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THE FUNDAMENTAL CONSTITUENTS OF ITERATION DIGRAPHS OF FINITE COMMUTATIVE RINGS

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Abstract. For a finite commutative ring R and a positive integer $k \geqslant 2$, we construct an iteration digraph $G(R,k)$ whose vertex set is R and for which there is a directed edge from $a \in R$ to $b \in R$ if $b = a^k$. Let $R = R_1 \oplus \ldots \oplus R_s$, where $s > 1$ and R_i is a finite commutative local ring for $i \in \{1,\ldots,s\}$. Let N be a subset of $\{R_1,\ldots,R_s\}$ (it is possible that N is the empty set $\emptyset$). We define the fundamental constituents $G_N^*(R,k)$ of $G(R,k)$ induced by the vertices which are of the form $\{(a_1,\ldots,a_s) \in R\colon a_i \in \mathrm{D}(R_i)$ if $R_i \in N$, otherwise $a_i \in \mathrm{U}(R_i), i = 1,\ldots,s\}$, where $\mathrm{U}(R)$ denotes the unit group of R and $\mathrm{D}(R)$ denotes the zero-divisor set of R. We investigate the structure of $G_N^*(R,k)$ and state some conditions for the trees attached to cycle vertices in distinct fundamental constituents to be isomorphic.

Keywords: iteration digraph; fundamental constituent; digraphs product

MSC 2010: 05C05, 11A07, 13M05

1. Introduction

Let R be a finite commutative ring. The graph $G(R,k)$ ($k \geqslant 2$ is a positive integer) is a digraph whose vertices are the elements of R and for which there is a directed edge from $a \in R$ to $b \in R$ if $b = a^k$. It is well known that if R is a finite commutative ring with identity 1, then R can be uniquely expressed as a direct sum of local rings:

$$R = R_1 \oplus \ldots \oplus R_s, \quad s \geqslant 1,$$

where R_i is a local ring for $i = 1,\ldots,s$ (see [1, Theorem 3.1.4]). Let N be a subset of $\{R_1,\ldots,R_s\}$ (it is possible that N is the empty set $\emptyset$). We define the subdigraph

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$G_N^*(R,k)$ of $G(R,k)$ induced by the vertices which are of the form

$$\{(a_1,\ldots,a_s)\in R\colon\ a_i\in \mathrm{D}(R_i) \text{ if } R_i\in N, \text{ otherwise } a_i\in \mathrm{U}(R_i),\ i=1,\ldots,s\},$$

where $\mathrm{U}(R)$ denotes the unit group of R and $\mathrm{D}(R)$ denotes the zero-divisor set of R. Then $G_N^*(R,k)$ is called a fundamental constituent of $G(R,k)$. Since the number of subsets of $\{R_1,\ldots,R_s\}$ is 2^s (including the empty set $\emptyset$), there are exactly 2^s fundamental constituents in $G(R,k)$, and the disjoint union of these 2^s fundamental constituents is precisely the digraph $G(R,k)$. The fundamental constituents of $G(\mathbb{Z}_n,k)$, where $\mathbb{Z}_n$ is the ring of integers modulo n, were introduced by Wilson in [8] and were investigated by Somer et al. in [5] and [6].

A *component* of a digraph is a directed subgraph which is a maximal connected subgraph of the associated undirected graph. If α is a vertex of a component in $G(R,k)$, we use $\mathrm{Com}_R(\alpha)$ to denote this component.

Suppose α is a vertex of $G(R,k)$. The indegree of α, denoted by $\mathrm{indeg}_R(\alpha)$, is the number of directed edges entering α. We will simply write $\mathrm{indeg}(\alpha)$ when it is understood that α is a vertex in $G(R,k)$. A digraph is regular if all its vertices have the same indegree, while the digraph $G(R,k)$ is said to be semiregular if there exists a positive integer d such that each vertex of $G(R,k)$ either has indegree 0 or d.

Cycles of length t are called t-cycles, and cycles of length one are called *fixed points*. For an *isolated fixed point* α, the indegree and outdegree (i.e., the number of edges leaving α) are both one. Attached to each cycle vertex α of $G(R,k)$ is a tree $T_R(\alpha)$ whose root is α and whose additional vertices are the noncycle vertices β for which $\beta^{k^i}=\alpha$ for some positive integers i, but $\beta^{k^{i-1}}$ is not a cycle vertex. Moreover, we specify two particular subdigraphs $G_1(R,k)$ and $G_2(R,k)$ of $G(R,k)$, i.e., $G_1(R,k)$ is induced by all the vertices of $\mathrm{U}(R)$, and $G_2(R,k)$ is induced by all the vertices of $\mathrm{D}(R)$.

Similarly to the proof of [3, Theorem 29], it is easy to show the following lemma.

Lemma 1.1. *Let R be a finite commutative ring. Let $\beta\in \mathrm{U}(R)$ be a cycle vertex of $G(R,k)$, $k\geqslant 2$. Then the tree $T_R(1)$ is isomorphic to the tree $T_R(\beta)$.*

Given two digraphs Γ_1 and Γ_2, let $\Gamma_1\times\Gamma_2$ denote the digraph whose vertices are the ordered pairs (a_1,a_2), where a_i is an arbitrary vertex of Γ_i for $i=1,2$. In addition, there is a directed edge in $\Gamma_1\times\Gamma_2$ from (a_1,a_2) to (b_1,b_2) if and only if there is a directed edge in Γ_1 from a_1 to b_1 and there is a directed edge in Γ_2 from a_2 to b_2. In general, if $S\cong S_1\oplus\ldots\oplus S_t$, where S, $S_1,\ldots,S_t$ are rings, then $G(S,k)\cong G(S_1,k)\times\ldots\times G(S_t,k)$. The following lemma is obvious.

Lemma 1.2. *Let Γ_i be digraphs, $i=1,2,3,4$, where $\Gamma_1\cong\Gamma_2$ and $\Gamma_3\cong\Gamma_4$. Then $\Gamma_1\times\Gamma_3\cong\Gamma_2\times\Gamma_4$.*

Lemma 1.3 ([4, Theorem 2]). *Let R be a finite local ring with an identity element 1 which is not necessarily commutative. Let M be the unique maximal ideal of R. Then $|R| = p^{nr}$, $|M| = p^{(n-1)r}$, $M^n = \{0\}$ and $\operatorname{char}(R) = p^k$, where $\operatorname{char}(R)$ is the characteristic of R, p is a prime, n, r, k are positive integers and $1 \leqslant k \leqslant n$.*

2. The fundamental constituents of $G(R,k)$

In the following two examples, we denote by $\mathbb{F}_q$ the finite field of order q.

Example 2.1. Let $R = \mathbb{F}_4 \oplus \mathbb{Z}_4$, where $\mathbb{F}_4 = \{0, 1, a, a^2\}$ with $o(a) = 3$. There are precisely 4 fundamental constituents $G^*_{N_i}(R,3)$ in $G(R,3)$, where $N_1 = \{\mathbb{F}_4\}$, $N_2 = \{\mathbb{F}_4, \mathbb{Z}_4\}$, $N_3 = \{\mathbb{Z}_4\}$, and $N_4 = \emptyset$. Figure 1 shows the fundamental constituents of $G(R,3)$.

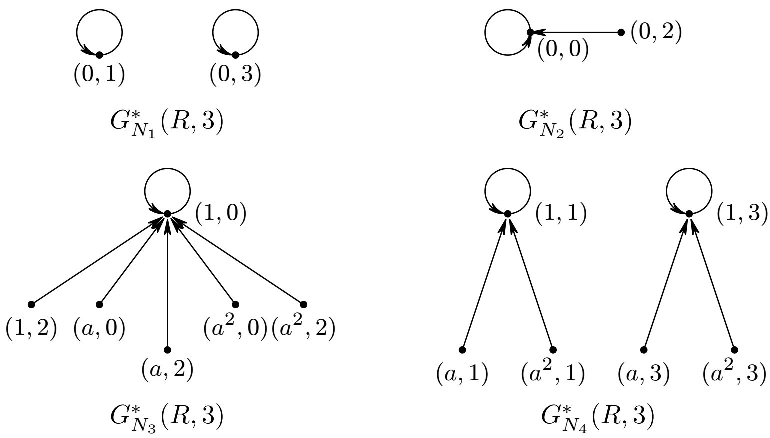

Figure 1. The four fundamental constituents of $G(\mathbb{F}_4 \oplus \mathbb{Z}_4, 3)$.

Example 2.2. Let $R = \mathbb{F}_4 \oplus \mathbb{F}_3 \oplus \mathbb{F}_5$, where $\mathbb{F}_4 = \{0, 1, a, a^2\}$ with $o(a) = 3$. There are precisely 8 fundamental constituents $G^*_{N_i}(R,2)$ in $G(R,2)$, where $N_1 = \{\mathbb{F}_4, \mathbb{F}_3, \mathbb{F}_5\}$, $N_2 = \{\mathbb{F}_3, \mathbb{F}_5\}$, $N_3 = \{\mathbb{F}_4, \mathbb{F}_5\}$, $N_4 = \{\mathbb{F}_5\}$, $N_5 = \{\mathbb{F}_4, \mathbb{F}_3\}$, $N_6 = \{\mathbb{F}_3\}$, $N_7 = \{\mathbb{F}_4\}$, and $N_8 = \emptyset$. Figure 2 shows the fundamental constituents of $G(R,2)$.

Remark 2.1. Let R be the direct sum of finite commutative local rings $R_1, \ldots, R_s$. If N is a subset of $\{R_1, \ldots, R_s\}$, then by the definition of digraphs products, we have

$$G^*_N(R,k) = \prod_{R_i \in N} G_2(R_i,k) \times \prod_{R_j \notin N} G_1(R_j,k).$$

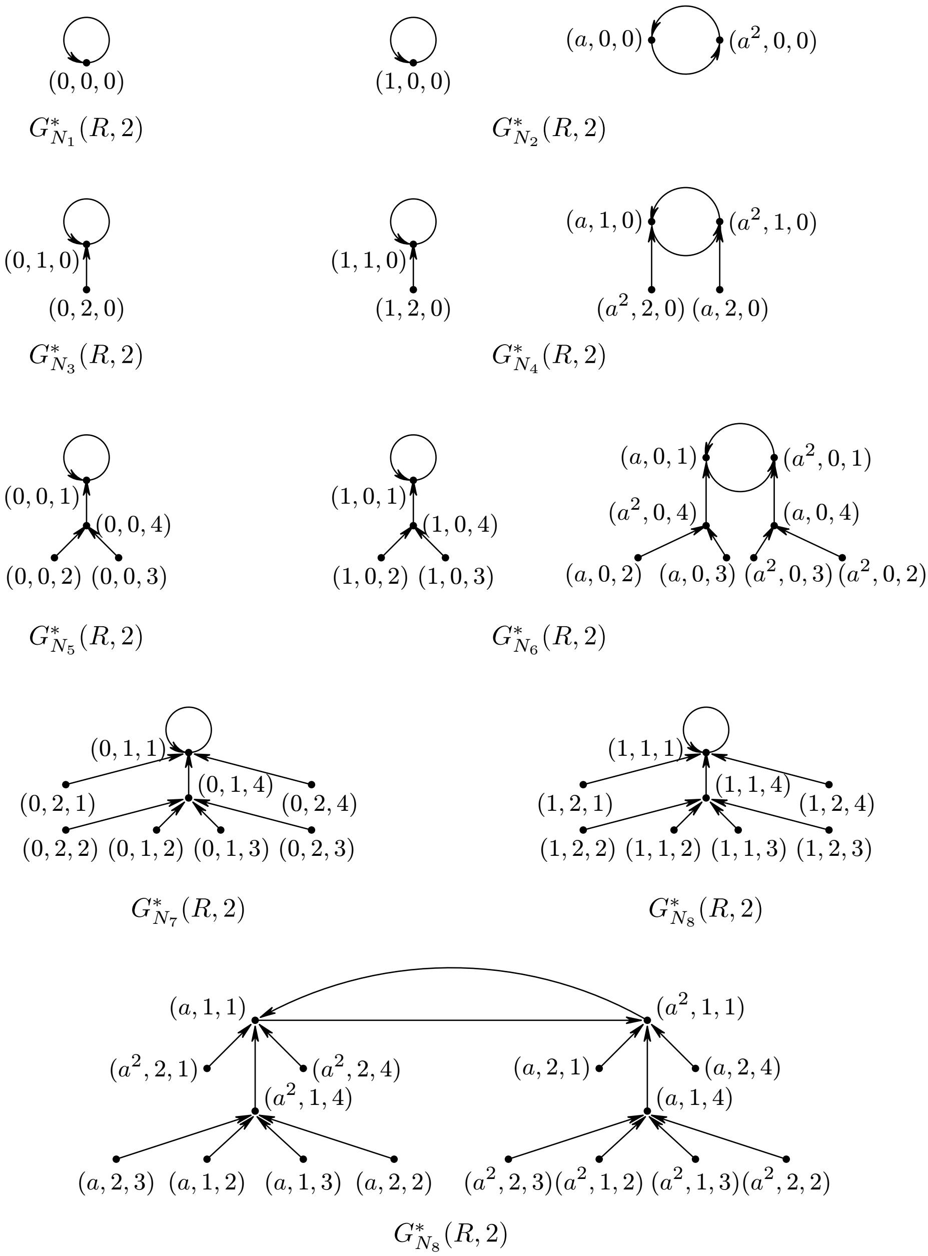

Figure 2. The eight fundamental constituents of $G(\mathbb{F}_4 \oplus \mathbb{F}_3 \oplus \mathbb{F}_5, 2)$.

In particular, if $N = \{R_1, \ldots, R_s\}$, then $G^*_N(R,k)$ has precisely one component, i.e.,

$$G^*_N(R,k) = G_2(R_1,k) \times \ldots \times G_2(R_s,k) = \mathrm{Com}_R(0).$$

If $N = \emptyset$, then $G_N^*(R,k) = G_1(R,k)$. Note that $G_2(R_i,k) = \mathrm{Com}_{R_i}(0)$, since R_i is a local ring. We can write $G_N^*(R,k) = \prod\limits_{R_i \in N} \mathrm{Com}_{R_i}(0) \times \prod\limits_{R_j \notin N} G_1(R_j,k)$.

Somer and Křížek obtained the semiregularity and regularity of the fundamental constituents of $G(\mathbb{Z}_n,k)$, see [5, Theorems 5.1, 5.4]. Analogously, we present the following Theorems 2.1 and 2.2 for any finite commutative ring.

Theorem 2.1. *Let R be the direct sum of finite commutative local rings $R_1,\ldots,R_s$. Let N be a subset of $\{R_1,\ldots,R_s\}$. Then $G_N^*(R,k)$ is semiregular if and only if $G_2(R_i,k)$ is semiregular for $R_i \in N$.*

Proof. Suppose that $G_N^*(R,k)$ is semiregular. By way of contradiction, we can assume without loss of generality that $R_1 \in N$ while $G_2(R_1,k)$ is not semiregular. Then there exists a vertex g of $G_2(R_1,k)$ with $\mathrm{indeg}_{R_1}(0) \neq \mathrm{indeg}_{R_1}(g) > 0$. Now let $\gamma = (g,a_2,\ldots,a_s) \in G_N^*(R,k)$, where for $j \in \{2,\ldots,s\}$, $a_j = 0$ if $R_j \in N$, while $a_j = 1$ if $R_j \notin N$. Further, let $\beta = (b_1,\ldots,b_s) \in G_N^*(R,k)$, where for $i \in \{1,\ldots,s\}$, $b_i = 0$ if $R_i \in N$, while $b_i = 1$ if $R_i \notin N$. Then we can see that both the vertices γ and β have positive indegree in $G_N^*(R,k)$. However, $\mathrm{indeg}_R(\gamma) \neq \mathrm{indeg}_R(\beta)$, since

$$\mathrm{indeg}_{R_1}(g) \neq \mathrm{indeg}_{R_1}(0) = \mathrm{indeg}_{R_1}(b_1),$$

but $\mathrm{indeg}_{R_j}(a_j) = \mathrm{indeg}_{R_j}(b_j)$ for $j \in \{2,\ldots,s\}$. This is impossible, as $G_N^*(R,k)$ is semiregular and $\gamma,\beta \in G_N^*(R,k)$. Hence, $G_2(R_i,k)$ is semiregular for $R_i \in N$.

Conversely, assume that $G_2(R_i,k)$ is semiregular for $R_i \in N$. Let $\alpha = (a_1,\ldots,a_s)$ and $\beta = (b_1,\ldots,b_s)$ be two distinct vertices in $G_N^*(R,k)$ such that $\mathrm{indeg}_R(\alpha) > 0$ and $\mathrm{indeg}_R(\beta) > 0$. By assumption, $\mathrm{indeg}_{R_i}(a_i) = \mathrm{indeg}_{R_i}(b_i)$ for $R_i \in N$. Moreover, by a proof similar to that of [7, Theorem 3.2], we derive that $G_1(R,k)$ is semiregular for any finite commutative ring R, so $\mathrm{indeg}_{R_j}(a_j) = \mathrm{indeg}_{R_j}(b_j)$ for $R_j \notin N$. Thus

$$\mathrm{indeg}_R(\alpha) = \prod_{i=1}^{s} \mathrm{indeg}_{R_i}(a_i) = \prod_{i=1}^{s} \mathrm{indeg}_{R_i}(b_i) = \mathrm{indeg}_R(\beta).$$

Hence, $G_N^*(R,k)$ is semiregular. □

Theorem 2.2. *Let R be the direct sum of finite commutative local rings $R_1,\ldots,R_s$. Let N be a subset of $\{R_1,\ldots,R_s\}$. Then $G_N^*(R,k)$ is regular if and only if R_i is a field for $R_i \in N$ and $\gcd(|\mathrm{U}(R_j)|,k) = 1$ for $R_j \notin N$.*

Proof. Suppose that $G_N^*(R,k)$ is regular. Then for $\alpha = (a_1,\ldots,a_s) \in G_N^*(R,k)$, the indegree of α is 1. Hence, $\mathrm{indeg}_{R_i}(a_i) = 1$ for any $a_i \in \mathrm{D}(R_i)$ if $R_i \in N$. Thus, $\mathrm{D}(R_i)$ has only one element, i.e., 0. So R_i is a field for $R_i \in N$. On the other hand, we also have $\operatorname{indeg}_{R_j}(a_j) = 1$ for any $a_j \in \mathrm{U}(R_j)$ if $R_j \notin N$, which implies that $G_1(R_j, k)$ is regular if $R_j \notin N$. So $\gcd(|\mathrm{U}(R_j)|, k) = 1$ for $R_j \notin N$.

Conversely, assume that R_i is a field for $R_i \in N$ and $\gcd(|\mathrm{U}(R_j)|, k) = 1$ for $R_j \notin N$. Let $\alpha = (a_1, \ldots, a_s) \in G_N^*(R, k)$. So we have $a_i = 0$ and hence $\operatorname{indeg}_{R_i}(a_i) = 1$ if $R_i \in N$. Moreover, since $\gcd(|\mathrm{U}(R_j)|, k) = 1$ for $R_j \notin N$, R_j is regular. So $\operatorname{indeg}_{R_j}(a_j) = 1$ if $R_j \notin N$. Therefore, the indegree of α in $G_N^*(R, k)$ is equal to $\prod_{i=1}^{s} \operatorname{indeg}_{R_i}(a_i) = 1$. Consequently, $G_N^*(R, k)$ is regular. □

Somer and Křížek showed in [6, Theorem 6.1] that all trees attached to cycle vertices in a fundamental constituent of $G(\mathbb{Z}_n, k)$ are isomorphic. In general, we have the following result.

Theorem 2.3. *Let R be the direct sum of finite commutative local rings $R_1, \ldots, R_s$. Let N be a subset of $\{R_1, \ldots, R_s\}$. If α and β are two cycle vertices in $G_N^*(R, k)$, then the tree $T_R(\alpha)$ is isomorphic to the tree $T_R(\beta)$.*

Proof. By Remark 2.1, if $N = \emptyset$, then $G_N^*(R, k) = G_1(R, k)$, and the assertion follows from Lemma 1.1. If $N = \{R_1, \ldots, R_s\}$, then $G_N^*(R, k) = \mathrm{Com}_R(0)$. Since the only cycle vertex in $\mathrm{Com}_R(0)$ is 0, there is only one tree in $G_N^*(R, k)$, and the theorem follows trivially.

Now suppose $\emptyset \neq N \neq \{R_1, \ldots, R_s\}$. We can suppose without loss of generality that $N = \{R_1, R_2, \ldots, R_t\}$, where $1 \leqslant t \leqslant s - 1$. Then by Remark 2.1,

$$G_N^*(R, k) = G_2(R_1, k) \times \ldots \times G_2(R_t, k) \times G_1(R_{t+1}, k) \times \ldots \times G_1(R_s, k).$$

Therefore, if $\alpha = (a_1, \ldots, a_s)$ is a cycle vertex in $G_N^*(R, k)$, then it is evident that $a_1 = \ldots = a_t = 0$, while a_j is a cycle vertex in $G_1(R_j, k)$ for $j \in \{t+1, \ldots, s\}$. In particular, $\gamma = (b_1, \ldots, b_s)$ is also a cycle vertex in $G_N^*(R, k)$, where $b_1 = \ldots = b_t = 0$ and $b_{t+1} = \ldots = b_s = 1$. Hence, we will complete the proof by showing that $T_R(\alpha) \cong T_R(\gamma)$. By the definition of digraphs products, it is easy to show that

$$\begin{aligned} T_R(\alpha) &= T_{R_1}(0) \times \ldots \times T_{R_t}(0) \times T_{R_{t+1}}(a_{t+1}) \times \ldots \times T_{R_s}(a_s), \\ T_R(\gamma) &= T_{R_1}(0) \times \ldots \times T_{R_t}(0) \times T_{R_{t+1}}(1) \times \ldots \times T_{R_s}(1). \end{aligned}$$

By Lemma 1.1, $T_{R_j}(a_j) \cong T_{R_j}(1)$ for any cycle vertex a_j in $G_1(R_j, k)$. So we can conclude by Lemma 1.2 that $T_R(\alpha) \cong T_R(\gamma)$, as desired. □

In Figure 1, we observe that trees attached to cycle vertices in different fundamental constituents are not isomorphic, whereas from Figure 2 we can see that the fundamental constituents $G_{N_3}^*(R, 2)$ and $G_{N_4}^*(R, 2)$ (as well as $G_{N_5}^*(R, 2)$ and

$G^*_{N_6}(R,2)$, $G^*_{N_7}(R,2)$ and $G^*_{N_8}(R,2)$) have isomorphic nontrivial trees attached to their cycle vertices. In the following theorem, we present some conditions for trees attached to cycle vertices in different fundamental constituents to be isomorphic.

Theorem 2.4. *Let R be the direct sum of finite commutative local rings $R_1,\dots,R_s$. There are two distinct subsets N_1 and N_2 of $\{R_1,\dots,R_s\}$ such that $G^*_{N_1}(R,k)$ and $G^*_{N_2}(R,k)$ have isomorphic trees attached to their cycle vertices, provided one of the following conditions holds:*

(1) *There exists $i\in\{1,\dots,s\}$ such that $\mathrm{Com}_{R_i}(0)\cong\mathrm{Com}_{R_i}(1)$.*
(2) *There exist $i,j\in\{1,\dots,s\}$ $(i\neq j)$ such that $\mathrm{Com}_{R_i}(0)\cong\mathrm{Com}_{R_j}(0)$ and $\mathrm{Com}_{R_i}(1)\cong\mathrm{Com}_{R_j}(1)$.*
(3) *There exist $i,j\in\{1,\dots,s\}$ $(i\neq j)$ such that $\mathrm{Com}_{R_i}(0)\cong\mathrm{Com}_{R_j}(1)$ and $\mathrm{Com}_{R_i}(1)\cong\mathrm{Com}_{R_j}(0)$.*

Proof. (1) We can suppose without loss of generality that $\mathrm{Com}_{R_1}(0)\cong\mathrm{Com}_{R_1}(1)$. Let $N_1=\{R_1,\dots,R_s\}$ and $N_2=\{R_2,\dots,R_s\}$ (if $s=1$ then let $N_2=\emptyset$). Clearly,

$$G^*_{N_1}(R,k)=\mathrm{Com}_{R_1}(0)\times\ldots\times\mathrm{Com}_{R_s}(0),$$
$$G^*_{N_2}(R,k)=G_1(R_1,k)\times\mathrm{Com}_{R_2}(0)\times\ldots\times\mathrm{Com}_{R_s}(0).$$

We observe that $G^*_{N_1}(R,k)$ has precisely one component. In addition,

$$L_{N_2}=\mathrm{Com}_{R_1}(1)\times\mathrm{Com}_{R_2}(0)\times\ldots\times\mathrm{Com}_{R_s}(0)$$

is one component of $G^*_{N_2}(R,k)$. By the hypothesis and Lemma 1.2, we have $G^*_{N_1}(R,k)\cong L_{N_2}$. Hence, the result follows by Theorem 2.3.

(2) We can suppose without loss of generality that

$$\mathrm{Com}_{R_1}(0)\cong\mathrm{Com}_{R_2}(0),\quad \mathrm{Com}_{R_1}(1)\cong\mathrm{Com}_{R_2}(1).$$

Let $N_1=\{R_1\}$, $N_2=\{R_2\}$. Then clearly

$$L_{N_1}=\mathrm{Com}_{R_1}(0)\times\mathrm{Com}_{R_2}(1)\times\prod_{j=3}^{s}\mathrm{Com}_{R_j}(1)$$

is one component of $G^*_{N_1}(R,k)$, while

$$L_{N_2}=\mathrm{Com}_{R_1}(1)\times\mathrm{Com}_{R_2}(0)\times\prod_{j=3}^{s}\mathrm{Com}_{R_j}(1)$$

is one component of $G^*_{N_2}(R,k)$. By the hypothesis and Lemma 1.2, $L_{N_1} \cong L_{N_2}$. Accordingly, the result follows by Theorem 2.3.

(3) We can suppose without loss of generality that

$$\mathrm{Com}_{R_1}(0) \cong \mathrm{Com}_{R_2}(1), \quad \mathrm{Com}_{R_1}(1) \cong \mathrm{Com}_{R_2}(0).$$

Let $N_1 = \{R_1, R_2\}$, $N_2 = \emptyset$. Then clearly

$$L_{N_1} = \mathrm{Com}_{R_1}(0) \times \mathrm{Com}_{R_2}(0) \times \prod_{j=3}^{s} \mathrm{Com}_{R_j}(1)$$

is one component of $G^*_{N_1}(R,k)$, while

$$L_{N_2} = \mathrm{Com}_{R_1}(1) \times \mathrm{Com}_{R_2}(1) \times \prod_{j=3}^{s} \mathrm{Com}_{R_j}(1)$$

is one component of $G^*_{N_2}(R,k)$. By the hypothesis and Lemma 1.2, $L_{N_1} \cong L_{N_2}$. Thus, by Theorem 2.3, the result follows. □

Theorem 2.5. *Let R be the direct sum of finite commutative local rings $R_1, \ldots, R_s$. If there exists $i \in \{1, \ldots, s\}$ such that $\mathrm{Com}_{R_i}(0) \cong \mathrm{Com}_{R_i}(1)$, then for any subset N of $\{R_1, \ldots, R_s\}$, there exists a subset N_0 ($N_0 \neq N$) of $\{R_1, \ldots, R_s\}$ such that $G^*_N(R,k)$ and $G^*_{N_0}(R,k)$ have isomorphic trees attached to their cycle vertices.*

Proof. We can suppose without loss of generality that $\mathrm{Com}_{R_1}(0) \cong \mathrm{Com}_{R_1}(1)$. If $R_1 \in N$, let $N_0 = N - \{R_1\}$. If $R_1 \notin N$, let $N_0 = N \cup \{R_1\}$. Then by the proof of Theorem 2.4 (1), the result follows. □

Finally, we state some conditions for $\mathrm{Com}_R(0) \cong \mathrm{Com}_R(1)$, where R is a finite commutative local ring.

Theorem 2.6. *Let R be a finite commutative local ring with a unique maximal ideal M. Then $\mathrm{Com}_R(0) \cong \mathrm{Com}_R(1)$ if one of the following conditions holds:*

(1) $|R| = 2|M| = 2^n$, $n \leqslant 2 \leqslant k$ *and* $2 \mid k$.
(2) $|R| = 2|M| = 2^n$, $n = 3$ *and* $4 \mid k$.
(3) $|R| = 2|M| = 2^n$, $n \geqslant 4$ *and* $2^{n-2} \mid k$.
(4) $|R| = p|M| = p^n$, *p is an odd prime,* $n \geqslant 1$, $p-1 \mid k-1$ *and* $p^{n-1} \mid k$.

P r o o f. (1) If $n=1$, then $R=\mathbb{F}_2$. So $G_1(R,k)\cong G_2(R,k)$, i.e., $\mathrm{Com}_R(0)\cong\mathrm{Com}_R(1)$ for $k\geqslant 2$.

If $n=2$, then $R=\mathbb{Z}_4$ if $\mathrm{char}(R)=2^2$. Otherwise, if $\mathrm{char}(R)=2$, then by [4, Theorem 3], R is isomorphic to the ring of upper triangular matrices R^* over $\mathbb{F}_2$, where

$$R^*=\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix}\right\}.$$

Obviously, $R^*\cong\mathbb{Z}_2[x]/\langle x^2\rangle$ and R^* is commutative. Hence, for $\alpha\in R$, either $\alpha^k=0$ or $\alpha^k=1$ if $2\mid k$. Thus $G(R,k)$ has precisely two components, one with fixed point 0 and the other with fixed point 1, and both components are isomorphic.

(2) Suppose $n=3$ and $4\mid k$. Clearly $\alpha^k=0$ or $\alpha^k=1$ for $\alpha\in R$, since $|M|=|\mathrm{U}(R)|=4$. Hence, $\mathrm{Com}_R(0)\cong\mathrm{Com}_R(1)$.

(3) Suppose that $n\geqslant 4$ and $2^{n-2}\mid k$. By assumption, $|M|=|\mathrm{U}(R)|=2^{n-1}$ and by Lemma 1.3, $M^n=\{0\}$. Note that $k\geqslant n$, since $n\geqslant 4$ and $2^{n-2}\mid k$. We see that $\alpha^k=0$ for $\alpha\in M$. Furthermore, by the work of Gilmer in [2], if $|S|=2^t$, where S is a local ring and $t\geqslant 4$, then $\mathrm{U}(S)$ is not a cyclic group. So $\mathrm{U}(R)\cong C_{2^{n_1}}\times\ldots\times C_{2^{n_s}}$, where $s\geqslant 2$, $1\leqslant n_i\leqslant n-2$, $C_{2^{n_i}}$ is a cyclic group with order 2^{n_i} for $i\in\{1,\ldots,s\}$, and $n_1+\ldots+n_s=n-1$. Therefore, $\beta^{2^{n-2}}=1$ for $\beta\in\mathrm{U}(R)$. Moreover, since $2^{n-2}\mid k$, we have $\beta^k=1$ for $\beta\in\mathrm{U}(R)$. Thus $G(R,k)$ has precisely two components, and the two components are isomorphic.

(4) By hypothesis, $|\mathrm{U}(R)|=p^{n-1}(p-1)$. So $\mathrm{U(R)}\cong H_1\times H_2$, where H_1 and H_2 are abelian groups, $|H_1|=p^{n-1}$ and $|H_2|=p-1$. Thus, $\alpha^{p^{n-1}}=1$ and hence $\alpha^k=1$ for $\alpha\in H_1$, since $p^{n-1}\mid k$. Therefore, $G(H_1,k)$ has exactly one component and $\mathrm{indeg}_{H_1}(1)=p^{n-1}$. On the other hand, for $\beta\in H_2$ we have $\beta^{p-1}=1$ and hence $\beta^k=\beta^{k-1}\beta=\beta$, since $p-1\mid k-1$. So we can conclude that each vertex of $G(H_2,k)$ is an isolated fixed point. By the definition of digraphs products, we have

$$G_1(R,k)=G(\mathrm{U(R)},k)\cong G(H_1,k)\times G(H_2,k).$$

So $G_1(R,k)$ has precisely $p-1$ components and each cycle is of length one, while the indegree of each cycle vertex is p^{n-1}. Moreover, by Lemma 1.3, $M^n=\{0\}$. Since $p^{n-1}\mid k$, we derive that $k>n$. Thus for $\gamma\in M$, $\gamma^k=0$. So $\mathrm{indeg}_R(0)=|M|=p^{n-1}$. Hence we can see that $G(R,k)$ has precisely p components, and all these components are isomorphic. $\square$

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