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A GENERALIZATION OF THE FINITENESS PROBLEM OF THE LOCAL COHOMOLOGY MODULES

Ahmad Abbasi, Hajar Roshan-Shekalgourabi, Rasht

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Abstract. Let R be a commutative Noetherian ring and $\mathfrak{a}$ an ideal of R. We introduce the concept of $\mathfrak{a}$-weakly Laskerian R-modules, and we show that if M is an $\mathfrak{a}$-weakly Laskerian R-module and s is a non-negative integer such that $\operatorname{Ext}^j_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-weakly Laskerian for all $i < s$ and all j, then for any $\mathfrak{a}$-weakly Laskerian submodule X of $H^s_{\mathfrak{a}}(M)$, the R-module $\operatorname{Hom}_R(R/\mathfrak{a}, H^s_{\mathfrak{a}}(M)/X)$ is $\mathfrak{a}$-weakly Laskerian. In particular, the set of associated primes of $H^s_{\mathfrak{a}}(M)/X$ is finite. As a consequence, it follows that if M is a finitely generated R-module and N is an $\mathfrak{a}$-weakly Laskerian R-module such that $H^i_{\mathfrak{a}}(N)$ is $\mathfrak{a}$-weakly Laskerian for all $i < s$, then the set of associated primes of $H^s_{\mathfrak{a}}(M, N)$ is finite. This generalizes the main result of S. Sohrabi Laleh, M. Y. Sadeghi, and M. Hanifi Mostaghim (2012).

Keywords: local cohomology module; weakly Laskerian module; $\mathfrak{a}$-weakly Laskerian module; associated prime

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1. Introduction

Throughout the paper, R is a commutative Noetherian ring with identity and all modules are unitary. Also, $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of R and $V(\mathfrak{a})$ is the set of all prime ideals of R containing $\mathfrak{a}$. Let M be an R-module. For each $i \geqslant 0$, the i-th local cohomology module of M with respect to $\mathfrak{a}$ is defined as

$$H^i_{\mathfrak{a}}(M) = \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M).$$

For the basic properties of local cohomology the reader can refer to [5] of Brodmann and Sharp. An important problem in commutative algebra is to determine when the set of associated primes of the local cohomology module $H^i_{\mathfrak{a}}(M)$ is finite. Huneke [10] raised the following conjecture:

If M is a finitely generated R-module, then the set of associated primes of $H_{\mathfrak{a}}^{i}(M)$ is finite for every ideal $\mathfrak{a}$ and every $i \geqslant 0$.

Singh [17] and Katzman [11] have given counterexamples to this conjecture. However, this problem has been studied by many authors and it is shown that this conjecture is true in many situations; for examples see [3], [4], [6], [8] and [12]. In particular, it is shown in [13, Theorem B] that if for a finitely generated R-module M and an integer s, the local cohomology modules $H_{\mathfrak{a}}^{i}(M)$ are finitely generated for all $i < s$, then the set $\mathrm{Ass}_R H_{\mathfrak{a}}^{s}(M)$ is finite. After a few months, it has been shown in [4, Theorem 2.2] that under this assumptions, $\mathrm{Ass}_R H_{\mathfrak{a}}^{s}(M)/N$ is finite for any finitely generated submodule N of $H_{\mathfrak{a}}^{s}(M)$. There are several papers devoted to the extension of the above results to more general situations. For example, Dibaei and Yassemi in [6, Theorem 2.1] showed that for a finitely generated R-module M and an integer s, if $\mathrm{Ext}_R^{j}(R/\mathfrak{a}, H_{\mathfrak{a}}^{i}(M))$ is finitely generated for all j and $i < s$, then $\mathrm{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{s}(M))$ is finitely generated and so the set $\mathrm{Ass}_R H_{\mathfrak{a}}^{s}(M)$ is finite. The generalizations of this result to the class of minimax modules, weakly Laskerian modules and $\mathfrak{a}$-minimax modules have been proved in [3], [8] and [1], respectively. This paper is concerned with what might be considered a generalization of the above-mentioned results. To do this, in Section 2 we introduce a new Serre class of R-modules which contains all weakly Laskerian modules. We define M to be an $\mathfrak{a}$-weakly Laskerian R-module if for any submodule N of M, the set $\mathrm{Ass}_R \Gamma_{\mathfrak{a}}(M/N)$ is finite, and we give some properties of $\mathfrak{a}$-weakly Laskerian modules.

In Section 3, we will prove that if M is an $\mathfrak{a}$-weakly Laskerian R-module and s is a non-negative integer such that $\mathrm{Ext}_R^{j}(R/\mathfrak{a}, H_{\mathfrak{a}}^{i}(M))$ is $\mathfrak{a}$-weakly Laskerian for all $i < s$ and all j, then for any $\mathfrak{a}$-weakly Laskerian submodule X of $H_{\mathfrak{a}}^{s}(M)$, the R-module $\mathrm{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{s}(M)/X)$ is $\mathfrak{a}$-weakly Laskerian. In particular, the set of associated primes of $H_{\mathfrak{a}}^{s}(M)/X$ is finite. This is a generalization of [4, Theorem 2.2], [6, Theorem 2.1], [3, Theorem 2.2], [13, Theorem B(β)], [1, Theorem 4.2] and [16, Proposition 3.1].

It is shown in [14] that if M and N are finitely generated R-modules such that $\mathrm{Supp}_R M \subseteq V(\mathfrak{a})$ and $H_{\mathfrak{a}}^{i}(N)$ is minimax for all $i < s$, then the set of associated prime ideals of the generalized local cohomology module $H_{\mathfrak{a}}^{s}(M, N)$ is finite. As a consequence of the main result, in Theorem 3.5 we extend this result to any finitely generated R-module M not necessarily $\mathfrak{a}$-torsion and with the $\mathfrak{a}$-weakly Laskerian condition on N and $H_{\mathfrak{a}}^{i}(N)$ instead of finitely generated and minimax conditions.

2. $\mathfrak{a}$-weakly Laskerian modules

Recall that for an R-module M, the *Goldie dimension of* M is defined as the cardinal number of the set of indecomposable submodules of $E(M)$ which appear in a decomposition of $E(M)$ into a direct sum of indecomposable submodules. We use $\operatorname{Gdim} M$ to denote the Goldie dimension of M. For a prime ideal $\mathfrak{p}$, let $\mu^0(\mathfrak{p}, M)$ denote the 0-th Bass number of M with respect to the prime ideal $\mathfrak{p}$. It is known that $\mu^0(\mathfrak{p}, M) > 0$ if and only if $\mathfrak{p} \in \operatorname{Ass}_R M$. So, by definition of the Goldie dimension it follows that

$$\operatorname{Gdim} M = \sum_{\mathfrak{p} \in \operatorname{Ass}_R M} \mu^0(\mathfrak{p}, M).$$

Also, for any ideal $\mathfrak{a}$ of R, the $\mathfrak{a}$-*relative Goldie dimension of* M, which is introduced in [7], is defined as

$$\operatorname{Gdim}_{\mathfrak{a}} M := \sum_{\mathfrak{p} \in \operatorname{Ass}_R M \cap V(\mathfrak{a})} \mu^0(\mathfrak{p}, M).$$

In [18], Zöschinger introduced the class of minimax modules. An R-module M is called *minimax* if there is a finite submodule N of M such that M/N is Artinian. It is shown in [18] that when R is a Noetherian ring, an R-module M is minimax if and only if for any submodule N of M, $\operatorname{Gdim} M/N < \infty$. Later, authors in [1] introduced the concept of an $\mathfrak{a}$-minimax module. An R-module M is said to be $\mathfrak{a}$-minimax if for any submodule N of M, $\operatorname{Gdim}_{\mathfrak{a}} M/N < \infty$.

On the other hand, an R-module M is called *weakly Laskerian* if for any submodule N of M, the set $\operatorname{Ass}_R M/N$ is finite.

Note that $\operatorname{Gdim}_{\mathfrak{a}} M = \operatorname{Gdim} \Gamma_{\mathfrak{a}}(M)$ by [7, Lemma 2.6]. This motivates the following definition:

Definition 2.1. An R-module M is said to be $\mathfrak{a}$-weakly Laskerian if the set of associated primes of the $\mathfrak{a}$-torsion submodule of any quotient module of M is finite; i.e., for any submodule N of M, the set $\operatorname{Ass}_R \Gamma_{\mathfrak{a}}(M/N)$ is finite.

We claim that the class of $\mathfrak{a}$-weakly Laskerian modules is strictly larger than the class of weakly Laskerian modules. To see this, consider the $\mathbb{Z}$-module $M = \bigoplus_{p \in \Omega} \mathbb{Z}/p\mathbb{Z}$, where Ω is the set of all prime integers. It is easy to see that $\operatorname{Ass}_{\mathbb{Z}}(M) = \{p\mathbb{Z};\ p \in \Omega\}$. So, M has infinitely many associated prime ideals. Hence, M is not weakly Laskerian, but if q is a fixed prime integer, then $\operatorname{Ass}_{\mathbb{Z}} \Gamma_{q\mathbb{Z}}(M/N) = \{q\mathbb{Z}\}$ for any submodule N of M. Therefore, M is a $q\mathbb{Z}$-weakly Laskerian $\mathbb{Z}$-module.

Let $\mathfrak{a}$ be an ideal of R and M an R-module. Any weakly Laskerian module is $\mathfrak{a}$-weakly Laskerian. So, any Noetherian and any Artinian R-module is $\mathfrak{a}$-weakly Laskerian. Also, any $\mathfrak{a}$-minimax R-module is $\mathfrak{a}$-weakly Laskerian. In particular, any minimax R-module is $\mathfrak{a}$-weakly Laskerian. If either $\mathfrak{a}=0$ or M is $\mathfrak{a}$-torsion, then M is $\mathfrak{a}$-weakly Laskerian if and only if M is weakly Laskerian. If $\mathfrak{b}$ is a second ideal of R such that $\mathfrak{a} \subseteq \mathfrak{b}$ and M is $\mathfrak{a}$-weakly Laskerian, then M is $\mathfrak{b}$-weakly Laskerian.

Now, we state an important property of the class of $\mathfrak{a}$-weakly Laskerian modules.

Proposition 2.2. *Let $\mathfrak{a}$ be an ideal of R and let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. Then M is $\mathfrak{a}$-weakly Laskerian if and only if both M' and M'' are $\mathfrak{a}$-weakly Laskerian.*

Proof. We may assume that M' is a submodule of M and $M'' = M/M'$. If M is $\mathfrak{a}$-weakly Laskerian, it is easy to see that M' and M'' are $\mathfrak{a}$-weakly Laskerian. Now, suppose that M' and M/M' are $\mathfrak{a}$-weakly Laskerian. Let N be an arbitrary submodule of M. Then the exact sequence

$$0 \to \frac{M'+N}{N} \to \frac{M}{N} \to \frac{M}{M'+N} \to 0$$

induces the exact sequence

$$0 \to \Gamma_{\mathfrak{a}}\Big(\frac{M'}{M' \cap N}\Big) \to \Gamma_{\mathfrak{a}}\Big(\frac{M}{N}\Big) \to \Gamma_{\mathfrak{a}}\Big(\frac{M}{M'+N}\Big).$$

Now, since $\mathrm{Ass}_R\Gamma_{\mathfrak{a}}(M/N) \subseteq \mathrm{Ass}_R\Gamma_{\mathfrak{a}}(M'/M' \cap N) \cup \mathrm{Ass}_R\Gamma_{\mathfrak{a}}(M/M'+N)$ and the sets $\mathrm{Ass}_R\Gamma_{\mathfrak{a}}(M'/M' \cap N)$ and $\mathrm{Ass}_R\Gamma_{\mathfrak{a}}(M/M'+N)$ are finite, it follows that the set $\mathrm{Ass}_R\Gamma_{\mathfrak{a}}(M/N)$ is finite, and so M is $\mathfrak{a}$-weakly Laskerian. □

Corollary 2.3. *Let $\mathfrak{a}$ be an ideal of R.*

(i) *The class of $\mathfrak{a}$-weakly Laskerian modules is closed under taking submodules, quotients and extensions, i.e., it is a Serre subcategory of the category of all R-modules. In particular, any finite sum of $\mathfrak{a}$-weakly Laskerian modules is $\mathfrak{a}$-weakly Laskerian.*

(ii) *Let M and N be two R-modules. If M is finitely generated and N is $\mathfrak{a}$-weakly Laskerian, then $\mathrm{Ext}^i_R(M,N)$ and $\mathrm{Tot}^R_i(M,N)$ are $\mathfrak{a}$-weakly Laskerian for all $i \geqslant 0$.*

Corollary 2.4. *If M is an $\mathfrak{a}$-weakly Laskerian R-module, then $\Gamma_{\mathfrak{a}}(M)$ is weakly Laskerian.*

Proposition 2.5. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R. Let M be an $\mathfrak{a}$-weakly Laskerian R-module such that $\mathrm{Ass}_R M \subseteq V(\mathfrak{b})$. Then $H^i_{\mathfrak{b}}(M)$ is $\mathfrak{a}$-weakly Laskerian for all $i \geqslant 0$.*

P r o o f. For $i=0$, $H^0_{\mathfrak b}(M)=\Gamma_{\mathfrak b}(M)$ is $\mathfrak a$-weakly Laskerian by Proposition 2.2. Since $\mathrm{Ass}_R M/\Gamma_{\mathfrak b}(M)\subseteq \mathrm{Ass}_R M$, it follows from $\mathrm{Ass}_R M\subseteq V(\mathfrak b)$ that $M=\Gamma_{\mathfrak b}(M)$. Hence, by [5, Corollary 2.1.7 (ii)], $H^i_{\mathfrak b}(M)=0$ for all $i>0$. So, $H^i_{\mathfrak b}(M)$ is $\mathfrak a$-weakly Laskerian for all $i\geqslant 0$, as required. $\square$

Remark 2.6. It is clear that for any ideals $\mathfrak a$ and $\mathfrak b$ of R,

$$\begin{aligned}\mathrm{Ass}_R\Gamma_{\mathfrak a}(\mathrm{Hom}_R(R/\mathfrak b,M)) &= \mathrm{Ass}_R\mathrm{Hom}_R(R/\mathfrak b,\Gamma_{\mathfrak a}(M))\\ &= \mathrm{Ass}_R\Gamma_{\mathfrak a}(M)\cap V(\mathfrak b).\end{aligned}$$

So, if $(0:_M \mathfrak b)$ is an $\mathfrak a$-weakly Laskerian R-module and $\mathrm{Supp}_R M\cap V(\mathfrak a)\subseteq V(\mathfrak b)$, then the set $\mathrm{Ass}_R\Gamma_{\mathfrak a}(M)$ is finite.

The following theorem is useful for the proof of the main result of the paper.

Theorem 2.7. *Let $\mathfrak a$ be an ideal of R, M a finitely generated R-module and N an arbitrary R-module. Suppose that for some $s\geqslant 0$, $\mathrm{Ext}^i_R(M,N)$ is $\mathfrak a$-weakly Laskerian for all $i\leqslant s$. Then for any finitely generated R-module L with $\mathrm{Supp}_R L\subseteq \mathrm{Supp}_R M$, $\mathrm{Ext}^i_R(L,N)$ is $\mathfrak a$-weakly Laskerian for all $i\leqslant s$.*

P r o o f. By Gruson's Theorem, since $\mathrm{Supp}L\subseteq \mathrm{Supp}M$, there exists a finite filtration

$$0=L_0\subset L_1\subset\ldots\subset L_n=L$$

of submodules of L such that each of the factors L_j/L_{j-1} is a homomorphic image of a direct sum of finitely many copies of M. In view of the short exact sequence $0\to L_{j-1}\to L_j\to L_j/L_{j-1}\to 0$ for $j=1,\ldots,n$ and induction on n, it suffices to prove the case when $n=1$. So, for some positive integer t and some finitely generated R-module K we have an exact sequence

$$0\to K\to M^t\to L\to 0.$$

This induces the long exact sequence

$$\ldots\to \mathrm{Ext}^{s-1}_R(K,N)\to \mathrm{Ext}^s_R(L,N)\to \mathrm{Ext}^s_R(M^t,N)\to\ldots\ .$$

Now, we use induction on s. If $s=0$, the result holds by Corollary 2.3. So, assume that $\mathrm{Ext}^i_R(L',N)$ is $\mathfrak a$-weakly Laskerian for all $i<s$ and all finitely generated R-modules L' with $\mathrm{Supp}_R L'\subseteq \mathrm{Supp}_R M$. Since $\mathrm{Supp}_R K\subseteq \mathrm{Supp}_R M$, by induction hypothesis on s we have $\mathrm{Ext}^{s-1}_R(K,N)$ is $\mathfrak a$-weakly Laskerian. As $\mathrm{Ext}^s_R(M^t,N)\cong \mathrm{Ext}^s_R(M,N)^t$, the result follows from Corollary 2.3. $\square$

Corollary 2.8. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R, let M be a finitely generated R-module and N an arbitrary R-module. Then the following conditions are equivalent:*

(i) $\operatorname{Ext}^i_R(M/\mathfrak{b}M, N)$ *is $\mathfrak{a}$-weakly Laskerian for all $i \geqslant 0$;*
(ii) $\operatorname{Ext}^i_R(M/\mathfrak{c}M, N)$ *is $\mathfrak{a}$-weakly Laskerian for all $i \geqslant 0$ and all ideals $\mathfrak{c} \supseteq \mathfrak{b}$;*
(iii) $\operatorname{Ext}^i_R(L/\mathfrak{b}L, N)$ *is $\mathfrak{a}$-weakly Laskerian for all $i \geqslant 0$ and any finitely generated R-module L with $\operatorname{Supp}_R L \subseteq \operatorname{Supp}_R M$;*
(iv) $\operatorname{Ext}^i_R(M/\mathfrak{p}M, N)$ *is $\mathfrak{a}$-weakly Laskerian for all $i \geqslant 0$ and every minimal prime ideal $\mathfrak{p}$ over $\mathfrak{b}$.*

P r o o f. In view of Theorem 2.7, it suffices to show that (iv) implies (i). Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal primes over $\mathfrak{b}$ and $T = M/\mathfrak{p}_1M \oplus \ldots \oplus M/\mathfrak{p}_nM$. Then by Corollary 2.3, $\operatorname{Ext}^i_R\Big(\bigoplus_{j=1}^n M/\mathfrak{p}_jM, N\Big) \cong \bigoplus_{j=1}^n \operatorname{Ext}^i_R(M/\mathfrak{p}_jM, N)$ is $\mathfrak{a}$-weakly Laskerian. Since

$$\operatorname{Supp}_R\left(\bigoplus_{j=1}^n M/\mathfrak{p}_jM\right) = \operatorname{Supp}_R M/\mathfrak{b}M,$$

it follows from Theorem 2.7 that $\operatorname{Ext}^i_R(M/\mathfrak{b}M, N)$ is $\mathfrak{a}$-weakly Laskerian, as required. $\square$

Corollary 2.9. *Let $\mathfrak{a}$ be an ideal of R, M a finitely generated R-module and N an arbitrary R-module. Then the set*

$$\mathfrak{B} = \{\mathfrak{b} \trianglelefteq R;\ \operatorname{Ext}^i_R(M/\mathfrak{b}M, N) \text{ is } \mathfrak{a}\text{-weakly Laskerian for all } i \geqslant 0\}$$

is closed under multiplication of ideals.

P r o o f. Let $\mathfrak{b}, \mathfrak{c} \in \mathfrak{B}$. The exact sequence

$$0 \to \mathfrak{b}M/\mathfrak{bc}M \to M/\mathfrak{bc}M \to M/\mathfrak{b}M \to 0$$

induces the long exact sequence

$$\ldots \to \operatorname{Ext}^i_R(M/\mathfrak{b}M, N) \to \operatorname{Ext}^i_R(M/\mathfrak{bc}M, N) \to \operatorname{Ext}^i_R(\mathfrak{b}M/\mathfrak{bc}M, N) \to \ldots$$

for all $i \geqslant 0$. So, by the assumption and Corollaries 2.3 and 2.8, it follows that $\mathfrak{bc} \in \mathfrak{B}$. $\square$

3. Finiteness of the associated primes of local cohomology modules

In this section, we use the arguments of the previous section and we prove our main theorem. Next, we give a finiteness result about the generalized local cohomology modules that generalizes the main result of [14].

Proposition 3.1. *Let $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ be ideals of R with $\mathfrak{b} \subseteq \mathfrak{c}$ and let M be an R-module. Let s be a non-negative integer such that $\operatorname{Ext}^s_R(R/\mathfrak{c}, M)$ and $\operatorname{Ext}^j_R(R/\mathfrak{c}, H^i_{\mathfrak{b}}(M))$ are $\mathfrak{a}$-weakly Laskerian for all $i < s$ and all j. Then for any submodule X of $H^s_{\mathfrak{b}}(M)$ such that $\operatorname{Ext}^1_R(R/\mathfrak{c}, X)$ is $\mathfrak{a}$-weakly Laskerian and any finitely generated R-module T with $\operatorname{Supp_R T} \subseteq \operatorname{V}(\mathfrak{c})$, the R-module $\operatorname{Hom}_R(T, H^s_{\mathfrak{b}}(M)/X)$ is $\mathfrak{a}$-weakly Laskerian. In particular, the set $\operatorname{Ass}_R H^s_{\mathfrak{b}}(M) \cap V(\mathfrak{c})$ is finite.*

P r o o f. The exact sequence

$$0 \to X \to H^s_{\mathfrak{b}}(M) \to H^s_{\mathfrak{b}}(M)/X \to 0$$

induces the following exact sequence:

$$\ldots \to \operatorname{Hom}_R(T, H^s_{\mathfrak{b}}(M)) \to \operatorname{Hom}_R(T, H^s_{\mathfrak{b}}(M)/X) \to \operatorname{Ext}^1_R(T, X) \to \ldots .$$

So, by Corollary 2.3 and Gruson's theorem, it is enough to show that $\operatorname{Hom}_R(R/\mathfrak{c}, H^s_{\mathfrak{b}}(M))$ is $\mathfrak{a}$-weakly Laskerian. For do this, we use induction on s. If $s = 0$, the assertion is clear since

$$\operatorname{Hom}_R(R/\mathfrak{c}, H^0_{\mathfrak{b}}(M)) = \operatorname{Hom}_R(R/\mathfrak{c}, M).$$

Now, suppose that $s > 0$ and that the claim has been proved for $s - 1$. Since $\operatorname{Ext}^j_R(R/\mathfrak{c}, \Gamma_{\mathfrak{b}}(M))$ is $\mathfrak{a}$-weakly Laskerian for all $j \geqslant 0$, in view of the exact sequence

$$0 \to \Gamma_{\mathfrak{b}}(M) \to M \to M/\Gamma_{\mathfrak{b}}(M) \to 0,$$

it follows that $\operatorname{Ext}^s_R(R/\mathfrak{c}, M/\Gamma_{\mathfrak{b}}(M))$ is $\mathfrak{a}$-weakly Laskerian and

$$H^i_{\mathfrak{b}}(M) \cong H^i_{\mathfrak{b}}(M/\Gamma_{\mathfrak{b}} M).$$

So, we may assume that $\Gamma_{\mathfrak{b}}(M) = 0$. Let E be the injective hull of M and let $K = E/M$. Hence, $H^0_{\mathfrak{b}}(E) = 0$. Therefore, $\operatorname{Hom}_R(R/\mathfrak{c}, E) = 0$. So, we obtain the isomorphisms $\operatorname{Ext}^{i-1}_R(R/\mathfrak{c}, L) \cong \operatorname{Ext}^i_R(R/\mathfrak{c}, M)$ and $H^{i-1}_{\mathfrak{b}}(L) \cong H^i_{\mathfrak{b}}(M)$ for all $i > 0$. Now, by induction hypothesis it follows that $\operatorname{Hom}_R(R/\mathfrak{c}, H^{s-1}_{\mathfrak{b}}(L))$ is $\mathfrak{a}$-weakly Laskerian and so $\operatorname{Hom}_R(R/\mathfrak{c}, H^s_{\mathfrak{b}}(M))$ is $\mathfrak{a}$-weakly Laskerian. □

Now we are in position to state our main result which is a generalization of [4, Theorem 2.2], [3, Theorem 2.2] and [1, Theorem 4.2].

Theorem 3.2. *Let M be an $\mathfrak{a}$-weakly Laskerian R-module and s be a non-negative integer such that $\operatorname{Ext}^j_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is $\mathfrak{a}$-weakly Laskerian for all $i < s$ and all j. Then for any $\mathfrak{a}$-weakly Laskerian submodule X of $H^s_{\mathfrak{a}}(M)$, $\operatorname{Hom}_R(R/\mathfrak{a}, H^s_{\mathfrak{a}}(M)/X)$ is an $\mathfrak{a}$-weakly Laskerian R-module. In particular, the set of associated primes of $H^s_{\mathfrak{a}}(M)/X$ is finite.*

Proof. It follows from Proposition 3.1 when $\mathfrak{a} = \mathfrak{b} = \mathfrak{c}$. Note that

$$\operatorname{Ass}_R(\operatorname{Hom}_R(R/\mathfrak{a}, H^s_{\mathfrak{a}}(M)/X)) = \operatorname{Ass}_R(H^s_{\mathfrak{a}}(M)/X).$$

□

Remark 3.3. We recall that an R-module M is said to be an FSF module if there is a finitely generated submodule N of M such that $\operatorname{Supp}_R M/N$ is a finite set (see [16]). It is easy to see that the class of weakly Laskerian modules includes the class of FSF modules and also the class of FSF modules includes the class of minimax modules. So, the main theorem of [16] is a direct consequence of [8, Corollary 2.7]. Moreover, clearly any R-module with finite support is weakly Laskerian. Hence, the following result is a generalization of [8, Corollary 2.7] and [13, Theorem B(β)].

Corollary 3.4. *Let M be an $\mathfrak{a}$-weakly Laskerian R-module and s a non-negative integer such that $H^i_{\mathfrak{a}}(M)$ is weakly Laskerian for all $i < s$. Then the set of associated primes of $H^s_{\mathfrak{a}}(M)$ is finite.*

Now, we state our final result about finiteness of the set of associated prime ideals of generalized local cohomology modules. The notion of the *generalized local cohomology* of two R-modules on a local ring $(R, \mathfrak{m})$ was first introduced by J. Herzog in [9]. Afterward Bijan-Zadeh [2] generalized it to a system of ideals of an arbitrary commutative Noetherian ring. For each $i \in \mathbb{N}_0$, the functor $H^i_{\mathfrak{a}}(-,-)$ is defined by

$$H^i_{\mathfrak{a}}(M, N) = \varinjlim_{n \in \mathbb{N}} \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N)$$

for all R-modules M and N. Clearly, this notion is a generalization of the ordinary local cohomology module $H^i_{\mathfrak{a}}(N)$, which corresponds to the case that $M = R$.

In [15] Mafi shows that if $\mathfrak{a}$ is an ideal of R, and M is a finitely generated R-module, then for every R-module N and any positive integer s we have

$$\operatorname{Ass}_R(H^s_{\mathfrak{a}}(M, N)) \subseteq \bigcup_{i=0}^{s} \operatorname{Ass}_R(\operatorname{Ext}^i_R(M, H^{s-i}_{\mathfrak{a}}(N))).$$

By virtue of this result we prove the following theorem which is a generalization of [14, Theorem 7].

Theorem 3.5. *Let M be a finitely generated R-module and N an $\mathfrak{a}$-weakly Laskerian R-module. Assume that s is a non-negative integer such that $H^i_{\mathfrak{a}}(N)$ is $\mathfrak{a}$-weakly Laskerian for all $i < s$. Then the set of associated primes of $H^s_{\mathfrak{a}}(M, N)$ is finite.*

Proof. It is enough to show that $\mathrm{Ext}^i_R(M, H^{s-i}_{\mathfrak{a}}(N))$ has finitely many associated prime ideals for all $0 \leqslant i \leqslant s$. By the assumption and Corollary 2.3, $\mathrm{Ext}^j_R(M, H^i_{\mathfrak{a}}(N))$ is $\mathfrak{a}$-weakly Laskerian for all $i < s$ and all j. Therefore, we see $\mathrm{Ass}_R(\mathrm{Ext}^i_R(M, H^{s-i}_{\mathfrak{a}}(N)))$ is finite for all $1 \leqslant i \leqslant s$ by the fact that

$$\Gamma_{\mathfrak{a}}(\mathrm{Ext}^j_R(M, H^i_{\mathfrak{a}}(N))) = \mathrm{Ext}^j_R(M, H^i_{\mathfrak{a}}(N))$$

for all i and j. Also, by Theorem 3.2, $\mathrm{Ass}_R H^s_{\mathfrak{a}}(N)$ is finite. So, it follows from

$$\mathrm{Ass}_R \mathrm{Hom}_R(M, H^s_{\mathfrak{a}}(N)) = \mathrm{Supp}_R M \cap \mathrm{Ass}_R H^s_{\mathfrak{a}}(N)$$

that $\mathrm{Ass}_R \mathrm{Hom}_R(M, H^s_{\mathfrak{a}}(N))$ is a finite set. This completes the proof. □

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Authors' address: Ahmad Abbasi, Hajar Roshan Shekalgourabi, Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, P. O. Box 41335–19141, Rasht, Iran, e-mail: `aabbasi@guilan.ac.ir`, `hroshan@guilan.ac.ir`.