e_h, \eta_h), (\mathbf{v}, q)) = (\bar{\lambda}_h - \lambda_H) b(\bar{\mathbf{u}}_h - \mathbf{u}_H, \mathbf{v}). \tag{4.14}$$

By (2.4), (3.10), and (4.14), we can find

$$|G_{\lambda_H}((e_h, \eta_h), (\mathbf{v}, q))| \leqslant C(|\bar{\lambda}_h - \lambda_H|\, \|\bar{\mathbf{u}} - \mathbf{u}_H\|_{-1}). \tag{4.15}$$

It is reasonable to assume that $e_h \perp M_h(\lambda_1)$. Using Sobolev embedding theorem, Theorem 3.2, and (4.4), we obtain

$$\|e_h\|_1 + \|\eta_h\|_0 \leqslant CH^4. \tag{4.16}$$

From Theorem 3.2 and the triangle inequality, we get

$$\begin{aligned} \|\mathbf{u} - \tilde{\mathbf{u}}_h\|_1 + \|p - \tilde{p}_h\|_0 &\leqslant \|e_h\|_1 + \|\eta_h\|_0 + \|\mathbf{u} - \bar{\mathbf{u}}\|_1 + \|p - \bar{p}_h\|_0 \\ &\leqslant CH^4 + Ch. \end{aligned} \tag{4.17}$$

Note that $\min\limits_{\alpha \in \mathbb{R}}(\|\mathbf{u} - \alpha\mathbf{u}_h\|_1 + \|p - \alpha p_h\|_0) \leqslant \|\mathbf{u} - \tilde{\mathbf{u}}_h\|_1 + \|p - \tilde{p}_h\|_0$, we find the desired result (4.10).

Next, using (4.15) and Lemma 4.3, we have

$$\begin{aligned}(4.18)\quad &\frac{B_h((\mathbf{u}_h,p_h),(\mathbf{u}_h,p_h))}{b(\mathbf{u}_h,\mathbf{u}_h)}-\lambda\\ &=\frac{B((\mathbf{u}_h-\mathbf{u},p_h-p),(\mathbf{u}_h-\mathbf{u},p_h-p))+G(p_h,p_h)}{b(\mathbf{u}_h,\mathbf{u}_h)}-\lambda\frac{(\mathbf{u}_h-\mathbf{u},\mathbf{u}_h-\mathbf{u})}{b(\mathbf{u}_h,\mathbf{u}_h)}.\end{aligned}$$

Taking norm and applying (2.8) and (3.5), we come to

$$|\lambda-\lambda_h|\leqslant C\big(\|\mathbf{u}-\mathbf{u}_h\|_1^2+\|p_h-\Pi p_h\|_0^2+\|p-p_h\|_0^2\big). \tag{4.19}$$

Next, using (3.6) and (4.10) and the triangle inequality, we obtain (4.11). The proof is completed. □

R e m a r k 4.2. In [16], the error estimates of the eigenvector and eigenvalue are as follows:

$$\|\mathbf{u}-\mathbf{u}_h\|_1+\|p-p_h\|_0\leqslant C(h+H^2) \tag{4.20}$$

and

$$|\lambda-\lambda_h|\leqslant C(h^2+H^4). \tag{4.21}$$

This means that the asymptotically optimal accuracy is obtained by taking $h=H^2$, but with our accelerated two-grid scheme, the asymptotically optimal accuracy is obtained by taking $h=H^4$. Obviously, the scheme here accelerates the convergence.

5. Numerical experiments

In this section we present numerical experiments to check the numerical theory developed in the previous sections and illustrate the efficiency of the accelerated two-grid method based on local polynomial pressure projection. Our method is characterized by using linear polynomial functions for both the velocity and pressure fields. An attractive feature of the stabilization approach is the flexibility in the choice of the stabilized operator Π. Now, the stabilized term is defined by local Gauss integration. In detail, the stabilized term can be rewritten as

$$G(p,q)=\nu\sum_{T\in T_h}\Big(\int_{T,2}p\cdot q\,\mathrm{d}x\,\mathrm{d}y-\int_{T,1}p\cdot q\,\mathrm{d}x\,\mathrm{d}y\Big)\quad\forall\,p,q\in W_h,$$

where $\int_{T,i} g(x,y)\,\mathrm{d}x\,\mathrm{d}y$ denotes an appropriate Gauss integral over T which is exact for polynomials of degree $i = 1, 2$. In particular, the trial function $p \in W_h$ must be projected to the piecewise constant space R_0 defined below when $i = 1$ for any $q \in W_h$. Indeed, Becker et al. have found that the stabilized methods of [18] are identical from a numerical point of view for the low-order approximations in [4], [3].

In this section we report test problems for the Stokes eigenvalue problem with the stabilized mixed finite element method to demonstrate the efficiency of our algorithm. The finite element discretization uses the P_1-P_1 pair for the velocity and pressure based on the pressure projection stabilization. The accuracy and the numerical stability of our method is checked, then we compare the results obtained by our method with those obtained by the two-grid method of [16]. Our algorithms are implemented using the public domain finite element software FreeFem++: Version 2.19.1, `http://www.freefem.org/`.

In our numerical experiments, Ω is the unit square domain $[0,1] \times [0,1]$ in $\mathbb{R}^2$. The domain Ω is uniformly divided by the triangulations of mesh size H and h in Figure 1, respectively. We denote by $\mathbf{U}$ the array of the velocity and by P the array

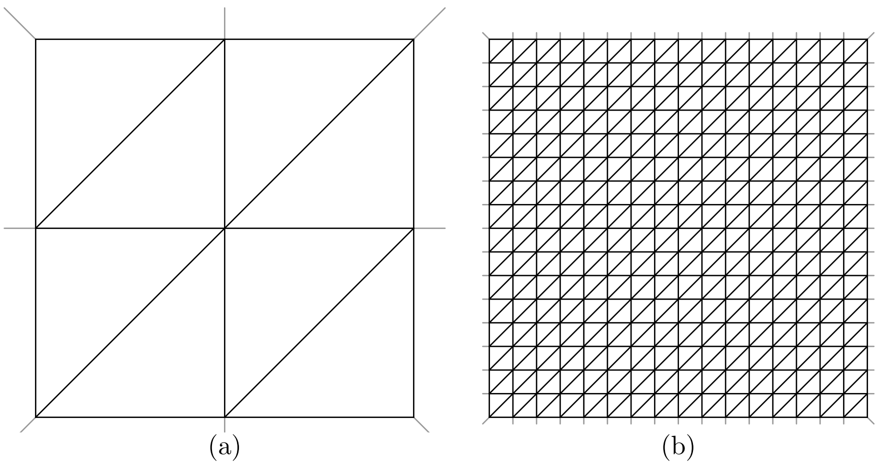

Figure 1. (a) Coarse grid division at $H = \frac{1}{2}$, (b) Fine grid division at $h = \frac{1}{16}$.

of the pressure. It is easy to see that (4.7) can be written in matrix form

$$\begin{bmatrix} A & -B \\ B^T & G \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ P \end{bmatrix} = \lambda_h \begin{bmatrix} E & O \\ O & O \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ P \end{bmatrix}, \tag{5.1}$$

where the matrices A, B, and E are deduced in the usual manner, using the bases for $\mathbf{V}_h$ and W_h, from the bilinear forms $a(\cdot,\cdot)$, $d(\cdot,\cdot)$ and $b(\cdot,\cdot)$, respectively, and B^T is the transpose of matrix B. Then the matrix G is deduced in the usual manner, using the bases for W_h, from the term $G(p_h, q)$. The coefficient matrix on the left-hand side of (5.1) is solved by LU decomposition, because all its leading principal minors are non-zero. The coefficient matrix on the right-hand side of (5.1) is solved by the conjugate gradient method with a fixed tolerance 10^{-6}, because its block matrix E is symmetric positive definite. The inverse power method is used for solving the generalized eigenvalue problem. This procedure is implemented on the coarse mesh for two-grid methods.

Here, we just consider the first eigenvalue of the Stokes eigenvalue problem for the sake of simplicity. The exact solution of this problem is unknown. Thus, we take the numerical solution by the standard Galerkin method (P_2-P_1 element) computed on a very fine mesh grid points ($h = 1/128$) as the exact solution for the purpose of comparison. This yields $\lambda = 52.3447$ as an accurate approximation of the first exact eigenvalue. Note that in these computations we set $\nu = 1$.

When solving the linear problem with a mesh size h, we need the solution λ_H and $\mathbf{u}_H$ generated on a coarse mesh. To do this we interpolate the solution λ_H and $\mathbf{u}_H$ onto the grid with mesh size h. Finally, the solution of accelerated two-grid method is obtained by one simple eigenvalue problem on the coarse mesh and one time interpolation on the fine one.

Our goal in this test is to validate the merit of the accelerated two-grid method as compared with the two-grid method [16]. We first show the convergence rate of our accelerated two-grid scheme. According to Theorem 4.1, the results shown in Table 1 consist of eigenvalue error estimates. Then, we apply both schemes on the same uniform coarse and fine grid satisfying $H^2 = h$. (By taking $H^2 = h$, the scheme from [16] can obtain asymptotically optimal accuracy.) Also, in order to show that our accelerated scheme can improve the results on a large class of coarse and fine grids, we choose mesh sizes satisfying $h = H/2$, a common occurrence in the mesh refinement process. Here, λ_h denotes the approximate eigenvalues obtained by the one grid scheme on the fine grid, λ_t and λ_{at} are the approximate eigenvalues obtained by the two-grid scheme from [16] and our accelerated scheme, respectively.

$1/H$	$1/h$	$\lvert\lambda_{at} - \lambda\rvert/\lambda$	Rate
2	16	8.489×10^{-1}	
3	81	4.081×10^{-2}	7.49
4	256	3.982×10^{-3}	8.09

Table 1. Convergence rate test on uniform grid for the P_1-P_1 pair at $h = H^4$.

From Table 2, we can see that the accelerated two-grid scheme outperforms in all cases. Although the accelerated scheme cannot obtain asymptotically optimal accuracy when $H = h^{1/2}$, we can still get a better approximate eigenvalue. For grids obtained by the mesh refinement procedure ($H = 2h$), our accelerated scheme still works better. Moreover, we give the plots of numerical solutions of the two schemes at the mesh $1/h = 64$ and $1/H = 8$ in Figure 2 in detail. Figure 2 shows the stability of the two schemes regardless of different isovalue of the graphics.

Coarse $1/H$	Fine $1/h$	λ_h	λ_t	λ_{at}	$\lvert\lambda_t - \lambda\rvert/\lambda$	$\lvert\lambda_{at} - \lambda\rvert/\lambda$
4	16	52.3055	53.9969	53.7477	3.156×10^{-2}	2.680×10^{-2}
8	64	52.4244	52.4574	52.4253	2.153×10^{-3}	1.540×10^{-3}
16	256	52.3497	52.3521	52.3497	1.411×10^{-4}	9.505×10^{-5}
$1/H$	$1/h$					
4	8	57.395	57.695	57.4303	1.022×10^{-1}	9.715×10^{-2}
8	16	53.6201	53.6393	53.6204	2.473×10^{-2}	2.437×10^{-2}
16	32	52.6638	52.6651	52.6638	6.121×10^{-3}	6.095×10^{-3}
32	64	52.4244	52.4245	52.4244	1.525×10^{-3}	1.523×10^{-3}
64	128	52.3646	52.3646	52.3646	3.806×10^{-4}	3.805×10^{-4}

Table 2. Results on Ω for the first eigenvalue $\lambda = 52.3447$ for the P_1-P_1 pair.

6. Conclusions

In this paper, we present an accelerated two-grid algorithm for the Stokes eigenvalue problem discretized by mixed finite element scheme based on the pressure projection stabilization. We show that when the coarse grid and the fine grid satisfy $H = O(h^{1/4})$, the accelerated two-grid algorithm can achieve the same accuracy of the mixed finite element solution. Finally, numerical tests show that the accelerated two-grid stabilized mixed finite element method is numerically efficient for solving the Stokes eigenvalue problem. Obviously, this method can be extended to the case of three dimensions easily. And there are some open questions including the possible extension of the method to other linear and nonlinear eigenvalue problems.

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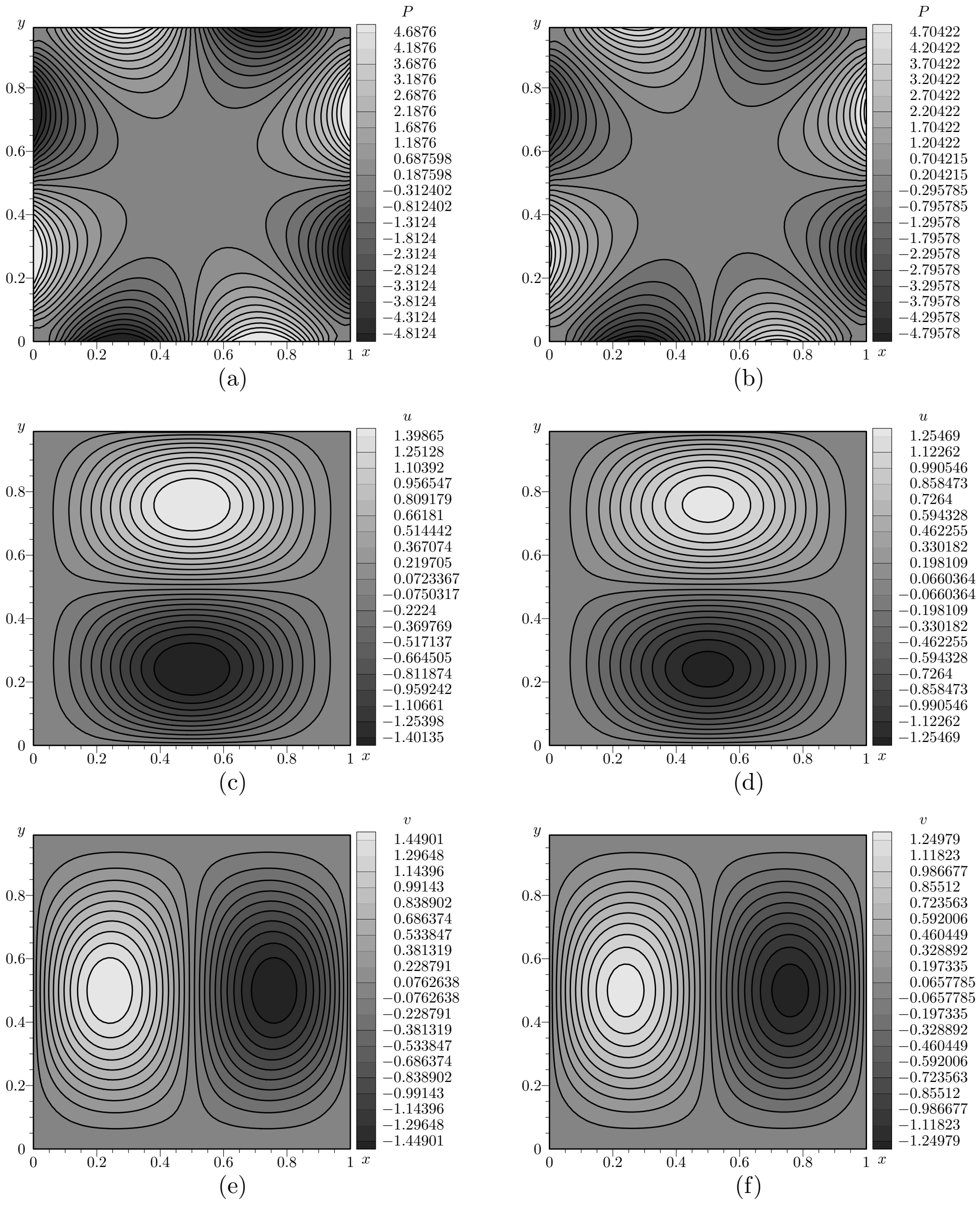

Figure 2. Plot of the pressure and velocity at $h = 1/64$: numerical solution of two-grid method (left) and numerical solution of accelerated two-grid method (right) with p_h, u_{1h}, u_{2h}

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