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SOBOLEV EMBEDDINGS FOR RIESZ POTENTIALS OF FUNCTIONS IN GRAND MORREY SPACES OF VARIABLE EXPONENTS OVER NON-DOUBLING MEASURE SPACES

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Abstract. Our aim in this paper is to deal with the boundedness of the Hardy-Littlewood maximal operator on grand Morrey spaces of variable exponents over non-doubling measure spaces. As an application of the boundedness of the maximal operator, we establish Sobolev's inequality for Riesz potentials of functions in grand Morrey spaces of variable exponents over non-doubling measure spaces. We are also concerned with Trudinger's inequality and the continuity for Riesz potentials.

Keywords: grand Morrey space; variable exponent; non-doubling measure; metric measure space; Riesz potential; maximal operator; Sobolev's inequality; Trudinger's exponential inequality; continuity

MSC 2010: 31B15, 46E35

1. Introduction

The space introduced by Morrey [37] in 1938 has become a useful tool in the study of the existence and regularity of partial differential equations (see also [39]). The maximal operator is a classical tool in harmonic analysis and studying Sobolev functions and partial differential equations and plays a central role in the study of differentiation, singular integrals, smoothness of functions and so on (see [4], [29], [44], etc.). Boundedness properties of the maximal operator and Riesz potentials of functions in Morrey spaces were investigated in [1], [5] and [38]. The same problem for the maximal operator and Riesz potentials of functions in Morrey spaces with non-doubling measure was studied in [41] (see also [23] and [40], etc.).

In the meantime, variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth condition. For a survey, see [9]. The boundedness of the maximal operator on variable exponent Lebesgue spaces $L^{p(\cdot)}$ was studied in [6], [7] and [25]. In [8], Sobolev's inequality for variable exponent Lebesgue spaces $L^{p(\cdot)}$ was studied. Then such properties were investigated on variable exponent Morrey spaces in [3], [22], [19], [21] and [36], and on variable exponent Morrey spaces with non-doubling measure in [30].

Grand Lebesgue spaces were introduced in [27] for the sake of studying the Jacobian. The grand Lebesgue spaces play an important role also in the theory of partial differential equations (see [16], [28] and [42], etc.). The generalized grand Lebesgue spaces appeared in [20], where the existence and uniqueness of the non-homogeneous N-harmonic equations $\operatorname{div}(|\nabla u|^{N-2}\nabla u) = \mu$ were studied. The boundedness of the maximal operator on the grand Lebesgue spaces was studied in [14]. The boundedness of the maximal operator and Sobolev's inequality for grand Morrey spaces with doubling measure were also studied in [32]. See also [15] and [31], etc.

Our first aim in this paper is to establish the boundedness of the maximal operator on grand Morrey spaces of variable exponents over non-doubling measure spaces. As an application of the boundedness of the maximal operator, making use of Hedberg's trick [26], we shall give Sobolev type inequalities for Riesz potentials of functions in these spaces.

The famous Trudinger inequality ([45]) insists that Sobolev functions in $W^{1,N}(G)$ satisfy finite exponential integrability, where G is an open bounded set in $\mathbb{R}^N$ (see also [2] and [46]). Great progress on Trudinger type inequalities has been made for Riesz potentials of order $\alpha (0 < \alpha < N)$ in the limiting case $\alpha p = N$ (see e.g. [10], [11], [12], [13], [43]). Trudinger type exponential integrability was investigated on variable exponent Lebesgue spaces $L^{p(\cdot)}$ in [17], [19] and [18] and on variable exponent Morrey spaces in [36]. For related results, see e.g. [33], [34] and [35].

Our second aim in this paper is to establish Trudinger's type exponential integrability for Riesz potentials of functions in grand Morrey spaces of variable exponents over non-doubling measure spaces. Further, in the final section, we are concerned with the continuity for Riesz potentials in our setting.

2. Preliminaries

By a quasi-metric measure space, we mean a triple (X, ϱ, μ), where X is a set, ϱ is a quasi-metric on X and μ is a complete measure on X. Here, we say that ϱ is a quasi-metric on X if ϱ satisfies the following conditions:

(ϱ1) $\varrho(x,y) \geqslant 0$ and $\varrho(x,y) = 0$ if and only if $x = y$;

(ϱ2) there exists a constant $a_0 \geqslant 1$ such that $\varrho(x,y) \leqslant a_0\varrho(y,x)$ for all $x, y \in X$;

(ϱ3) there exists a constant $a_1 > 0$ such that $\varrho(x,y) \leqslant a_1(\varrho(x,z) + \varrho(z,y))$ for all $x, y, z \in X$.

We denote $B(x,r)=\{y\in X\colon \varrho(x,y)<r\}$ and $d_X=\sup\{\varrho(x,y)\colon x,y\in X\}$. In this paper, we assume that $0<d_X<\infty$ and $0<\mu(B(x,r))<\infty$ for all $x\in X$ and $r>0$. This implies $\mu(X)<\infty$.

We say that a measure μ is lower Ahlfors q-regular if there exists a constant $c_0>0$ such that

$$\mu(B(x,r))\geqslant c_0 r^q \tag{2.1}$$

for all $x\in X$ and $0<r<d_X$. Further, μ is said to be a doubling measure if there exists a constant $c_1>0$ such that $\mu(B(x,2r))\leqslant c_1\mu(B(x,r))$ for every $x\in X$ and $0<r<d_X$. By the doubling property, if $0<r\leqslant R<d_X$, then there exist constants $C_Q>0$ and $Q\geqslant 0$ such that

$$\frac{\mu(B(x,r))}{\mu(B(x,R))}\geqslant C_Q\Big(\frac{r}{R}\Big)^Q \tag{2.2}$$

for all $x\in X$ (see e.g. [24]).

For $\alpha>0$, $k\geqslant 1$ and a locally integrable function f on X, we define the Riesz potential $U_{\alpha,k}f$ of order α by

$$U_{\alpha,k}f(x)=\int_X\frac{\varrho(x,y)^\alpha}{\mu(B(x,k\varrho(x,y)))}f(y)\,\mathrm{d}\mu(y).$$

Let $p(\cdot)$ be a measurable function on X such that

(P1) $1<p^-:=\inf_{x\in X}p(x)\leqslant\sup_{x\in X}p(x)=:p^+<\infty$

and

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x)-p(y)|\leqslant\frac{c_p}{\log(\mathrm{e}+1/\varrho(x,y))}\quad\text{for } x,y\in X$$

with a constant $c_p\geqslant 0$. Here note from (ϱ2) that

(P2′)

$$|p(x)-p(y)|\leqslant\frac{c_p'}{\log(\mathrm{e}+1/\varrho(y,x))}\quad\text{for } x,y\in X$$

with a constant $c_p'\geqslant 0$.

For a locally integrable function f on X, set

$$\|f\|_{L^{p(\cdot)}(X)}=\inf\Big\{\lambda>0\colon\int_X\Big(\frac{|f(y)|}{\lambda}\Big)^{p(y)}\,\mathrm{d}\mu(y)\leqslant 1\Big\}.$$

For $0<\varepsilon<p^--1$, set

$$p_\varepsilon(x)=p(x)-\varepsilon.$$

For $\nu > 0$, $\theta > 0$ and $k \geqslant 1$, we denote by $L^{p(\cdot)-0,\nu,\theta;k}(X)$ the class of locally integrable functions f on X satisfying

$$\|f\|_{L^{p(\cdot)-0,\nu,\theta;k}(X)} = \sup_{\substack{x\in X,\, 0<r<d_X \\ 0<\varepsilon<p^- -1}} \varepsilon^\theta \Bigl(\frac{r^\nu}{\mu(B(x,kr))}\Bigr)^{1/p_\varepsilon(x)} \|f\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} < \infty.$$

Throughout this paper, let C denote various constants independent of the variables in question. The proportion $g \sim h$ means that $C^{-1}h \leqslant g \leqslant Ch$ for some constant $C > 0$.

Lemma 2.1. *Let $k \geqslant 1$. If μ is lower Ahlfors q-reqular, then*

$$\mu(B(x,kr))^{p_\varepsilon(y)} \sim \mu(B(x,kr))^{p_\varepsilon(x)}$$

whenever $y \in B(x,r)$.

Proof. Since $p_\varepsilon(\cdot)$ satisfies the condition (P2), we see from (2.1) that

$$\begin{aligned}\Bigl(\frac{\mu(B(x,kr))}{\mu(X)}\Bigr)^{-|p_\varepsilon(x)-p_\varepsilon(y)|} &\leqslant \exp\Bigl(\frac{c_p}{\log(\mathrm{e}+1/\varrho(x,y))}\log\frac{\mu(X)}{\mu(B(x,kr))}\Bigr)\\ &\leqslant \exp\Bigl(\frac{c_p}{\log(\mathrm{e}+1/r)}\log\frac{\mu(X)}{c_0(kr)^q}\Bigr) \leqslant C\end{aligned}$$

whenever $y \in B(x,r)$. Hence, we obtain the required result. □

Lemma 2.2. *Let $k \geqslant 1$. If μ is lower Ahlfors q-regular and $0 < \varepsilon_0 < p^- - 1$, then*

$$\sup_{x\in X,\, 0<r<d_X,\, 0<\varepsilon<\varepsilon_0} \varepsilon^\theta \Bigl(\frac{r^\nu}{\mu(B(x,kr))}\Bigr)^{1/p_\varepsilon(x)} \|f\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} \sim \|f\|_{L^{p(\cdot)-0,\nu,\theta;k}(X)}$$

for all $f \in L^1_{\rm loc}(X)$.

Proof. We may assume that

$$\sup_{x\in X,\, 0<r<d_X,\, 0<\varepsilon<\varepsilon_0} \varepsilon^\theta \Bigl(\frac{r^\nu}{\mu(B(x,kr))}\Bigr)^{1/p_\varepsilon(x)} \|f\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} \leqslant 1.$$

Then it follows from Lemma 2.1 that

$$\frac{1}{\mu(B(x,kr))}\int_{B(x,r)} f(y)^{p_{\varepsilon_0/2}(y)}\,\mathrm{d}\mu(y) \leqslant Cr^{-\nu}$$

for all $x \in X$ and $0 < r < d_X$. To complete the proof, it is sufficient to show that there exists a constant $C > 0$ such that

$$\frac{1}{\mu(B(x,kr))} \int_{B(x,r)} f(y)^{p_{\varepsilon_1}(y)} \, \mathrm{d}\mu(y) \leqslant Cr^{-\nu}$$

for all $\varepsilon_0 \leqslant \varepsilon_1 < p^- - 1$. For this, see that

$$\begin{aligned} &\frac{1}{\mu(B(x,kr))} \int_{B(x,r)} f(y)^{p_{\varepsilon_1}(y)} \, \mathrm{d}\mu(y) \\ &\qquad \leqslant 1 + \frac{1}{\mu(B(x,kr))} \int_{B(x,r)} f(y)^{p_{\varepsilon_0/2}(y)} \, \mathrm{d}\mu(y) \leqslant Cr^{-\nu}. \end{aligned}$$

Thus the required result is proved. □

Lemma 2.3. *If μ is lower Ahlfors q-regular, then*

$$\|1\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} \sim \mu(B(x,r))^{1/p_\varepsilon(x)}$$

for all $x \in X$, $0 < r < d_X$ and $0 < \varepsilon < p^- - 1$.

P r o o f. By Lemma 2.1 we have

$$\int_{B(x,r)} \Big(\frac{1}{\mu(B(x,r))^{1/p_\varepsilon(x)}} \Big)^{p_\varepsilon(y)} \, \mathrm{d}\mu(y) \sim 1$$

for all $x \in X$, $0 < r < d_X$ and $0 < \varepsilon < p^- - 1$, as required. □

3. Boundedness of the maximal operator

From now on, we assume that μ is lower Ahlfors q-regular. For a locally integrable function f on X, we consider the maximal function $M_2 f$ defined by

$$M_2 f(x) = \sup_{r>0} \frac{1}{\mu(B(x,2r))} \int_{B(x,r)} |f(y)| \, \mathrm{d}\mu(y).$$

We first show the boundedness of the maximal operator on grand Morrey spaces of variable exponents over non-doubling measure spaces, as an extension of Meskhi [32, Theorem 3.1].

Let j_0 be the smallest integer satisfying $2^{j_0} > a_1$.

Theorem 3.1. *The maximal operator: $f \to M_2 f$ is bounded from $L^{p(\cdot)-0,\nu,\theta;2}(X)$ to $L^{p(\cdot)-0,\nu,\theta;2^{j_0+1}}(X)$, that is,*

$$\|M_2 f\|_{L^{p(\cdot)-0,\nu,\theta;2^{j_0+1}}(X)} \leqslant C\|f\|_{L^{p(\cdot)-0,\nu,\theta;2}(X)} \quad \text{for all } f \in L^{p(\cdot)-0,\nu,\theta;2}(X).$$

To show Theorem 3.1, we need the following results.

Lemma 3.2. *Let $k \geqslant 1$. Let f be a nonnegative function on X such that $\|f\|_{L^{p(\cdot)-0,\nu,\theta;k}(X)} \leqslant 1$. Then there exists a constant $C > 0$ such that*

$$\frac{1}{\mu(B(x,kr))}\int_{B(x,r)} g(y)\,\mathrm{d}\mu(y) \leqslant Cr^{-\nu/p_\varepsilon(x)}$$

for all $x \in X$, $0 < r < d_X$ and $0 < \varepsilon < p^- - 1$, where $g(y) = \varepsilon^\theta f(y)$.

Proof. Let f be a nonnegative function on X such that $\|f\|_{L^{p(\cdot)-0,\nu,\theta;k}(X)} \leqslant 1$. Then note that

$$\frac{1}{\mu(B(x,kr))}\int_{B(x,r)} g(y)^{p_\varepsilon(y)}\,\mathrm{d}\mu(y) \leqslant Cr^{-\nu}$$

for all $x \in X$, $0 < r < d_X$ and $0 < \varepsilon < p^- - 1$. Hence, we find

$$\begin{aligned}
&\frac{1}{\mu(B(x,kr))}\int_{B(x,r)} g(y)\,\mathrm{d}\mu(y)\\
&\quad\leqslant r^{-\nu/p_\varepsilon(x)} + \frac{1}{\mu(B(x,kr))}\int_{B(x,r)} g(y)\Big(\frac{g(y)}{r^{-\nu/p_\varepsilon(x)}}\Big)^{p_\varepsilon(y)-1}\,\mathrm{d}\mu(y)\\
&\quad\leqslant r^{-\nu/p_\varepsilon(x)} + Cr^{\nu(p_\varepsilon(x)-1)/p_\varepsilon(x)}\frac{1}{\mu(B(x,kr))}\int_{B(x,r)} g(y)^{p_\varepsilon(y)}\,\mathrm{d}\mu(y)\\
&\quad\leqslant Cr^{-\nu/p_\varepsilon(x)},
\end{aligned}$$

as required. □

We denote by χ_E the characteristic function of E.

Lemma 3.3. *Let $j \geqslant j_0$. Let f be a nonnegative function on X such that $\|f\|_{L^{p(\cdot)-0,\nu,\theta;2}(X)} \leqslant 1$. Set $g_j(y) = \varepsilon^\theta f(y)\chi_{B(x,2^{j+1}r)\setminus B(x,2^j r)}(y)$ for $0 < \varepsilon < p^- - 1$. Then there exists a constant $C > 0$ such that*

$$M_2 g_j(z) \leqslant C2^{-\nu j/p^+} r^{-\nu/p_\varepsilon(x)}$$

for all $z \in B(x,r)$ and $0 < \varepsilon < p^- - 1$.

Proof. Let $z \in B(x,r)$. Noting that $g_j(y)=0$ for $y \in B(z,(2^j/a_1-1)r)$, we have by Lemma 3.2 and (P2)

$$\begin{aligned} M_2 g_j(z) &= \sup_{t>(2^j/a_1-1)r} \frac{1}{\mu(B(z,2t))} \int_{B(z,t)} g_j(y)\,\mathrm{d}\mu(y) \\ &\leqslant C \sup_{t>(2^j/a_1-1)r} t^{-\nu/p_\varepsilon(z)} \\ &\leqslant C 2^{-\nu j/p^+} r^{-\nu/p_\varepsilon(x)}, \end{aligned}$$

as required. □

Lemma 3.4 (cf. [30, Theorem 3.1]). *Suppose that $p_0(\cdot)$ is a function on X such that*

$$1 < p_0^- := \inf_{x\in X} p_0(x) \leqslant \sup_{x\in X} p_0(x) =: p_0^+ < \infty$$

and

$$|p_0(x)-p_0(y)| \leqslant \frac{c_{p_0}}{\log(\mathrm{e}+1/\varrho(x,y))}$$

for all $x,y\in X$ and some constant $c_{p_0}\geqslant 0$. Then there exists a constant $c_0>0$ depending only on p_0^-, p_0^+, c_{p_0} and $\mu(X)$ such that

$$\|M_2 f\|_{L^{p_0(\cdot)}(X)} \leqslant c_0 \|f\|_{L^{p_0(\cdot)}(X)}$$

for all $f \in L^{p_0(\cdot)}(X)$.

Proof of Theorem 3.1. Let f be a nonnegative function on X such that $\|f\|_{L^{p(\cdot)-0,\nu,\theta;2}(X)} \leqslant 1$. Let $x\in X$, $0<r<d_X$ and $0<\varepsilon<(p^- -1)/2$ be fixed. Set $g(y)=\varepsilon^\theta f(y)$.

For positive integers $j \geqslant j_0$, set

$$g_j = g\chi_{B(x,2^{j+1}r)\setminus B(x,2^j r)}(y)$$

and $g_0 = g\chi_{B(x,2^{j_0}r)}(y)$.

Here we find by Lemmas 3.3 and 2.3 that

$$\begin{aligned} \|M_2 g_j\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} &\leqslant C 2^{-\nu j/p^+} r^{-\nu/p_\varepsilon(x)} \|1\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} \\ &\leqslant C 2^{-\nu j/p^+} r^{-\nu/p_\varepsilon(x)} \mu(B(x,r))^{1/p_\varepsilon(x)} \end{aligned}$$

for $j \geqslant j_0$. Since $p_\varepsilon^- > (p^- + 1)/2 > 1$, we see from Lemma 3.4 that

$$\begin{aligned}
&\|M_2 g\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} \\
&\quad \leqslant \|M_2 g_0\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} + \sum_{j=j_0}^{\infty} \|M_2 g_j\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} \\
&\quad \leqslant C\Big\{ \|g_0\|_{L^{p_\varepsilon(\cdot)}(B(x,2^{j_0}r))} + \mu(B(x,r))^{1/p_\varepsilon(x)} r^{-\nu/p_\varepsilon(x)} \sum_{j=j_0}^{\infty} 2^{-\nu j/p^+} \Big\} \\
&\quad \leqslant C\{\mu(B(x,2^{j_0+1}r))^{1/p_\varepsilon(x)} (2^{j_0} r)^{-\nu/p_\varepsilon(x)} + \mu(B(x,r))^{1/p_\varepsilon(x)} r^{-\nu/p_\varepsilon(x)}\} \\
&\quad \leqslant C\mu(B(x,2^{j_0+1}r))^{1/p_\varepsilon(x)} r^{-\nu/p_\varepsilon(x)},
\end{aligned}$$

so that

$$\sup_{x\in X,\, 0<r<d_X,\, 0<\varepsilon<(p^- -1)/2} \varepsilon^\theta \Big(\frac{r^\nu}{\mu(B(x,2^{j_0+1}r))}\Big)^{1/p_\varepsilon(x)} \|M_2 f\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} \leqslant C.$$

Hence, we obtain the required result by Lemma 2.2. □

4. Sobolev's inequality

Now we show the Sobolev type inequality for Riesz potentials in grand Morrey spaces of variable exponents over non-doubling measure spaces, as an extension of Meskhi [32, Theorems 5.3 and 5.4].

Theorem 4.1. *Suppose*

$$\frac{1}{p^*(x)} = \frac{1}{p(x)} - \frac{\alpha}{\nu} \geqslant \frac{1}{p^+} - \frac{\alpha}{\nu} > 0.$$

Then there exists a constant $C > 0$ such that

$$\|U_{\alpha,4} f\|_{L^{p^*(\cdot)-0,\nu,\theta;2^{j_0+1}}(X)} \leqslant C\|f\|_{L^{p(\cdot)-0,\nu,\theta;2}(X)}.$$

Proof. Let f be a nonnegative function on X such that $\|f\|_{L^{p(\cdot)-0,\nu,\theta;2}(X)} \leqslant 1$. Let $x \in X, 0 < r < d_X$ and $0 < \varepsilon < \min\{p^- - 1, ((p^*)^- - 1)/\gamma\}$ be fixed, where

$$\gamma = \sup_{z\in X, 0<\varepsilon<p^- -1} \frac{(p_\varepsilon)^*(z) p^*(z)}{p_\varepsilon(z) p(z)}.$$

For $z \in B(x,r)$ and $\delta > 0$, we write

$$\begin{aligned}
U_{\alpha,4} f(z) &= \int_{B(z,\delta)} \frac{\varrho(z,y)^\alpha}{\mu(B(z,4\varrho(z,y)))} f(y)\,\mathrm{d}\mu(y) \\
&\quad + \int_{X\setminus B(z,\delta)} \frac{\varrho(z,y)^\alpha}{\mu(B(z,4\varrho(z,y)))} f(y)\,\mathrm{d}\mu(y) \\
&= U_1(z) + U_2(z).
\end{aligned}$$

First we have

$$\begin{aligned}
U_1(z) &= \sum_{j=1}^{\infty} \int_{B(z,2^{-j+1}\delta)\setminus B(z,2^{-j}\delta)} \frac{\varrho(z,y)^\alpha}{\mu(B(z,4\varrho(z,y)))} f(y)\,\mathrm{d}\mu(y) \\
&\leqslant \sum_{j=1}^{\infty} \int_{B(z,2^{-j+1}\delta)} \frac{(2^{-j+1}\delta)^\alpha}{\mu(B(z,2^{-j+2}\delta))} f(y)\,\mathrm{d}\mu(y) \\
&\leqslant \sum_{j=1}^{\infty} (2^{-j+1}\delta)^\alpha M_2 f(z) \\
&\leqslant C\delta^\alpha M_2 f(z).
\end{aligned}$$

To estimate U_2, set $g(y) = \varepsilon^\theta f(y)$. Then we have by Lemma 3.2

$$\begin{aligned}
\varepsilon^\theta U_2(z) &= \sum_{j=1}^{\infty} \int_{X\cap(B(z,2^j\delta)\setminus B(z,2^{j-1}\delta))} \frac{\varrho(z,y)^\alpha}{\mu(B(z,4\varrho(z,y)))} g(y)\,\mathrm{d}\mu(y) \\
&\leqslant C\sum_{j=1}^{\infty} (2^j\delta)^\alpha \frac{1}{\mu(B(z,2^{j+1}\delta))} \int_{B(z,2^j\delta)} g(y)\,\mathrm{d}\mu(y) \\
&\leqslant C\sum_{j=1}^{\infty} (2^j\delta)^{\alpha-\nu/p_\varepsilon(z)} \\
&\leqslant C\delta^{\alpha-\nu/p_\varepsilon(z)}.
\end{aligned}$$

Hence

$$U_{\alpha,4} g(z) \leqslant C\{\delta^\alpha M_2 g(z) + \delta^{\alpha-\nu/p_\varepsilon(z)}\}.$$

Letting $\delta = M_2 g(z)^{-p_\varepsilon(z)/\nu}$, we establish

$$U_{\alpha,4} g(z) \leqslant C M_2 g(z)^{1-\alpha p_\varepsilon(z)/\nu}.$$

Now Theorem 3.1 gives

$$\begin{aligned}
&\frac{1}{\mu(B(x,2^{j_0+1}r))} \int_{B(x,r)} \{\varepsilon^\theta U_{\alpha,4} f(z)\}^{(p_\varepsilon)^*(z)}\,\mathrm{d}\mu(z) \\
&\qquad \leqslant \frac{C}{\mu(B(x,2^{j_0+1}r))} \int_{B(x,r)} \{M_2 g(z)\}^{p_\varepsilon(z)}\,\mathrm{d}\mu(z) \leqslant Cr^{-\nu}.
\end{aligned}$$

Here one sees that

$$(p_\varepsilon)^*(z) = p^*(z) - \frac{(p_\varepsilon)^*(z)p^*(z)}{p_\varepsilon(z)p(z)}\varepsilon.$$

Setting $\tilde{\varepsilon} = \gamma\varepsilon$, we have

$$\begin{aligned}\frac{1}{\mu(B(x,2^{j_0+1}r))} &\int_{B(x,r)} \{\tilde{\varepsilon}^\theta U_{\alpha,4}f(z)\}^{(p^*)_{\tilde{\varepsilon}}(z)}\,\mathrm{d}\mu(z) \\ &\leqslant C\left[\frac{1}{\mu(B(x,2^{j_0+1}r))}\int_{B(x,r)}\{\varepsilon^\theta U_{\alpha,4}f(z)\}^{(p_\varepsilon)^*(z)}\,\mathrm{d}\mu(z) + 1\right] \leqslant Cr^{-\nu}\end{aligned}$$

for all $x \in X$, $0 < r < d_X$ and $0 < \varepsilon < \min\{p^- - 1, ((p^*)^- - 1)/\gamma\}$, so that we obtain the required result by Lemma 2.2. □

5. Exponential integrability

In this section, we assume that

$$\operatorname*{ess\,sup}_{x\in X}\,(1/p(x) - \alpha/\nu) \leqslant 0. \tag{5.1}$$

Our aim in this section is to give an exponential integrability of Trudinger type. Recall that j_0 is the smallest integer satisfying $2^{j_0} > a_1$, where $a_1 > 0$ is the constant in (ϱ3). Set

$$k_0 = \max\{2a_0a_1(a_0+1), a_1^2(a_0 + 2^{j_0+1})/(2^{j_0} - a_1), 2\},$$

where $a_0 \geqslant 1$ is the constant in (ϱ2).

Theorem 5.1. *Let* $0 < \eta < \alpha$. *Suppose that* (5.1) *holds. Then there exist constants* $c_1, c_2 > 0$ *such that*

$$\frac{1}{\mu(B(z,2^{j_0}r))}\int_{B(z,r)} \exp(c_1 U_{\alpha,k_0}f(x)^{1/(\theta+1)})\,\mathrm{d}\mu(x) \leqslant c_2 r^{\eta-\alpha}$$

for all $z \in X$ *and* $0 < r < d_X$, *whenever* f *is a nonnegative measurable function on* X *satisfying* $\|f\|_{L^{p(\cdot)-0,\nu,\theta;1}(X)} \leqslant 1$.

To prove the theorem, we prepare some lemmas.

Lemma 5.2. *Let $k \geqslant 2$, $\theta > 0$ and $0 < \eta < \alpha$. Let f be a nonnegative function on X such that there exists a constant $C > 0$ such that*

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)} f(y)\,\mathrm{d}\mu(y) \leqslant Cr^{-\alpha}(\log(\mathrm{e}+1/r))^{\theta}. \tag{5.2}$$

Then there exists a constant $C > 0$ such that

$$\int_{X\setminus B(x,\delta)} \frac{\varrho(x,y)^{\eta}}{\mu(B(x,k\varrho(x,y)))} f(y)\,\mathrm{d}\mu(y) \leqslant C\delta^{\eta-\alpha}(\log(\mathrm{e}+1/\delta))^{\theta}$$

for $x \in X$ and $\delta > 0$.

Proof. Let f be a nonnegative function on X satisfying (5.2). We choose the smallest integer j_1 such that $2^{j_1}\delta \geqslant d_X$. We have by (5.2)

$$\begin{aligned}\int_{X\setminus B(x,\delta)} &\frac{\varrho(x,y)^{\eta}}{\mu(B(x,k\varrho(x,y)))} f(y)\,\mathrm{d}\mu(y) \\ &= \sum_{j=1}^{j_1}\int_{B(x,2^j\delta)\setminus B(x,2^{j-1}\delta)} \frac{\varrho(x,y)^{\eta}}{\mu(B(x,k\varrho(x,y)))} f(y)\,\mathrm{d}\mu(y) \\ &\leqslant \sum_{j=1}^{j_1}(2^j\delta)^{\eta}\frac{1}{\mu(B(x,2^{j-1}k\delta))}\int_{B(x,2^j\delta)} f(y)\,\mathrm{d}\mu(y) \\ &\leqslant C\sum_{j=1}^{j_1}(2^j\delta)^{\eta-\alpha}(\log(\mathrm{e}+1/(2^j\delta)))^{\theta} \\ &\leqslant C\sum_{j=1}^{j_1}\int_{2^{j-1}\delta}^{2^j\delta} t^{\eta-\alpha}(\log(\mathrm{e}+1/t))^{\theta}\frac{\mathrm{d}t}{t} \\ &\leqslant C\int_{\delta}^{2d_X} t^{\eta-\alpha}(\log(\mathrm{e}+1/t))^{\theta}\frac{\mathrm{d}t}{t}.\end{aligned} \tag{5.3}$$

Hence we find for $\eta < \alpha$

$$\int_{X\setminus B(x,\delta)} \frac{\varrho(x,y)^{\eta}}{\mu(B(x,k\varrho(x,y)))} f(y)\,\mathrm{d}\mu(y) \leqslant C\delta^{\eta-\alpha}(\log(\mathrm{e}+1/\delta))^{\theta},$$

as required. $\square$

Lemma 5.3. *Let* $0<\eta<\alpha$. *Let* f *be a nonnegative function on* X *satisfying* (5.2). *Define*

$$I_\eta f(x)=\int_X \frac{\varrho(x,y)^\eta}{\mu(B(x,k_0\varrho(x,y)))} f(y)\,\mathrm{d}\mu(y).$$

Then there exists a constant $C>0$ *such that*

$$\frac{1}{\mu(B(z,2^{j_0}r))}\int_{B(z,r)} I_\eta f(x)\,\mathrm{d}\mu(x)\leqslant Cr^{\eta-\alpha}(\log(\mathrm{e}+1/r))^\theta$$

for all $z\in X$ *and* $0<r<d_X$.

P r o o f. Write

$$\begin{aligned}I_\eta f(x)&=\int_{B(z,2^{j_0}r)}\frac{\varrho(x,y)^\eta}{\mu(B(x,k_0\varrho(x,y)))}f(y)\,\mathrm{d}\mu(y)\\&\quad+\int_{X\setminus B(z,2^{j_0}r)}\frac{\varrho(x,y)^\eta}{\mu(B(x,k_0\varrho(x,y)))}f(y)\,\mathrm{d}\mu(y)\\&=I_1(x)+I_2(x).\end{aligned}$$

Let $a=a_1(2^{j_0}a_0+1)$. By Fubini's theorem, we have

$$\begin{aligned}&\int_{B(z,r)} I_1(x)\,\mathrm{d}\mu(x)=\int_{B(z,2^{j_0}r)}\biggl(\int_{B(z,r)}\frac{\varrho(x,y)^\eta}{\mu(B(x,k_0\varrho(x,y)))}\,\mathrm{d}\mu(x)\biggr)f(y)\,\mathrm{d}\mu(y)\\&\leqslant\int_{B(z,2^{j_0}r)}\biggl(\int_{B(y,ar)}\frac{\varrho(x,y)^\eta}{\mu(B(x,k_0\varrho(x,y)))}\,\mathrm{d}\mu(x)\biggr)f(y)\,\mathrm{d}\mu(y)\\&=\int_{B(z,2^{j_0}r)}\biggl(\sum_{j=0}^\infty\int_{B(y,2^{-j}ar)\setminus B(y,2^{-j-1}ar)}\frac{\varrho(x,y)^\eta}{\mu(B(x,k_0\varrho(x,y)))}\,\mathrm{d}\mu(x)\biggr)f(y)\,\mathrm{d}\mu(y)\\&\leqslant\int_{B(z,2^{j_0}r)}\biggl(\sum_{j=0}^\infty\int_{B(y,2^{-j}ar)\setminus B(y,2^{-j-1}ar)}\frac{(2^{-j}a_0ar)^\eta}{\mu(B(x,2^{-j-1}a_0^{-1}k_0ar))}\,\mathrm{d}\mu(x)\biggr)f(y)\,\mathrm{d}\mu(y)\\&\leqslant\int_{B(z,2^{j_0}r)}\biggl(\sum_{j=0}^\infty\int_{B(y,2^{-j}ar)\setminus B(y,2^{-j-1}ar)}\frac{(2^{-j}a_0ar)^\eta}{\mu(B(y,2^{-j}ar))}\,\mathrm{d}\mu(x)\biggr)f(y)\,\mathrm{d}\mu(y)\\&\leqslant\int_{B(z,2^{j_0}r)}\biggl(\sum_{j=0}^\infty(2^{-j}a_0ar)^\eta\biggr)f(y)\,\mathrm{d}\mu(y),\end{aligned}$$

since $B(y, 2^{-j}ar) \subset B(x, 2^{-j-1}a_0^{-1}k_0ar)$ due to the fact that $k_0 \geqslant 2a_0a_1(a_0+1)$. Using $\eta > 0$ and (5.2), we have

$$\begin{aligned}
\int_{B(z,r)} I_1(x)\,\mathrm{d}\mu(x) &\leqslant C\int_{B(z,2^{j_0}r)}\left(\sum_{j=1}^{\infty}(2^{-j}r)^{\eta}\right)f(y)\,\mathrm{d}\mu(y)\\
&\leqslant Cr^{\eta}\int_{B(z,2^{j_0}r)} f(y)\,\mathrm{d}\mu(y)\\
&\leqslant Cr^{\eta}\mu(B(z,2^{j_0}r))(2^{j_0}r)^{-\alpha}(\log(\mathrm{e}+1/(2^{j_0}r)))^{\theta}\\
&\leqslant Cr^{\eta-\alpha}(\log(\mathrm{e}+1/r))^{\theta}\mu(B(z,2^{j_0}r)).
\end{aligned}$$

For $x \in B(z,r)$ and $y \in X \setminus B(z, 2^{j_0}r)$, we obtain

$$\varrho(z,y) \leqslant \frac{a_1 2^{j_0}}{2^{j_0}-a_1}\varrho(x,y) \tag{5.4}$$

and

$$\varrho(x,z) \leqslant \frac{a_0a_1}{2^{j_0}-a_1}\varrho(x,y) \tag{5.5}$$

Indeed, we have

$$\varrho(z,y) \leqslant a_1(\varrho(z,x)+\varrho(x,y)) \leqslant a_1(r+\varrho(x,y)) \leqslant a_1(2^{-j_0}\varrho(z,y)+\varrho(x,y)),$$

which yields (5.4). Also we have

$$\varrho(x,z) \leqslant a_0\varrho(z,x) \leqslant a_0 r \leqslant a_0 2^{-j_0}\varrho(z,y),$$

which implies (5.5) by (5.4). In view of (5.4), (5.5) and the fact that $k_0 \geqslant a_1^2 (a_0+2^{j_0+1})/(2^{j_0}-a_1)$, we have $B(z, 2\varrho(z,y)) \subset B(x, k_0\varrho(x,y))$. Further, we note that

$$\varrho(x,y) \leqslant a_1(a_0 2^{-j_0}+1)\varrho(z,y)$$

for $x \in B(z,r)$ and $y \in X \setminus B(z,2^{j_0}r)$. Therefore, we obtain

$$\begin{aligned}
I_2(x) &\leqslant C\int_{X\setminus B(z,2^{j_0}r)} \frac{\varrho(z,y)^{\eta}}{\mu(B(x,k_0\varrho(x,y)))}f(y)\,\mathrm{d}\mu(y)\\
&\leqslant C\int_{X\setminus B(z,2^{j_0}r)} \frac{\varrho(z,y)^{\eta}}{\mu(B(z,2\varrho(z,y)))}f(y)\,\mathrm{d}\mu(y)
\end{aligned}$$

for $x \in B(z,r)$. Hence we have by Lemma 5.2

$$I_2(x) \leqslant Cr^{\eta-\alpha}(\log(\mathrm{e}+1/r))^{\theta}.$$

Thus this lemma is proved. □

Proof of Theorem 5.1. Let f be a nonnegative measurable function on X satisfying $\|f\|_{L^{p(\cdot)-0,\nu,\theta;1}(X)} \leqslant 1$. Set $g(y)=\varepsilon^{\theta} f(y)$. Then we have by Lemma 3.2 and (5.1)

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)} g(y)\,\mathrm{d}\mu(y) \leqslant Cr^{-\nu/p_{\varepsilon}(x)} \leqslant Cr^{-\alpha p(x)/p_{\varepsilon}(x)}.$$

Here we take $\varepsilon=(p^{-}-1)(\log(\mathrm{e}+1/r))^{-1}$ and obtain

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)} f(y)\,\mathrm{d}\mu(y) \leqslant Cr^{-\alpha}(\log(\mathrm{e}+1/r))^{\theta} \tag{5.6}$$

for all $x\in X$ and $0<r<d_X$, which is nothing but (5.2). For $x\in B(z,r)$, $\delta>0$ and $0<\eta<\alpha$, we find

$$\begin{aligned} U_{\alpha,k_0}f(x) &= \int_{B(x,\delta)} \frac{\varrho(x,y)^{\alpha}}{\mu(B(x,k_0\varrho(x,y)))} f(y)\,\mathrm{d}\mu(y) \\ &\quad + \int_{X\setminus B(x,\delta)} \frac{\varrho(x,y)^{\alpha}}{\mu(B(x,k_0\varrho(x,y)))} f(y)\,\mathrm{d}\mu(y) \\ &\leqslant \delta^{\alpha-\eta} I_{\eta}f(x) + U_2(x). \end{aligned}$$

As in the proof of (5.3), it follows that

$$U_2(x) \leqslant C\int_{\delta}^{2d_X} (\log(\mathrm{e}+1/t))^{\theta}\,\frac{\mathrm{d}t}{t} \leqslant C(\log(\mathrm{e}+1/\delta))^{\theta+1},$$

which gives

$$U_{\alpha,k_0}f(x) \leqslant C\{\delta^{\alpha-\eta} I_{\eta}f(x) + (\log(\mathrm{e}+1/\delta))^{\theta+1}\}.$$

Here, letting,

$$\delta = \{I_{\eta}f(x)\}^{-1/(\alpha-\eta)}(\log(\mathrm{e}+I_{\eta}f(x)))^{(\theta+1)/(\alpha-\eta)},$$

we have the inequality

$$U_{\alpha,k_0}f(x) \leqslant C(\log(\mathrm{e}+I_{\eta}f(x)))^{\theta+1}.$$

Then, in view of Lemma 5.3, there exist constants $c_1, c_3>0$ such that

$$\begin{aligned} &\frac{1}{\mu(B(z,2^{j_0}r))}\int_{B(z,r)} \exp(c_1 U_{\alpha,k_0}f(x)^{1/(\theta+1)})\,\mathrm{d}\mu(x) \\ &\qquad \leqslant C\Big\{\frac{1}{\mu(B(z,2^{j_0}r))}\int_{B(z,r)} I_{\eta}f(x)\,\mathrm{d}\mu(x) + 1\Big\} \\ &\qquad \leqslant c_3 r^{\eta-\alpha}(\log(\mathrm{e}+1/r))^{\theta} \end{aligned}$$

for all $z\in X$ and $0<r<d_X$. Since $c_3 r^{\eta-\alpha}(\log(\mathrm{e}+1/r))^{\theta} \leqslant c_2 r^{\eta'-\alpha}$ for all $0<r<d_X$ and some constant $c_2>0$ when $0<\eta'<\eta$, the proof of the present theorem is completed. $\square$

In this section, we assume that there exist constants $C_1 > 0$ and $0 < \sigma \leqslant 1$ such that

$$\left|\frac{\varrho(x,y)^\alpha}{\mu(B(x,2\varrho(x,y)))} - \frac{\varrho(z,y)^\alpha}{\mu(B(z,2\varrho(z,y)))}\right| \leqslant C_1\Big(\frac{\varrho(x,z)}{\varrho(x,y)}\Big)^\sigma \frac{\varrho(x,y)^\alpha}{\mu(B(x,2\varrho(x,y)))} \tag{6.1}$$

whenever $\varrho(x,z) \leqslant \varrho(x,y)/2$.

Let $\omega(\cdot)$ be a positive function on $(0,\infty)$ satisfying the doubling condition

$$\omega(2r) \leqslant C_2\omega(r) \quad \text{for all } r > 0$$

and

$$\omega(s) \leqslant C_3\omega(t) \quad \text{whenever } 0 < s \leqslant t,$$

where C_2 and C_3 are positive constants. Then, in view of (2.2), one can find constants $Q > 0$ and $C_Q > 0$ such that

$$\omega(r) \geqslant C_Q r^Q \tag{6.2}$$

for all $0 < r < d_X$.

In this section, for $\theta > 0$ we consider the space $L^{p(\cdot)-0,\omega,\theta}(X)$ of locally integrable functions f on X satisfying

$$\|f\|_{L^{p(\cdot)-0,\omega,\theta}(X)} = \sup_{\substack{x\in X,0<r<d_X\\ 0<\varepsilon<p^- -1}} \varepsilon^\theta\Big(\frac{\omega(r)}{\mu(B(x,r))}\Big)^{1/p_\varepsilon(x)}\|f\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} < \infty.$$

Set

$$\Omega_*(x,r) = \int_0^r t^\alpha\omega(t)^{-1/p(x)}(\log(\mathrm{e}+1/t))^\theta\,\frac{\mathrm{d}t}{t}$$

and

$$\Omega^*(x,r) = \int_r^{2d_X} t^{\alpha-\sigma}\omega(t)^{-1/p(x)}(\log(\mathrm{e}+1/t))^\theta\,\frac{\mathrm{d}t}{t}$$

for $x \in X$ and $0 < r < d_X$.

Example 6.1. Let $\omega(r) = r^\nu(\log(\mathrm{e}+1/r))^\beta$. If $p^- \geqslant \nu/\alpha$ and $\operatorname*{ess\,sup}_{x\in X}(-\beta/p(x)+\theta+1) < 0$, then

$$\Omega_*(x,r) + r^\sigma\Omega^*(x,r) \leqslant C(\log(\mathrm{e}+1/r))^{-\beta/p(x)+\theta+1}$$

for $x \in X$ and $0 < r < d_X$.

Our final goal is to establish the following result, which deals with the continuity for Riesz potentials of functions in grand Morrey spaces of variable exponents over non-doubling measure spaces.

Theorem 6.2. *Suppose that* (6.1) *holds. Then there exists a constant* $C > 0$ *such that*

$$|U_{\alpha,2}f(x) - U_{\alpha,2}f(z)| \\ \leqslant C\{\Omega_*(x, \varrho(x,z)) + \Omega_*(z, \varrho(x,z)) + \varrho(x,z)^\sigma \Omega^*(x, \varrho(x,z))\}$$

for all $x, z \in X$, *whenever* f *is a nonnegative measurable function on* X *satisfying* $\|f\|_{L^{p(\cdot)-0,\omega,\theta}(X)} \leqslant 1$.

Before the proof of Theorem 6.2, we prepare some lemmas.

Since

$$\omega(r)^{-|p(x)-p(y)|} \leqslant Cr^{-Q|p(x)-p(y)|} \leqslant C$$

for all $y \in B(x,r)$ by (6.2) and (P2), we can show the following result in the same manner as Lemma 3.2 and (5.6).

Lemma 6.3. *Let* f *be a nonnegative function on* X *such that* $\|f\|_{L^{p(\cdot)-0,\omega,\theta}(X)} \leqslant 1$. *Then there exists a constant* $C > 0$ *such that*

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y) \, \mathrm{d}\mu(y) \leqslant C\omega(r)^{-1/p(x)} (\log(\mathrm{e} + 1/r))^\theta$$

for all $x \in X$ *and* $0 < r < d_X$.

Lemma 6.4. *Let* f *be a nonnegative function on* X *such that* $\|f\|_{L^{p(\cdot)-0,\omega,\theta}(X)} \leqslant 1$. *Then there exists a constant* $C > 0$ *such that*

$$\int_{B(x,\delta)} \frac{\varrho(x,y)^\alpha}{\mu(B(x, 2\varrho(x,y)))} f(y) \, \mathrm{d}\mu(y) \leqslant C\Omega_*(x,\delta)$$

and

$$\int_{G \setminus B(x,\delta)} \frac{\varrho(x,y)^{\alpha-\sigma}}{\mu(B(x, 2\varrho(x,y)))} f(y) \, \mathrm{d}\mu(y) \leqslant C\Omega^*(x,\delta)$$

for all $x \in X$ *and* $0 < \delta < d_X$.

Proof. Let f be a nonnegative function on X such that $\|f\|_{L^{p(\cdot)-0,\omega,\theta}(X)} \leqslant 1$. We show only the first case. As in the proof of (5.3), we have by Lemma 6.3

$$\begin{aligned}
&\int_{B(x,\delta)} \frac{\varrho(x,y)^\alpha}{\mu(B(x, 2\varrho(x,y)))} f(y) \, \mathrm{d}\mu(y) \\
&\quad = \sum_{j=1}^{\infty} \int_{B(x,2^{-j+1}\delta) \setminus B(x,2^{-j}\delta)} \frac{\varrho(x,y)^\alpha}{\mu(B(x, 2\varrho(x,y)))} f(y) \, \mathrm{d}\mu(y) \\
&\quad \leqslant \sum_{j=1}^{\infty} (2^{-j+1}\delta)^\alpha \frac{1}{\mu(B(x, 2^{-j+1}\delta))} \int_{B(x,2^{-j+1}\delta)} f(y) \, \mathrm{d}\mu(y)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\sum_{j=1}^{\infty}(2^{-j+1}\delta)^{\alpha}\omega(2^{-j+1}\delta)^{-1/p(x)}(\log(\mathrm{e}+1/(2^{-j+1}\delta)))^{\theta}\\
&\leqslant C\int_0^{\delta} t^{\alpha}\omega(t)^{-1/p(x)}(\log(\mathrm{e}+1/t))^{\theta}\,\frac{\mathrm{d}t}{t}=C\Omega_*(x,\delta),
\end{aligned}
$$

as required. □

Proof of Theorem 6.2. Let f be a nonnegative measurable function on X satisfying $\|f\|_{L^{p(\cdot)-0,\omega,\theta}(X)}\leqslant 1$. Write

$$
\begin{aligned}
U_{\alpha,2}f(x)-U_{\alpha,2}f(z)&=\int_{B(x,2\varrho(x,z))}\frac{\varrho(x,y)^{\alpha}}{\mu(B(x,2\varrho(x,y)))}f(y)\,\mathrm{d}\mu(y)\\
&\quad-\int_{B(x,2\varrho(x,z))}\frac{\varrho(z,y)^{\alpha}}{\mu(B(z,2\varrho(z,y)))}f(y)\,\mathrm{d}\mu(y)\\
&\quad+\int_{X\setminus B(x,2\varrho(x,z))}\Big(\frac{\varrho(x,y)^{\alpha}}{\mu(B(x,2\varrho(x,y)))}-\frac{\varrho(z,y)^{\alpha}}{\mu(B(z,2\varrho(z,y)))}\Big)f(y)\,\mathrm{d}\mu(y)
\end{aligned}
$$

for $x,z\in X$. Using Lemma 6.4, we have

$$
\int_{B(x,2\varrho(x,z))}\frac{\varrho(x,y)^{\alpha}}{\mu(B(x,2\varrho(x,y)))}f(y)\,\mathrm{d}\mu(y)\leqslant C\Omega_*(x,2\varrho(x,z))\leqslant C\Omega_*(x,\varrho(x,z))
$$

and

$$
\begin{aligned}
&\int_{B(x,2\varrho(x,z))}\frac{\varrho(z,y)^{\alpha}}{\mu(B(z,2\varrho(z,y)))}f(y)\,\mathrm{d}\mu(y)\\
&\qquad\leqslant\int_{B(z,a_1(a_0+2)\varrho(x,z))}\frac{\varrho(z,y)^{\alpha}}{\mu(B(z,2\varrho(z,y)))}f(y)\,\mathrm{d}\mu(y)\\
&\qquad\leqslant C\Omega_*(z,a_1(a_0+2)\varrho(x,z))\leqslant C\Omega_*(z,\varrho(x,z)).
\end{aligned}
$$

On the other hand, by (6.1) and Lemma 6.4, we have

$$
\begin{aligned}
&\int_{X\setminus B(x,2\varrho(x,z))}\Big|\frac{\varrho(x,y)^{\alpha}}{\mu(B(x,2\varrho(x,y)))}-\frac{\varrho(z,y)^{\alpha}}{\mu(B(z,2\varrho(z,y)))}\Big|f(y)\,\mathrm{d}\mu(y)\\
&\qquad\leqslant C_1\varrho(x,z)^{\sigma}\int_{X\setminus B(x,2\varrho(x,z))}\frac{\varrho(x,y)^{\alpha-\sigma}}{\mu(B(x,2\varrho(x,y)))}f(y)\,\mathrm{d}\mu(y)\\
&\qquad\leqslant C\varrho(x,z)^{\sigma}\Omega^*(x,2\varrho(x,z))\leqslant C\varrho(x,z)^{\sigma}\Omega^*(x,\varrho(x,z)).
\end{aligned}
$$

Thus we have the conclusion. □

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