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Qingjun Kong; Qingfeng Liu

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THE INFLUENCE OF WEAKLY-SUPPLEMENTED SUBGROUPS ON THE STRUCTURE OF FINITE GROUPS

Qingjun Kong, Tianjin, Qingfeng Liu, Rizhao

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The paper is dedicated to Professor John Cossey on his 70th birthday

Abstract. A subgroup H of a finite group G is weakly-supplemented in G if there exists a proper subgroup K of G such that $G = HK$. In the paper it is proved that a finite group G is p-nilpotent provided p is the smallest prime number dividing the order of G and every minimal subgroup of $P \cap G'$ is weakly-supplemented in $N_G(P)$, where P is a Sylow p-subgroup of G. As applications, some interesting results with weakly-supplemented minimal subgroups of $P \cap G'$ are obtained.

Keywords: weakly-supplemented subgroup; p-nilpotent group; supersolvable group

MSC 2010: 20D10, 20D20

1. Introduction

It is well known that a subgroup H of a finite group G is complemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. Such a subgroup K of G is called a complement to H in G. The existence of complements for certain subgroups of a finite group provides useful structural information. For instance, in [5], Hall proved that a finite group is solvable if and only if every Sylow subgroup of G is complemented. New criteria for the solvability of finite groups were obtained by Arad and Ward in [1]. They proved that a finite group is solvable if and only if every Sylow 2-subgroup and every Sylow 3-subgroup are complemented. In particular, Hall in [6] proved that a finite G is supersolvable with elementary abelian Sylow

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subgroups if and only if every subgroup of G is complemented in G. In a recent paper, Ballester-Bolinches and Xiuyun Guo [3] studied finite groups for which every minimal subgroup is complemented. They proved that such groups are just the finite supersolvable groups with elementary abelian Sylow subgroups. In this paper, we go on to investigate the p-nilpotency and supersolvability of finite groups by using weakly-supplemented subgroups in [8]. As applications, some interesting results with weakly-supplemented minimal subgroups of $P \cap G'$ are obtained.

For notation and conventions we refer to the book [10]. Unless otherwise stated, G will always be a finite group.

2. Basic definitions and preliminary results

In this section, we give one definition and some results that are needed in this paper.

Definition 2.1 ([8]). Let G be a finite group. A subgroup H of a finite group G is weakly-supplemented in G if there exists a proper subgroup K of G such that $G = HK$. Such a subgroup K of G is called a weak supplement to H in G.

It is clear that a weakly-supplemented subgroup cannot be contained in the Frattini subgroup.

Lemma 2.2 ([8]). *Let G be a group and N a normal subgroup of G.*

(1) *If $H \leqslant K \leqslant G$ and H is weakly-supplemented in G, then H is weakly-supplemented in K.*
(2) *If N is contained in H and H is weakly-supplemented in G, then H/N is weakly-supplemented in G/N.*
(3) *Let π be a set of primes. Let N a π'-subgroup and A be a π-subgroup of G. If A is weakly-supplemented in G, then AN/N is weakly-supplemented in G/N.*

Recall a group G is inner-supersolvable if G is not supersolvable but every proper subgroup of G is supersolvable. A group G is inner-p-nilpotent if G is not p-nilpotent but every proper subgroup of G is p-nilpotent. By the main result of [4], we have the following lemma.

Lemma 2.3. *Suppose that G is an inner-supersolvable group. Then there exists a normal $P \in \mathrm{Syl}_p(G)$ such that $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*

Lemma 2.4 ([7] IV, 5.4, page 434). *Suppose G is a group which is not p-nilpotent but whose all proper subgroups are p-nilpotent. Then G is a group which is not nilpotent but whose all proper subgroups are nilpotent.*

Lemma 2.5. *Every minimal subgroup of G is weakly-supplemented in G if and only if G is a supersolvable group and all Sylow subgroups of G are elementary abelian.*

P r o o f. Assume that the lemma is false and G is a minimal counterexample. We know that the hypothesis of the lemma is inherited by subgroups by Lemma 2.2 (1). By the choice of G, we know that G is an inner-supersolvable group. Then by Lemma 2.3 there exists a normal $P \in \mathrm{Syl}_p(G)$ such that $P/\Phi(P)$ is a chief factor of G and $\Phi(P) \leqslant \Phi(G)$. By the hypothesis of the lemma we know that $\Phi(G) = 1$, Thus $\Phi(P) = 1$, so P is elementary abelian and it is a minimal normal subgroup of G. If N is a subgroup of order p of P then by the hypothesis of the lemma we know that N is weakly-supplemented in G. Let K be a weak supplement of N in G, then $K < G$ and $G = NK = PK$. As $P \cap K \trianglelefteq G$, by the minimality of P we have $P \cap K = 1$, thus $P = N$.

Consider the quotient group G/P. It is easy to see that G/P satisfies the hypothesis of the lemma. By the choice of G we know that G/P is a supersolvable group, so G is supersolvable, a contradiction.

In the following, we prove that all Sylow subgroups of G are elementary abelian. Assume that P is a Sylow p-subgroup of G. By Lemma 2.2 (1) we know that P satisfies the hypothesis of the lemma, so $\Phi(P) = 1$ and P is elementary abelian.

Conversely, let p be the largest prime dividing $|G|$ and let P be a Sylow p-subgroup of G. Assume that Q is a p'-Hall subgroup of G and $N \leqslant P$ is a minimal normal subgroup of G. As G is supersolvable, we have that $|N| = p$. Consider the p'-group Q acts on the p-group P by conjugation, then N is a Q-invariant subgroup. As P is abelian and by the Theorem of Maschke, there exists a Q-invariant subgroup P_1 such that $P = P_1 \times N$, thus $P_1 \trianglelefteq G$ and $P_1Q \leqslant G$.

Next we consider the quotient group G/N. It is easy to see that G/N satisfies the hypothesis of the lemma and so any minimal subgroup of G/N is weakly-supplemented in G/N. Suppose that T is a minimal subgroup of G.

(1) If $N = T$, then P_1Q is a weak supplement of T in G;

(2) if $N \neq T$, we consider two cases: if $TN = G$, then N is a weak supplement of T in G; if $TN < G$, then TN/N is a minimal normal subgroup of G/N and TN/N is weakly-supplemented in G/N, so T is weakly-supplemented in G. □

Lemma 2.6. *Let N be a minimal normal subgroup of G. If every minimal subgroup of N is weakly-supplemented in G, then N is cyclic of prime order.*

P r o o f. By Lemma 2.5, N is supersolvable and so it is an elementary abelian p-group for some prime p. Let $\langle z\rangle$ be a minimal subgroup of N and let H be a weak supplement to $\langle z\rangle$ in G. Since $N = \langle z\rangle(N\cap H)$ and $N\cap H \vartriangleright H$, we have $N\cap H \vartriangleright G$.

Thus by the minimality of N we have $N \cap H = 1$, and therefore N is cyclic of order p. $\square$

3. Main results

In this section, we concentrate on the structure of a finite group under the assumption that some minimal subgroups of $P \cap G'$ are weakly-supplemented.

First we prove the following result about p-nilpotency.

Theorem 3.1. *Let G be a group and p a prime number dividing the order of G. If every minimal subgroup of $P \cap G'$ is weakly-supplemented in $N_G(P)$ and $N_G(P)$ is p-nilpotent, then G is p-nilpotent, where P is a Sylow p-subgroup of G.*

Proof. Assume that the theorem is not true and let G be a counterexample of the smallest order. Then we have the following fact:

$$(*) \qquad P \cap G' \leqslant Z(N_G(P)), \quad \text{where } Z(N_G(P)) \text{ is the center of } N_G(P).$$

In fact, by the hypothesis of the theorem that every minimal subgroup of $P \cap G'$ is weakly-supplemented in $N_G(P)$, we know that $P \cap G'$ is an elementary abelian group. If $P \cap G'{=}1$, then there is nothing to be proved. Now, since $P \cap G' \rhd P$, we may assume that N_1 is a minimal normal subgroup of $P \cap G'$ and $|N_1| = p$. Also by our hypothesis and Lemma 2.2 (1), there is a subgroup K of P such that $P = N_1K$ and $K < P$. Since $N_1 \cap K \trianglelefteq K$, it follows that $N_1 \cap K \trianglelefteq P$. By the minimality of N_1 we have that $N_1 \cap K = 1$. Noticing that $(P \cap G') \cap K$ is still a normal subgroup of P, therefore, by using similar arguments, we can prove that $P \cap G' = N_1 \times N_2 \times \ldots \times N_s$ and $N_i \leqslant Z(P)$. This shows that $P \cap G' \leqslant Z(P)$. By the hypothesis that $N_G(P)$ is p-nilpotent, we have $P \cap G' \leqslant Z(N_G(P))$. This establishes $(*)$.

Since G is not p-nilpotent, G has a subgroup H such that H is an inner-p-nilpotent group. By Lemma 2.4 and according to a result due to Schmidt [10, Theorem 9.1.9, Exercise 9.1.11], H has a normal Sylow p-subgroup H_p such that $H = H_pH_q$ for a Sylow q-subgroup H_q in $H (q \neq p)$. Moreover, $H_p = [H_p, H_q]$. Hence, it follows that $H_p \leqslant H' \leqslant G'$. On the other hand, without loss of generality, we may assume that H_p is contained in P. Hence $H_p \leqslant P \cap G'$.

Let $A = N_G(H_p)$. Since $H_p \leqslant P \cap G'$ and $P \cap G' \leqslant Z(N_G(P))$, we have that H_p is centralized by $N_G(P)$. In particular, $P \leqslant C_G(H_p)$. As $C_G(H_p) \rhd N_G(H_p) = A$ and $P \in \mathrm{Syl}_p(C_G(H_p))$, we have, by the Frattini argument, $A = N_G(H_p) = C_G(H_p)N_A(P)$. Since $H_p \leqslant Z(N_G(P))$ and $N_A(P) \leqslant N_G(P)$, we have $N_A(P) \leqslant C_G(H_p)$. It follows that $N_G(H_p) = C_G(H_p)$ and therefore $H = H_p \times H_q$, which is a contradiction. This proves the theorem. $\square$

The assumption that $N_G(P)$ is p-nilpotent in Theorem 3.1 cannot be removed. In fact, if we let $G = A_5$, the alternating group of degree 5, then it is easy to see that $N_G(P)$ is a subgroup of order 10 for every Sylow 5-subgroup P of G. Hence every minimal subgroup of order 5 in P is weakly-supplemented in $N_G(P)$ for the Sylow 5-subgroup P of G. However, $G = A_5$ is simple. Nonetheless, if we assume that p is the smallest prime number dividing the order of G, the assumption that $N_G(P)$ is p-nilpotent in Theorem 3.1 can be removed.

Theorem 3.2. *Let G be a group and p the smallest prime number dividing the order of G. If every minimal subgroup of $P \cap G'$ is weakly-supplemented in $N_G(P)$, then G is p-nilpotent, where P is a Sylow p-subgroup of G.*

Proof. If $N_G(P) = G$, then, by applying the well known Schur-Zassenhaus Theorem, there exists a Hall p'-subgroup K of G such that $G = PK$. For any prime $q \in \pi(K)$ and $Q \in \mathrm{Syl}_q(K)$, it is easy to show that the group $G_1 = PQ$ satisfies the hypothesis of our theorem. Hence, if $G_1 < G$, then by induction on the order of G we know that G_1 is p-nilpotent. Consequently, K is a normal p-complement of G. So we may assume that K is a q-group for some prime q. Now, the solvability of G implies that $G' < G$. Let T/G' be a Sylow q-group of G/G'. Then $P \cap G'$ is a Sylow p-subgroup of T and every minimal subgroup of $P \cap G'$ is weakly-supplemented in T by Lemma 2.2 (1). If $P \cap G'=1$, then T is a normal p-complement of G. On the other hand, if $P \cap G' \neq 1$, then we claim that T has a normal p-complement N. In fact, let $\langle a \rangle$ be a subgroup of order p in $P \cap G'$, then there is a subgroup K of T such that $T = \langle a \rangle K$ and $K < T$. We can get that $\langle a \rangle \cap K = 1$. Indeed, if $\langle a \rangle \cap K \neq 1$, then $\langle a \rangle \leqslant K$, it follows that $T = \langle a \rangle K = K$, a contradiction, so $\langle a \rangle \cap K = 1$. Since $[T : K] = p$ and p is the smallest prime number dividing the order of T, we know that K is a normal subgroup of T. Obviously every subgroup of K with order p must be a minimal subgroup of T. Then, by Lemma 2.2 (1), every subgroup of K with order p is weakly-supplemented in K. Using induction, we deduce that K has a normal p-complement N. It is clear that N is a normal p-complement in T. Since $T/G' \vartriangleright G/G'$, it is easy to see that N is a normal p-complement of G.

Thus, we conclude that $N_G(P) < G$. As $N_G(P)$ satisfies the hypothesis of the theorem, by induction we can assume that $N_G(P)$ is p-nilpotent. Now applying Theorem 3.1, we have that G is p-nilpotent and therefore the proof is complete. □

Corollary 3.3. *Let G be a group. If every minimal subgroup of $P \cap G'$ is weakly-supplemented in $N_G(P)$ for every Sylow subgroup P of G, then G has a Sylow tower of supersolvable type.*

Proof. We use induction on $|G|$. Let q be the smallest prime dividing $|G|$ and let Q be a Sylow q-subgroup of G. Then, by our hypothesis, every minimal subgroup of $Q \cap G'$ is weakly-supplemented in $N_G(Q)$. By applying Theorem 3.2, we see that G has a normal q-complement K. It is clear that every Sylow subgroup P of K must be a Sylow subgroup of G with $N_K(P) \leqslant N_G(P)$ and $K' \cap P \leqslant G' \cap P$. Now, by our hypotheses and Lemma 2.2 (1), we see that K also satisfies the hypotheses of our corollary. Thus, by using induction, we know that K has a Sylow tower of supersolvable type and so does G. The proof is now completed. □

As an application of Theorem 3.1, we prove

Theorem 3.4. *Let $\mathscr{R}$ be a formation containing $\mathscr{F}$, the class of supersolvable groups. Let H be a normal subgroup of a group G such that $G/H \in \mathscr{R}$. If for every Sylow subgroup P of H, every minimal subgroup of $P \cap G'$ is weakly-supplemented in $N_G(P)$, then G is in $\mathscr{R}$.*

Proof. Suppose that the theorem is false and let G be a minimal counterexample. By Lemma 2.2 (1) and Corollary 3.3, the normal subgroup H of G has a Sylow tower of supersolvable type. Let p be the largest prime number in $\pi(H)$ and $P \in \mathrm{Syl}_p(H)$. Then P must be a normal subgroup of G. Now let $\overline{G} = G/P$ and $\overline{H} = H/P$. Clearly, $\overline{G}/\overline{H} \simeq G/H \in \mathscr{R}$. Observe that $N_{\overline{G}}(\bar{Q}) = N_G(Q)P/P$ for every Sylow q-subgroup $\bar{Q} = QP/P$ of $\overline{H}$, where $Q \in \mathrm{Syl}_q(H)(q \neq p)$, and $(\overline{G})' = G'P/P$. We know that for every element $\bar{x}$ of order q in $\bar{Q} \cap (\overline{G})', \bar{x} = xP$ for some element $x \in Q \cap G'$. Thus, by our hypothesis, there exists a subgroup K of $N_G(Q)$ such that $N_G(Q) = \langle x \rangle K$ and $K < N_G(Q)$. It is clear that $N_{\overline{G}}(\bar{Q}) = \overline{\langle x \rangle}\overline{K}$. If $\langle x \rangle \cap KP \neq 1$, then $\langle x \rangle \leqslant KP$ and therefore $N_G(Q)P = KP$, which is contrary to $K < N_G(Q)$. Hence $\langle x \rangle \cap KP = 1$, and so $\overline{\langle x \rangle} \cap \overline{K} = 1$. Now we have proved that G/P satisfies the hypothesis of the theorem. Therefore, by the minimality of G, we have $G/P \in \mathscr{R}$.

Since G/G' is abelian and $\mathscr{F}$ is contained in $\mathscr{R}$, we have $G/G' \in \mathscr{F}$. It follows that $G/(G' \cap P) \in \mathscr{R}$ and, by our hypothesis, we know that every minimal subgroup of $G' \cap P$ is weakly-supplemented in G since P is normal in G and therefore $G' \cap P$ is an elementary abelian subgroup. Now, let N be a minimal normal subgroup of G such that $N \leqslant G' \cap P$. By Lemma 2.6 we know that N is a cyclic group of order p. We now denote by bars the images in $\overline{G} = G/N$. Then $\overline{G}$ has a normal subgroup $\overline{G' \cap P}$ such that $\overline{G}/\overline{G' \cap P}$ belongs to $\mathscr{R}$. Obviously, $(\overline{G})' \cap \overline{G' \cap P} = (G' \cap P)/N$. We now proceed to prove that every minimal subgroup of $(G' \cap P)/N$ is weakly-supplemented in $\overline{G}$. For this purpose, let $\overline{\langle x \rangle}$ be a minimal subgroup of $\overline{G' \cap P}$. Since $G' \cap P$ is an elementary abelian group, we know that there is an element $x \in G' \cap P$ with order p such that $\overline{\langle x \rangle} = \langle x \rangle N/N$. Since $\langle x \rangle$ is minimal in G, so by the hypothesis there exists a subgroup K of G such that $G = \langle x \rangle K$ and $\langle x \rangle \cap K = 1$. If $N \leqslant K$, then it is clear that $\overline{G} = \overline{\langle x \rangle}\overline{K}$ and $\overline{\langle x \rangle} \cap \overline{K} = 1$. If $N \nleqslant K$, then $G = NK$. It follows that $|(\langle x \rangle N) \cap K| = p$. Denote $(\langle x \rangle N) \cap K = A$. Then A is a minimal subgroup of $G' \cap P$

and $A \leqslant K$. By Lemma 2.2 (1), there is a subgroup K_1 of K such that $K = AK_1$ and $K_1 < K$. It is clear that $AN = \langle x \rangle N$ and therefore $\overline{G} = \overline{\langle x \rangle}\overline{K_1}$ and $\overline{K_1} < \overline{G}$. Hence, $\overline{G}$ satisfies the hypothesis of the theorem. By minimality of G, we have that $\overline{G} = G/N \in \mathscr{R}$.

Now by the hypothesis there is a proper subgroup M of G such that $G = NM$, hence $N \cap M = 1$. For if not, then $N \leqslant M$, $G = NM = M$, which is contrary to $M < G$. It follows that $G' = N(G' \cap M)$ and $G' \cap M \rhd M$. Since N is a cyclic group of order p, $\mathrm{Aut}(N)$ is a cyclic group of order $p-1$. Also, since $G/C_G(N) \leqslant \mathrm{Aut}(N)$, we have $G' \leqslant C_G(N)$. Hence $G' \cap M$ is normal in G. If $G' \cap M \neq 1$, let N_1 be a minimal subgroup of G with $N_1 \leqslant G' \cap M$. Consider the quotient group G/N_1. Since $(G/N_1)/(NN_1/N_1) \simeq G/NN_1 \simeq (G/N)/(NN_1/N) \in \mathscr{R}$ and noticing that every minimal subgroup of $(G/N_1)' \cap (NN_1/N_1) = NN_1/N_1$ is weakly-supplemented in G/N_1, by the minimality of G we have that $G/N_1 \in \mathscr{R}$. Hence $G = G/(N \cap N_1) \in \mathscr{R}$ by the definition of formation. Therefore we may assume that $G' \cap M = 1$. Then $G' = N$ and G/N is abelian. It follows that G is supersolvable and therefore $G \in \mathscr{R}$ since $\mathscr{F} \subseteq \mathscr{R}$, which is a contradiction. The proof of the theorem is now completed. □

Remark 3.5. Theorem 3.4 is true for any formation containing the class of supersolvable groups. But if the formation $\mathscr{R}$ does not contain $\mathscr{F}$ (the class of supersolvable groups), Theorem 3.4 is not true. For example, if we let $\mathscr{N}$ be the saturated formation of all nilpotent groups, then the symmetric group of degree three is a counterexample.

We can choose the normal subgroup H of G in Theorem 3.4 to get some results of special interest. For example, if we choose $H = G'$ in Theorem 3.4, we have the following results:

Corollary 3.6. *Let G be a group. If for every Sylow subgroup P of G', every minimal subgroup of P is weakly-supplemented in $N_G(P)$, then G is supersolvable.*

If G is assumed to be a solvable group, then the number of weakly-supplemented minimal subgroups in Theorem 3.4 can be further reduced. In fact, we have the following theorem.

Theorem 3.7. *Let $\mathscr{R}$ be a formation containing $\mathscr{F}$, the class of supersolvable groups. Let H be a normal subgroup of a solvable group G such that $G/H \in \mathscr{R}$. If every minimal subgroup of the Fitting subgroup $F(G' \cap H)$ of $G' \cap H$ is weakly-supplemented in G, then G belongs to $\mathscr{R}$.*

Remark 3.8. Since $F(G' \cap H) = G' \cap F(H) = (G' \cap P_1) \times (G' \cap P_2) \times \ldots \times (G' \cap P_k)$, we know that every minimal subgroup of $F(G' \cap H)$ in Theorem 3.7 is still a minimal subgroup of some $G' \cap P_i$, where P_i is the Sylow p_i-subgroup of $F(H)$ for some prime p_i.

Proof of Theorem 3.7. Assume that the theorem is false and let G be a counterexample of the smallest order. Since G/G' is abelian, we have that $G/G' \in \mathscr{R}$ since $\mathscr{F} \subseteq \mathscr{R}$ and so $G/(H \cap G') \in \mathscr{R}$. Hence, we can prove our theorem by replacing $G' \cap H$ by H and assume that $H \leqslant G'$.

We first prove that $\Phi(G) = 1$. If $\Phi(G) \neq 1$, then there is a prime number q dividing the order of $\Phi(G)$ and $Q \in \mathrm{Syl}_q(\Phi(G))$. Since Q is a characteristic subgroup of $\Phi(G)$ and $\Phi(G) \trianglerighteq G$, we know that Q is a normal subgroup of G. Observe that $(G/Q)' = G'Q/Q$, so we still have $HQ/Q \leqslant (G/Q)'$. Clearly, $(G/Q)/(HQ/Q) \simeq G/HQ \in \mathscr{R}$. By [7, Satz 3.5, page 270], $F(HQ/Q) = F(HQ)/Q$ and therefore by [2, Lemma 3.1], we have that $F(HQ) = F(H)Q$. It follows that $F(HQ/Q) = F(H)Q/Q$. Thus, for any minimal subgroup $\bar{A}$ of $F(HQ/Q)$ we can find a minimal subgroup $A \leqslant F(H)$ such that $\bar{A} = AQ/Q$. By the hypothesis of the theorem, there exists a subgroup K of G such that $G = AK$ and $K < G$. The minimality of A implies that K has a prime index in G and so K is a maximal subgroup of G. It follows that $Q \leqslant K$ and therefore $(K/Q) \cap (AQ/Q) = 1$. It is clear that $G/Q = (AQ/Q) \cdot K/Q$. Thus, we have shown that $G/Q \in \mathscr{R}$. If $Q \cap H \neq 1$, then let A be a minimal subgroup of $Q \cap H$. By the hypothesis, since $Q \cap H \leqslant F(H)$, there is a subgroup K of G such that $G = AK$ and $K < G$. But the fact that $A \leqslant Q \cap H \leqslant \Phi(G)$ implies that $G = K$, a contradiction. So $Q \cap H = 1$. Hence $G \simeq G/H \cap Q \in \mathscr{R}$, a contradiction. Thus $\Phi(G) = 1$.

Next, by applying a result of Deyu Li and Xiuyun Guo in [9, Lemma 2.3], we deduce that $F(G) = M_1 \times M_2 \times \ldots \times M_s \times N_1 \times N_2 \times \ldots \times N_t$ where M_i and $N_j (i = 1, 2, \ldots, s,\ j = 1, 2, \ldots, t)$ are minimal normal subgroups of G, $M_i \cap H = 1$ and $F(H) = N_1 \times \ldots \times N_t$.

Since every minimal subgroup of N_j is weakly-supplemented in G, by Lemma 2.6 N_j is a cyclic group of prime order ($j = 1, 2, \ldots, t$). Then it follows that $G/C_G(N_j)$ is an abelian group and therefore $G' \leqslant C_G(N_j)$. Hence $H \leqslant G' \leqslant C_G(F(H))$. The solvability of G implies that $H \cap C_G(F(H)) = C_H(F(H)) \leqslant F(H)$. It follows that $H = F(H) = C_H(F(H))$.

Now consider the quotient group G/N_j. Then it is clear that $H/N_j \leqslant (G/N_j)' = G'/N_j$ and $(G/N_j)/(H/N_j) \simeq G/H \in \mathscr{R}$. Since $H = F(H)$, we have that $F(H/N_j) = H/N_j$. Let $\overline{\langle x \rangle}$ be a minimal subgroup of H/N_j. It is easy to prove that there is a minimal subgroup $\langle x \rangle$ of H such that $\overline{\langle x \rangle} = \langle x \rangle N_j/N_j$. By the hypothesis there is a subgroup K of G such that $G = \langle x \rangle K$ and $K < G$. Then $\langle x \rangle \cap K = 1$; if not, we have that $\langle x \rangle \leqslant K$, then $G = \langle x \rangle K = K$, a contradiction. If $(|\langle x \rangle|, |N_j|) = 1$, then it is clear that $N_j \leqslant K$ and therefore $(\langle x \rangle N_j/N_j)(K/N_j) = G$ and $(\langle x \rangle N_j/N_j) \cap$

$(K/N_j) = 1$. If $(|\langle x\rangle|, |N_j|) \neq 1$, by using the arguments similar to those in the proof of Theorem 3.4, we know that $\overline{\langle x\rangle}$ has a weak supplement in G/N_j. The minimality of G implies that $G/N_j \in \mathscr{R}$. If $t \neq 1$, then $G \simeq G/(N_1 \cap N_2) \in \mathscr{R}$, a contradiction. Hence we may assume that $H = N_1$ is a minimal subgroup.

By the hypothesis, there is a subgroup K of G such that $G = HK$ and $K < G$. By the above we know that $H \cap K = 1$. Then $F(G) = H(K \cap F(G))$ and $K \cap F(G)$ is a normal subgroup of G. So if $K \cap F(G) \neq 1$, we may assume that $M_1 \leqslant K \cap F(G)$ and consider the quotient group G/M_1. Then it is clear that $HM_1/M_1 \leqslant (G/M_1)' = G'M_1/M_1$, $(G/M_1)/(HM_1/M_1) \simeq G/HM_1 \simeq (G/H)/(HM_1/H) \in \mathscr{R}$ and the minimal subgroup of HM_1/M_1 has a weak supplement K/M_1 in G/M_1. The minimality of G implies that $G/M_1 \in \mathscr{R}$ and therefore $G \simeq G/(H \cap M_1) \in \mathscr{R}$, a contradiction. Hence $F(G) = F(H) = N$ is a minimal subgroup. It follows that $G' \leqslant H = N$ since $G' \leqslant C_G(F(H)) = C_G(F(G)) \leqslant F(G)$. Now we have that G/N is an abelian group and therefore G is supersolvable. Hence $G \in \mathscr{R}$ since $\mathscr{F} \subseteq \mathscr{R}$. The proof of the theorem is complete. □

Similarly to Corollary 3.6, we can choose the normal subgroup H of G in Theorem 3.7 to get some results of special interest. For example, if we choose $H = G'$ in Theorem 3.7, we have the following result:

Corollary 3.9. *Let G be a solvable group. If every minimal subgroup of the Fitting subgroup $F(G')$ of G' is weakly-supplemented in G, then G is supersolvable.*

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Authors' addresses: Qingjun Kong (corresponding author), Department of Mathematics, Tianjin Polytechnic University, Tianjin 300387, People's Republic of China, e-mail: kqj2929@163.com; Qingfeng Liu, Department of Basic Sciences, Shandong Water Polytechnic, Rizhao 276826, People's Republic of China.