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DECOMPOSITION OF COMPLETE GRAPHS INTO $(0,2)$-PRISMS

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Abstract. R. Frucht and J. Gallian (1988) proved that bipartite prisms of order $2n$ have an α-labeling, thus they decompose the complete graph K_{6nx+1} for any positive integer x. We use a technique called the ϱ^+-labeling introduced by S. I. El-Zanati, C. Vanden Eynden, and N. Punnim (2001) to show that also some other families of 3-regular bipartite graphs of order $2n$ called generalized prisms decompose the complete graph K_{6nx+1} for any positive integer x.

Keywords: decompositions; prism; ϱ^+-labeling

MSC 2010: 05C51, 05C70, 05C78, 05B30

1. Introduction

All graphs considered in this paper are simple, finite and undirected. We use standard terminology and notation of graph theory.

Graph decompositions have been widely studied in many different settings. We say that a graph B has a *G-decomposition* if there are subgraphs $G_1, G_2, \ldots, G_s$ of B, all isomorphic to G, such that each edge of B belongs to exactly one G_i. If each G_i for $i \in \{1, \ldots, s\}$ contains all vertices of B, then we say that B has a *G-factorization.*

Recall that a *prism* is a graph of the form $C_m \times P_2$. As in [1] we generalize prisms and let the $(0,j)$-*prism* (pronounced "oh-jay prism") of order $2n$ for j even be the graph with two vertex disjoint cycles $R_n^i = v_0^i, \ldots, v_{n-1}^i$ for $i \in \{1,2\}$ of length n called *rims* and edges $v_0^1v_0^2, v_2^1v_2^2, v_4^1v_4^2, \ldots$ and $v_1^2v_{j+1}^1,\ v_3^2v_{3+j}^1, v_5^2v_{5+j}^1, \ldots$ called *spokes* of *type* 0 and *type* j, respectively (see Figure 1). It is easy to observe that an $(0,j)$-prism is a 3-regular graph and is isomorphic to an $(0,-j)$-prism, $(j,0)$-prism and $(-j,0)$-prism. We can therefore always assume that $0 \leqslant j \leqslant n/2$. In our terminology the usual prism is an $(0,0)$-prism.

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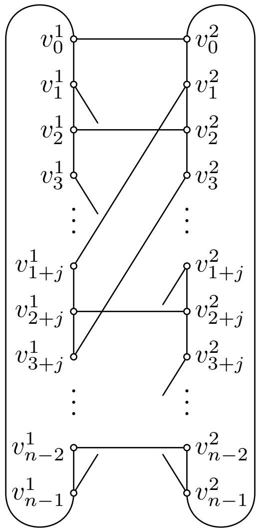

Figure 1. $(0, j)$-prism.

The problem of factorization of $K_{n,n}$ into $(0, j)$-prisms was solved in [1]. The problem of decomposition of $K_{n,n}$ into $(0, j)$-prisms was tackled in [2], [3]. In this paper we approach the decomposition problem of K_{2nx+1} into $(0, 2)$-prisms.

A labeling of a graph G with m edges is an injection ϱ from $V(G)$ into a set $S \subset \{0, 1, \ldots, 2m\}$. A. Rosa ([8]) introduced several types of graph labelings as tools for decompositions of complete graphs. The length of an edge xy is $l(x, y) = \min(\varrho(x) - \varrho(y), \varrho(y) - \varrho(x))$ where the subtraction is taken $\pmod{2m+1}$. If the set of all lengths of the m edges is equal to $\{1, 2, \ldots, m\}$ and $S \subset \{0, 1, 2, \ldots, 2m\}$, then ϱ is a *rosy labeling* (called a ϱ-valuation by A. Rosa [8]). If $S \subset \{0, 1, 2, \ldots, m\}$ instead then ϱ is a *graceful labeling* (called a β-valuation by A. Rosa [8]).

A graceful labeling ϱ is an *α-labeling* if there exists a number α such that for every edge $xy \in E(G)$ with $\varrho(x) < \varrho(y)$ it holds that $\varrho(x) \leqslant \alpha < \varrho(y)$.

Obviously, G must be bipartite to allow an α-labeling. Of the labelings mentioned above, graceful labelings have gained most attention, and cycle-related graphs are among the most studied (see [5], [6], [7]). R. W. Frucht and J. Gallian proved the following result.

Theorem 1 (R. W. Frucht and J. Gallian [5]). *Every $(0, 0)$-prism of order $4k$ has an α-labeling and every $(0, 0)$-prism of order $2(2k + 1)$ is graceful.*

Obviously, labelings are important tools in graph decomposition, as follows from two results by A. Rosa.

Theorem 2 (A. Rosa [8]). *If a graph G with m edges has a rosy labeling, then there is a G-decomposition of K_{2m+1} into $2m+1$ copies of G.*

Theorem 3 (A. Rosa [8]). *If a bipartite graph G with m edges has an α-labeling, then there is a G-decomposition of K_{2mx+1} for any positive integer x.*

In [4] S. I. El-Zanati, N. Punnim and C. Vanden Eynden defined ϱ^+-*labeling*. A labeling of a bipartite graph G with bipartition X, Y is called a ϱ^+-labeling if it is a rosy labeling with the additional property that for every edge $xy \in E(G)$ with $x \in X$, $y \in Y$ it holds that $\varrho^+(x) < \varrho^+(y)$.

The difference between these labelings is that while in an α-labeling we require all vertices in X to have labels smaller than every vertex in Y, in a ϱ^+-labeling we require that all neighbors of each given vertex $y \in Y$ have their labels smaller than $\varrho^+(y)$. Moreover, we can use labels from the set $\{0, 1, \ldots, 2m\}$ while in an α-labeling only from the set $\{0, 1, \ldots, m\}$.

Theorem 4 (S. I. El-Zanati, N. Punnim and C. Vanden Eynden [4]). *If a bipartite graph G with m edges has a ϱ^+-labeling, then there is a G-decomposition of K_{2mx+1} for any positive integer x.*

While $(0, 0)$-prisms exist for any cycle length, $(0, 2)$-prisms only exist when the rims are of an even length, and therefore are always bipartite. In this paper we will prove that $(0, 2)$-prisms have a ϱ^+-labeling and therefore there is a decomposition of K_{2mx+1} into $(0, 2)$-prisms of size $m = 3n$ for any positive integer x.

2. Decomposition

Theorem 5. *Every $(0, 2)$-prism of order $2n$ has a ϱ^+ labeling.*

Proof. Recall that we denote the vertices of G by v_i^j where $i \in \mathbb{Z}_n$, $j \in \mathbb{Z}_2$ and the size of G is $3n$. We need to consider three cases:

Case 1: $n \equiv 4 \pmod 6$

First we label the rim R_n^1 and spokes $v_i^1 v_i^2$ as follows:

$\varrho(v_{2i}^1) = n + i$,
$\varrho(v_{2i}^2) = n - 2i - 1$,
$\varrho(v_{2i+1}^1) = n - 2i - 2$ for $i \in \{0, 1, \ldots, \frac{n-4}{3}\}$,
$\varrho(v_{(2n-2)/3}^1) = \frac{4n-1}{3}$,
$\varrho(v_{(2n-2)/3}^2) = \frac{n-1}{3}$,
$\varrho(v_{(2n+1)/3}^1) = \frac{n-7}{3}$,

and $\varrho(v^1_{2i}) = n+i$, $\varrho(v^2_{2i}) = n-2i-2$,
$\varrho(v^1_{2i+1}) = n-2i-3$ for $i \in \{\frac{n+2}{3}, \frac{n+5}{3}, \dots, \frac{n-2}{2}\}$.

Notice that we obtain all lengths of the edges from the set $\{1, 2, \dots, \frac{3n}{2}\}$. Now we need to label the vertices v^2_{2i+1} for $i \in \{0, 1, \dots, \frac{n-2}{2}\}$.

Let $\varrho(v^2_{2i+1}) = \frac{5n}{2} + i + 1$ for $i \in \{-1, 1, 3, \dots, \frac{n-13}{3},\}$ and $\varrho(v^2_{2i+1}) = \frac{11n}{2} - 6 - 5i$ for $i \in \{0, 2, 4, \dots, \frac{n-16}{3}\}$. Notice that $\varrho(v^2_{(2n-23)/3}) < \varrho(v^2_{(2n-29)/3})$ and we obtain all edge lengths from the set $\{\frac{3n}{2}+1, \frac{3n}{2}+2, \dots, \frac{5n}{2}-8, \frac{5n}{2}\}$.

Putting

$\varrho(v^2_{(2n-17)/3}) = \frac{5n}{2} + \frac{n-10}{3} + 2 = \frac{17n-8}{6}$,
$\varrho(v^2_{(2n-11)/3}) = \frac{5n}{2} + 5 + \frac{n+2}{3} = \frac{17n+34}{6}$,
$\varrho(v^2_{(2n-5)/3}) = \frac{5n}{2} + 3 + \frac{n-7}{3} = \frac{17n+4}{6}$,
$\varrho(v^2_{(2n+1)/3}) = \frac{5n}{2} - 2 + \frac{n-13}{3} = \frac{17n-38}{6}$,

we obtain all edge lengths from the set $\{\frac{5n}{2}-7, \frac{5n}{2}-6, \dots, \frac{5n}{2}-1, \frac{5n}{2}+1, \frac{5n}{2}+2, \dots, \frac{5n}{2}+5\}$.

Observe that $\varrho(v^2_{(2n+1)/3}) \neq \varrho(v^2_{2i+1})$ for $i \in \{-1, 1, 3, \dots, \frac{n-13}{3}\}$.

We need to consider now two subcases:

Case 1.1: $n \equiv 4 \pmod{12}$

Let $\varrho(v^2_{2i+1}) = \frac{5n}{2} + i + 1$ for $i \in \{\frac{n+5}{3}, \frac{n+11}{3}, \dots, \frac{n-4}{2}\}$, and $\varrho(v^2_{2i+1}) = \frac{11n}{2} - 8 - 5i$ for $i \in \{\frac{n+2}{3}, \frac{n+8}{3}, \dots, \frac{n-6}{2}\}$.

Notice that $\varrho(v^2_{n-3}) < \varrho(v^2_{n-5})$.

Case 1.2: $n \equiv 10 \pmod{12}$

Let $\varrho(v^2_{2i+1}) = \frac{5n}{2} + i + 1$ for $i \in \{\frac{n+5}{3}, \frac{n+11}{3}, \dots, \frac{n-6}{2}\}$, and $\varrho(v^2_{2i+1}) = \frac{11n}{2} - 8 - 5i$ for $i \in \{\frac{n+2}{3}, \frac{n+8}{3}, \dots, \frac{n-8}{2}\}$.

Putting $\varrho(v^2_{n-3}) = 3n+2$ we obtain three missing lengths $3n, 3n-1, 3n-2$.

Notice that $\varrho(v^2_{n-5}) < \varrho(v^2_{n-3}) < \varrho(v^2_{n-7})$.

Case 2: $n \equiv 2 \pmod 6$

First we label the rim R^1_n and spokes $v^1_i v^2_i$ as follows: $\varrho(v^1_{2i}) = n+i$, $\varrho(v^2_{2i}) = n-2i-1$, $\varrho(v^1_{2i+1}) = n-2i-2$ for $i \in \{0, 1, \dots, \frac{n-5}{3}\}$, $\varrho(v^1_{(2n-4)/3}) = \frac{4n-2}{3}$, $\varrho(v^2_{(2n-4)/3}) = \frac{n+1}{3}$, $\varrho(v^1_{(2n-1)/3}) = \frac{n-5}{3}$, and $\varrho(v^1_{2i}) = n+i$, $\varrho(v^2_{2i}) = n-2i-2$, $\varrho(v^1_{2i+1}) = n-2i-3$ for $i \in \{\frac{n+1}{3}, \frac{n+4}{3}, \dots, \frac{n-4}{2}\}$.

Further, $\varrho(v^1_{n-2}) = \frac{3n-2}{2}$, $\varrho(v^2_{n-2}) = 6n$, $\varrho(v^1_{n-1}) = 0$.

Notice that we obtain all edge lengths from the set $\{1, 2, \dots, \frac{3n}{2}\}$. We need to label now the vertices v^2_{2i+1} for $i \in \{0, 1, \dots, \frac{n-2}{2}\}$.

Let $\varrho(v^2_{2i+1}) = \frac{5n}{2} + i + 1$ for $i \in \{-1, 1, 3, \dots, \frac{n-11}{3}\}$ and $\varrho(v^2_{2i+1}) = \frac{11n}{2} - 6 - 5i$ for $i \in \{0, 2, 4, \dots, \frac{n-14}{3}\}$. Notice that $\varrho(v^2_{(2n-19)/3}) < \varrho(v^2_{(2n-25)/3})$ and we obtain all edge lengths from the set $\{\frac{3n}{2}+1, \frac{3n}{2}+2, \dots, \frac{5n}{2}-6, \frac{5n}{2}+1\}$.

Putting

$\varrho(v^2_{(2n-13)/3}) = \frac{23n+32}{6}$,

$\varrho(v^2_{(2n-7)/3}) = \frac{23n-28}{6}$,
$\varrho(v^2_{(2n-1)/3}) = \frac{23n+14}{6}$,
$\varrho(v^2_{(2n+5)/3}) = \frac{23n-40}{6}$,
$\varrho(v^2_{(2n+11)/3}) = \frac{17n+14}{6}$,
we obtain all edge lengths from the set $\{\frac{5n}{2}-5, \frac{5n}{2}-4, \ldots, \frac{5n}{2}, \frac{5n}{2}+2, \frac{5n}{2}+3, \ldots, \frac{5n}{2}+10\}$.

Observe that $\varrho(v^2_{2i+1}) < \varrho(v^2_{(2n+11)/3})$ for $i \in \{-1, 1, 3, \ldots, (n-11)/3\}$ and $\varrho(v^2_{(2n-13)/3}) < \varrho(v^2_{2i+1})$ for $i \in \{0, 2, 4, \ldots, \frac{n-14}{3}\}$.

We need to consider now two subcases:

Case 2.1: $n \equiv 8 \pmod{12}$

Let $\varrho(v^2_{2i+1}) = \frac{5n}{2}+i+1$ for $i \in \{\frac{n+7}{3}, \frac{n+13}{3}, \ldots, \frac{n-6}{2}\}$, and $\varrho(v^2_{2i+1}) = \frac{11n}{2}-8-5i$ for $i \in \{\frac{n+10}{3}, \frac{n+16}{3}, \ldots, \frac{n-4}{2}\}$.

Notice that $\varrho(v^2_{n-3}) < \varrho(v^2_{n-5})$.

Observe also that $\varrho(v^2_{(2n+5)/3}) > \varrho(v^2_{(2n+23)/3})$, $\varrho(v^2_{(2n+11)/3}) < \varrho(v^2_{(2n+17)/3})$ and $\varrho(v^2_{(2n+11)/3}) \neq \varrho(v^2_{2i+1})$ for $i \in \{\frac{n+10}{3}, \frac{n+16}{3}, \ldots, \frac{n-4}{2}\}$.

Case 2.2: $n \equiv 2 \pmod{12}$

Let $\varrho(v^2_{2i+1}) = \frac{5n}{2}+i+1$ for $i \in \{\frac{n+7}{3}, \frac{n+13}{3}, \ldots, \frac{n-8}{2}\}$, and $\varrho(v^2_{2i+1}) = \frac{11n}{2}-8-5i$ for $i \in \{\frac{n+10}{3}, \frac{n+16}{3}, \ldots, \frac{n-6}{2}\}$.

Putting $\varrho(v^2_{n-3}) = 3n+2$ we obtain three missing lengths $3n, 3n-1, 3n-2$.

Notice that $\varrho(v^2_{n-5}) < \varrho(v^2_{n-3}) < \varrho(v^2_{n-7})$.

Case 3: $n \equiv 0 \pmod 6$

First we label the rim R^1_n and spokes $v^1_i v^2_i$ as follows: $\varrho(v^1_{2i}) = n+i$, $\varrho(v^2_{2i}) = n-2i-1$, $\varrho(v^1_{2i+1}) = n-2i-2$ for $i \in \{0, 1, \ldots, \frac{n-3}{3}\}$, $\varrho(v^1_{2i}) = n+i+1$, $\varrho(v^2_{2i}) = n-2i-1$, $\varrho(v^1_{2i+1}) = n-2i-2$ for $i \in \{\frac{n}{3}, \frac{n+3}{3}, \ldots, \frac{n-2}{2}\}$.

Notice that we obtain all edge lengths from the set $\{1, 2, \ldots, \frac{3n}{2}\}$. We need to label now the vertices v^2_{2i+1} for $i \in \{0, 1, \ldots, \frac{n-2}{2}\}$.

Let $\varrho(v^2_{2i+1}) = \frac{5n}{2}+i+1$ for $i \in \{-1, 1, 3, \ldots, \frac{n-9}{3}\}$, and $\varrho(v^2_{2i+1}) = \frac{11n}{2}-6-5i$ for $i \in \{0, 2, 4, \ldots, \frac{n-12}{3}\}$. Let $\varrho(v^2_{(2n-9)/3}) = \frac{23n+24}{6}$.

Case 3.1: $n \equiv 0 \pmod{12}$

Let $\varrho(v^2_{2i+1}) = \frac{5n}{2}+i+2$ for $i \in \{\frac{n}{3}, \frac{n+6}{3}, \ldots, \frac{n-4}{2}\}$, and $\varrho(v^2_{2i+1}) = \frac{11n}{2}-7-5i$ for $i \in \{\frac{n-3}{3}, \frac{n+3}{3}, \ldots, \frac{n-6}{2}\}$.

Notice that $\varrho(v^2_{n-3}) < \varrho(v^2_{n-5})$.

Case 3.2: $n \equiv 6 \pmod{12}$

Let $\varrho(v^2_{2i+1}) = \frac{5n}{2}+i+2$ for $i \in \{\frac{n}{3}, \frac{n+6}{3}, \ldots, \frac{n-6}{2}\}$, $\varrho(v^2_{2i+1}) = \frac{11n}{2}-7-5i$ for $i \in \{\frac{n-3}{3}, \frac{n+3}{3}, \ldots, \frac{n-8}{2}\}$.

Putting $\varrho(v^2_{n-3}) = 3n+2$ we obtain three missing lengths $3n, 3n-1, 3n-2$.

Notice that $\varrho(v_{n-5}^2) < \varrho(v_{n-3}^2) < \varrho(v_{n-7}^2)$. Let

$$\gamma(v) = \begin{cases} \varrho(v) + 1 & \text{for } n \not\equiv 0 \pmod 6, \\ \varrho(v) & \text{for } n \equiv 0 \pmod 6. \end{cases}$$

Notice that γ is a ϱ^+-labeling. □

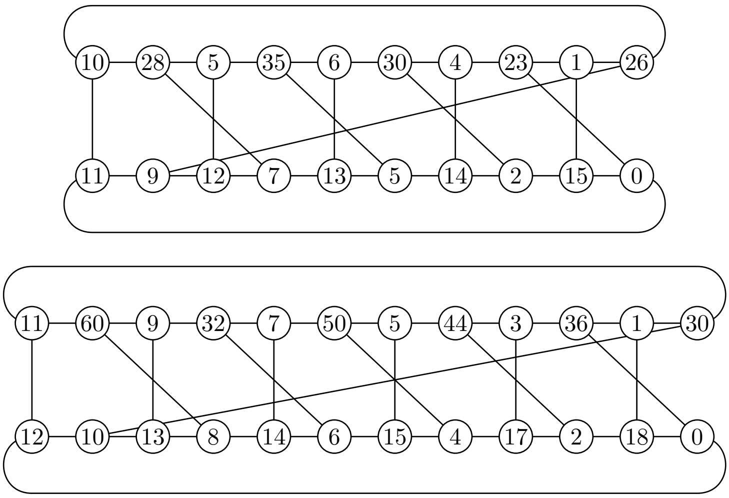

Figure 2. ϱ^+-labelings for $(0, 2)$-prisms of order 20 and 24.

Corollary 6. *Every $(0, 2)$-prism of order $2n$ decomposes the complete graph K_{6nx+1} for any positive integer x.*

Proof. Follows directly from Theorems 4 and 5. □

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