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SPECTRAL RADIUS INEQUALITIES FOR POSITIVE COMMUTATORS

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Abstract. We establish several inequalities for the spectral radius of a positive commutator of positive operators in a Banach space ordered by a normal and generating cone. The main purpose of this paper is to show that in order to prove the quasi-nilpotency of the commutator we do not have to impose any compactness condition on the operators under consideration. In this way we give a partial answer to the open problem posed in the paper by J. Bračič, R. Drnovšek, Y. B. Farforovskaya, E. L. Rabkin, J. Zemánek (2010). Inequalities involving an arbitrary commutator and a generalized commutator are also discussed.

Keywords: cone; positive operator; commutator; spectral radius

MSC 2010: 47A10, 47B47, 47B65

1. Introduction

In this paper we study the properties of the spectral radius of the commutator of two positive operators. In particular, we are interested in establishing conditions implying the quasi-nilpotency of the commutator. Recall that the spectral radius of a bounded linear operator A in a Banach space E is the number

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}.$$

Denote by $\mathcal{L}(E)$ the space of all bounded linear operators from E to E. It is well known that the spectral radius of two commuting bounded linear operators is subadditive and submultiplicative, that is, if $A, B \in \mathcal{L}(E)$ and $AB - BA = 0$, then

$$r(A+B) \leqslant r(A) + r(B) \tag{1.1}$$

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and

$$r(AB) \leqslant r(A)r(B) \tag{1.2}$$

(see for example [16]).

It has been proved in [19] (see also [6], [21]) that the same conclusions hold for positive A and B whose commutator $AB - BA$ is also positive with respect to a normal and generating cone of a Banach space. For the convenience of the reader we recall here some basic facts from the cone theory in Banach spaces.

A nonempty subset K, $K \neq \{0\}$, of a real Banach space E is called a cone if K is closed, convex and

(i) $\lambda x \in K$ for all $x \in K$ and $\lambda \geqslant 0$,
(ii) $x, -x \in K$ implies $x = 0$.

It is well known that every cone induces a partial ordering in E as follows: for $x, y \in E$ we write $x \preceq y$ if and only if $y - x \in K$.

A cone K is said to be *normal* if there exists $\gamma > 0$ such that if $0 \preceq x \preceq y$, then $\|x\| \leqslant \gamma \|y\|$. The smallest number γ satisfying this condition is called the normal constant of K. Obviously, $\gamma \geqslant 1$.

A cone K is said to be *generating* (or reproducing) if $E = K - K$. We say that $A \in \mathcal{L}(E)$ is *positive* if $A(K) \subset K$. For more details on the cone theory we refer the reader to [3] and [10]. In [19] one can find the following result which will play a significant role in our considerations.

Theorem 1.1 ([19]). *Let K be a normal and generating cone in a Banach space E. If A, B, and $AB - BA$ are positive, then* (1.1) *and* (1.2) *hold.*

In this paper we will apply Theorem 1.1 to discuss an open problem posed in [1]. In [1], the authors studied the size of the spectrum of a positive commutator of positive operators acting in Banach lattices. In particular, they were interested in the case when $AB - BA$ is *quasi-nilpotent*, that is, $r(AB - BA) = 0$. The main result of [1] is the following theorem (see [1], Theorem 2.2).

Theorem 1.2 ([1]). *Let A and B be positive compact operators on a Dedekind complete Banach lattice E such that the commutator $C = AB - BA$ is also positive. Then C is a quasi-nilpotent operator having a triangularizing chain of closed order ideals of E. Moreover, C belongs to the radical of the Banach algebra generated by A and B. If, in addition, at least one of the operators A and B is irreducible, then $C = 0$, and so $AB = BA$.*

Recall that by a positive operator in a Banach lattice E we mean a mapping which leaves the cone $E^+ = \{x \in E\colon\ x \geqslant 0\}$ invariant. For more details on Banach lattices we refer the reader to [18]. One of the open problems stated in [1] is whether the compactness assumption in Theorem 1.2 can be dropped. In a recent paper [5], Drnovšek proved that it is enough to assume that only one of the operators A or B is compact (see [5], Theorem 3.1 and Corollary 3.2, and [9], Theorem 4.5). The main purpose of the present paper is to show that in order to prove the quasi-nilpotency of $AB - BA$ we do not have to impose any compactness condition on the operators under consideration. Our result is stated in the setting of Banach spaces but we show that it is also valid for Banach lattices. Moreover, we prove several inequalities for the spectral radius of the commutator of positive operators that refine and generalize some results from the literature. We complete the paper by discussing the case of an arbitrary commutator, that is, not necessarily positive, and a generalized commutator.

In what follows we will make use of some properties of the local spectral radius. Recall that the number

$$r(A,x) = \limsup_{n\to\infty} \|A^n x\|^{1/n} \tag{1.3}$$

is called the *local spectral radius* of A at x. In general, $r(A,x)$ is equal to $r(A)$ for many vectors in E, see [2]. For positive operators, Förster and Nagy [7] pointed out the following result.

Lemma 1.1 ([7]). *If K is a generating cone in E and $A \in \mathcal{L}(E)$ is positive, then*

$$r(A) = \max\{r(A,x)\colon\ x \in K\}.$$

Note that a generating cone can have empty interior (see [10], page 6, [21], page 15). The following monotonicity property of the spectral radius can be found in [14].

Lemma 1.2 ([14], Theorem 5.3, page 76). *If K is a normal and generating cone in E, $A, B \in \mathcal{L}(E)$, B is positive and $-Bx \preceq Ax \preceq Bx$ for every $x \in K$, then $r(A) \leqslant r(B)$.*

As a direct consequence of Lemma 1.2 we get the following result.

Corollary 1.1. *If K is a normal and generating cone in E, $A, B \in \mathcal{L}(E)$ are positive and $Ax \preceq Bx$ for every $x \in K$, then $r(A) \leqslant r(B)$.*

For more properties of the local spectral radius, including sufficient conditions implying the equality $r(A) = r(A,x)$, we refer to [2], [7], [8], [15], [17], [19] and [20].

2. The spectral radius of a positive commutator

In this section we establish some inequalities for the spectral radius of positive commutators.

Theorem 2.1. *Let K be a normal and generating cone in a Banach space E. If A, B and $AB - BA$ are positive, then*

$$r(AB - BA) \leqslant r(AB) \leqslant r(A)r(B) \tag{2.1}$$

and

$$r(AB + BA) = 2r(AB). \tag{2.2}$$

Proof. Observe that $0 \preceq (AB - BA)x \preceq ABx$ for every $x \in K$. Since K is normal and generating, by Corollary 1.1 and Theorem 1.1 we get

$$r(AB - BA) \leqslant r(AB) \leqslant r(A)r(B).$$

Moreover, for every $x \in K$, $0 \preceq 2BAx \preceq (AB + BA)x \preceq 2ABx$. This, in view of Corollary 1.1 again and the well-known equality $r(AB) = r(BA)$, implies (2.2). □

Remark 2.1. Clearly, if $r(AB) = 0$ or $r(A) = 0$ or $r(B) = 0$, then $AB - BA$ is quasinilpotent. Observe that we do not require either A or B to be compact. This means that Theorem 2.1 partially generalizes Theorem 1.2. Indeed, if E is a Banach lattice, then E^+ is a generating and normal cone in E. In fact, E^+ is a closed subset of E as the preimage of $\{0\}$ under the continuous mapping $E \ni x \to x^- \in E^+$ (see [18], Proposition II.5.2, page 83). Moreover, for every $x \in E$ we have $x = x^+ - x^-$, and if $0 \preceq x \preceq y$, then $|x| \leqslant |y|$. This, by definition of a lattice norm, implies $\|x\| \leqslant \|y\|$.

Remark 2.2. In [4], Drnovšek and Kandić by a different kind of reasoning proved that if A, B, and $AB - BA$ are positive operators on a Banach lattice, then

$$r(AB - BA) \leqslant r(A)r(B).$$

Remark 2.3. In [11], Kittaneh proved that if A and B are bounded linear operators in a Hilbert space, then

$$r(AB \pm BA) \leqslant \frac{1}{2}(\|AB\| + \|BA\| + \sqrt{(\|AB\| - \|BA\|)^2 + 4\|A^2\|\,\|B^2\|}).$$

Since $r(A) \leqslant \|A\|$ for every $A \in \mathcal{L}(E)$, Theorem 2.1 shows that for positive operators with positive commutator sharper estimates are valid.

Next we will show that under some additional assumptions on A and B one can obtain more precise estimates of $r(AB - BA)$.

Theorem 2.2. *Let A, B, and $AB - BA$ be positive with respect to a normal and generating cone K in a Banach space E. If there exists a positive operator D such that $A - D$ and $BD - DB$ are positive, then*

$$r(AB - BA) \leqslant r((A - D)B) \leqslant r(A - D)r(B). \tag{2.3}$$

P r o o f. We have $0 \preceq (AB - BA)x \preceq [(A - D)B - B(A - D)]x \preceq (A - D)Bx$ for every $x \in K$. Now (2.3) follows from Corollary 1.1 and Theorem 1.1. $\square$

We illustrate Theorem 2.2 by an example.

Example 2.1. In the space $C[0, 1]$ of all continuous functions on $[0, 1]$ with the norm

$$\|x\| = \max_{t\in[0,1]} |x(t)|,$$

consider the operators

$$Ax(t) = (t + 1) \int_0^t x(s)\,\mathrm{d}s + (1 - t)x(t) + x(0)$$

and

$$Bx(t) = \frac{1}{2} \int_0^t x(s)\,\mathrm{d}s + \Bigl(1 - \frac{1}{2}t\Bigr)x(t).$$

It is easy to show that

$$(AB - BA)x(t) = \frac{1}{2}\biggl[\int_0^t \biggl((t - s) \int_0^s x(\tau)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s + \int_0^t t(t - s)x(s)\,\mathrm{d}s\biggr].$$

Observe that A, B, and $AB - BA$ are positive with respect to the cone

$$K = \{x \in C[0, 1]\colon\ x(t) \geqslant 0,\ t \in [0, 1]\}.$$

Let $Dx(t) = (1 - t)x(t) + x(0)$. Then

$$(A - D)x(t) = (t + 1) \int_0^t x(s)\,\mathrm{d}s$$

and

$$(BD - DB)x(t) = \frac{1}{2} \int_0^t (t - s)x(s)\,\mathrm{d}s.$$

Since D, $A - D$, and $BD - DB$ are positive, Theorem 2.2 gives

$$r(AB - BA) \leqslant r(A - D)r(B).$$

Clearly, $r(A - D) = 0$. Thus $AB - BA$ is quasi-nilpotent. Note that in this case A and B are far from being compact and quasi-nilpotent.

Theorem 2.3. *Let A, B, and $AB - BA$ be positive with respect to a normal and generating cone in a Banach space E. If there exists $\alpha > 0$ such that $A - \alpha B$ is positive, then*

$$r(AB - BA) \leqslant \frac{1}{2\alpha} r((A - \alpha B)(A + \alpha B)) \leqslant \frac{1}{2\alpha} r(A - \alpha B) r(A + \alpha B). \tag{2.4}$$

Proof. Observe that $AB - BA = \frac{1}{2\alpha}[(A - \alpha B)(A + \alpha B) - (A + \alpha B)(A - \alpha B)]$. This means that $(A - \alpha B)(A + \alpha B) - (A + \alpha B)(A - \alpha B)$ is positive. Since $A - \alpha B$ is positive, (2.4) follows from Corollary 1.1 and Theorem 1.1. □

Remark 2.4. Theorem 2.3 with $\alpha = 1$ can be regarded as a modification of [13], Corollary 1. Namely, in [13] Kittaneh proved that if A and B are bounded linear operators in a Hilbert space H, and $A + B$ is positive with respect to the inner product in H, that is $\langle (A + B)x, x \rangle \geqslant 0$ for all $x \in H$, then

$$\|AB - BA\| \leqslant \frac{1}{2} \|A + B\| \, \|A - B\|.$$

Observe that the definition of positivity employed in [13] is different from the one we use. In general, these two positivity notions are not related. To illustrate this, consider $H = \mathbb{R}^2$ with the standard inner product $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 + x_2 y_2$ and the cone $K = \{(x_1, x_2) \in H \colon x_1, x_2 \geqslant 0\}$. Then $A(x_1, x_2) = (x_1 + x_2, -x_1)$ is positive with respect to the given inner product and it is not positive with respect to K, while the operator $A(x_1, x_2) = (x_1 + x_2, x_1)$ is positive with respect to K and it is not positive with respect to the given inner product. However, $A(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)$ with $a, b, c, d \geqslant 0$ and $(b + c)^2 - 4ad \leqslant 0$ fulfils both the definitions of positivity mentioned above.

Theorem 2.4. *If A, B, and $AB - BA$ are positive with respect to a normal and generating cone K in a Banach space E, and there exist $\alpha, \beta \geqslant 0$ such that $A - \alpha I$ and $B - \beta I$ are positive, then*

$$r(AB - BA) \leqslant r((A - \alpha I)(B - \beta I)) \leqslant r(A - \alpha I) r(B - \beta I).$$

Proof. It is enough to observe that

$$AB - BA = (A - \alpha I)(B - \beta I) - (B - \beta I)(A - \alpha I).$$

□

Example 2.2. Let functions φ and ψ be non-negative and continuous on $[0,1]$ and assume that φ is decreasing. In the space $C[0,1]$ consider the operators

$$Ax(t) = \int_0^t x(s)\,\mathrm{d}s + \alpha x(t)$$

and

$$Bx(t) = \varphi(t)x(t) + \psi(t)x(0) + \beta x(t),$$

with α, $\beta > 0$. Then

$$(AB - BA)x(t) = \int_0^t (\varphi(s) - \varphi(t))x(s)\,\mathrm{d}s + x(0)\int_0^t \psi(s)\,\mathrm{d}s.$$

Observe that A, B, $AB - BA$, $A - \alpha I$, and $B - \beta I$ are positive with respect to the cone

$$K = \{x \in C[0,1]\colon\ x(t) \geqslant 0,\ t \in [0,1]\}.$$

By Theorem 2.4, $r(AB - BA) \leqslant r(A - \alpha I)r(B - \beta I)$. Clearly $r(A - \alpha I) = 0$, and therefore we get $r(AB - BA) = 0$.

3. Further inequalities

In this section we discuss some inequalities for the spectral radius of an arbitrary commutator, that is, not necessarily positive, and for a generalized commutator.

First consider the case when $AB - BA$ is an arbitrary commutator. In order to estimate its spectral radius we can use Lemma 1.2. It follows from Lemma 1.2 that if K is a normal and generating cone and for $AB - BA$ there exists a positive operator $D \in \mathcal{L}(E)$ such that

$$-Dx \preceq (AB - BA)x \preceq Dx$$

for every $x \in K$, then $r(AB - BA) \leqslant r(D)$. We will illustrate this case by the following example.

Example 3.1. In $C[0,1]$ consider the multiplication operator

$$Ax(t) = tx(t)$$

and a Volterra composition operator

$$Bx(t) = \int_0^{\sqrt{t}} x(s)\,\mathrm{d}s.$$

Then

$$(AB - BA)x(t) = \int_0^{\sqrt{t}} (t-s)x(s)\,\mathrm{d}s,$$

and it is clear that $AB - BA$ is not positive with respect to the cone

$$K = \{x \in C[0,1] \colon\ x(t) \geqslant 0,\ t \in [0,1]\}.$$

Let $Dx(t) = \sqrt{t}\int_0^{\sqrt{t}} x(s)\,\mathrm{d}s$. Then D is positive with respect to K and

$$-Dx \preceq (AB - BA)x \preceq Dx$$

for every $x \in K$. Therefore $r(AB - BA) \leqslant r(D)$. We will show that $r(D) = \frac{1}{3}$. For $x_0(t) \equiv 1$ we have $Dx_0(t) = t$, $D^2x_0(t) = \frac{1}{2}t^{3/2}$, and by induction on n,

$$D^n x_0(t) = \frac{2^{(n-2)(n-1)/2}}{2\cdot 5\cdot \ldots \cdot (2^{n-2} + 2^{n-1} - 1)} t^{(2^n-1)/2^{n-1}}$$

for $n \geqslant 2$. Hence

$$\|D^n x_0\| = \frac{2^{(n-2)(n-1)/2}}{2\cdot 5\cdot \ldots \cdot (2^{n-2} + 2^{n-1} - 1)}$$

for $n \geqslant 2$. Clearly, $\|D^n x_0\| > 0$ for all $n \in \mathbb{N}$. It is easy to show that

$$\begin{aligned}
\lim_{n\to\infty} \frac{\|D^{n+1}x_0\|}{\|D^n x_0\|} &= \lim_{n\to\infty} \frac{2^{(n-1)n/2}\cdot 2\cdot 5\cdot 11\cdot \ldots \cdot (2^{n-2}+2^{n-1}-1)}{2^{(n-2)(n-1)/2}\cdot 2\cdot 5\cdot 11\cdot \ldots \cdot (2^{n-2}+2^{n-1}-1)(2^{n-1}+2^n-1)} \\
&= \lim_{n\to\infty} \frac{2^{n-1}}{2^{n-1}+2^n-1} = \frac{1}{3}.
\end{aligned}$$

In this case we have

$$\limsup_{n\to\infty} \|D^n x_0\|^{1/n} = \lim_{n\to\infty} \frac{\|D^{n+1}x_0\|}{\|D^n x_0\|}.$$

Since K is a normal and generating cone and x_0 belongs to the interior of K, we get

$$r(D) = r(D, x_0) = \limsup_{n\to\infty} \|D^n x_0\|^{1/n}$$

(see for example [14], [17]). Therefore $r(D) = \frac{1}{3}$.

We complete this section with one result for a generalized commutator, that is, for an operator of the form $A\Gamma - \Gamma B$, where A, B, $\Gamma \in \mathcal{L}(E)$.

Theorem 3.1. *Let K be a normal and generating cone and A, B, $\Gamma \in \mathcal{L}(E)$. Suppose that Γ is a positive operator and there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that*

$$\alpha_1 x \preceq Ax \preceq \alpha_2 x$$

and

$$\beta_1 x \preceq Bx \preceq \beta_2 x$$

for every $x \in K$. Then

$$r(A\Gamma - \Gamma B) \leqslant \alpha r(\Gamma), \tag{3.1}$$

where $\alpha = \max\{\alpha_2 - \beta_1, \beta_2 - \alpha_1\}$. In particular, if $A = B$, then

$$r(A\Gamma - \Gamma A) \leqslant \alpha r(\Gamma).$$

Proof. For every $x \in K$ we have

$$-\alpha\Gamma x \preceq (\alpha_1 - \beta_2)\Gamma x \preceq (A\Gamma - \Gamma B)x \preceq (\alpha_2 - \beta_1)\Gamma x \preceq \alpha\Gamma x.$$

Now (3.1) follows from Lemma 1.2. □

A similar result for a unitarily invariant norm of $A\Gamma - \Gamma B$ can be found in [12].

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