Nihat Yagmur; Halit Orhan

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FEKETE-SZEGŐ PROBLEM FOR SUBCLASSES OF GENERALIZED UNIFORMLY STARLIKE FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

Nihat Yagmur, Erzincan, Halit Orhan, Erzurum

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Abstract. The authors obtain the Fekete-Szegő inequality (according to parameters s and t in the region $s^2+st+t^2<3$, $s\neq t$ and $s+t\neq 2$, or in the region $s^2+st+t^2>3$, $s\neq t$ and $s+t\neq 2$) for certain normalized analytic functions $f(z)$ belonging to k-$\mathrm{UST}^n_{\lambda,\mu}(s,t,\gamma)$ which satisfy the condition

$$\Re\left\{\frac{(s-t)z(D^n_{\lambda,\mu}f(z))'}{D^n_{\lambda,\mu}f(sz)-D^n_{\lambda,\mu}f(tz)}\right\}>k\left|\frac{(s-t)z(D^n_{\lambda,\mu}f(z))'}{D^n_{\lambda,\mu}f(sz)-D^n_{\lambda,\mu}f(tz)}-1\right|+\gamma,\quad z\in\mathcal{U}.$$

Also certain applications of the main result a class of functions defined by the Hadamard product (or convolution) are given. As a special case of this result, the Fekete-Szegő inequality for a class of functions defined through fractional derivatives is obtained.

Keywords: Fekete-Szegő problem; Sakaguchi function; uniformly starlike function; symmetric point

MSC 2010: 30C45, 30C50

1. Introduction

Let $\mathcal{A}$ denote the family of functions f of the form

$$f(z)=z+\sum_{j=2}^{\infty}a_jz^j \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U}=\{z\colon |z|<1\}$. Further, let $\mathcal{S}$ denote the class of functions which are univalent in $\mathcal{U}$. If f and g are analytic in $\mathcal{U}$, we say

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that f is subordinate to g, written symbolically as $f \prec g$ or $f(z) \prec g(z)$, $z \in \mathcal{U}$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathcal{U}$ with $w(0)=0$ and $|w(z)|<1$ in $\mathcal{U}$ such that $f(z)=g(w(z)), z \in \mathcal{U}$. In particular, if the function $g(z)$ is univalent in $\mathcal{U}$, then we have that $f(z) \prec g(z)$, $z \in \mathcal{U}$, if and only if $f(0)=g(0)$ and $f(\mathcal{U}) \subseteq g(\mathcal{U})$. A function $f(z) \in \mathcal{A}$ is said to be in the class of k-starlike functions of order γ, denoted by k-UST(γ), if

$$\Re\Big(\frac{zf'(z)}{f(z)}\Big) > k\Big|\frac{zf'(z)}{f(z)} - 1\Big| + \gamma \tag{1.2}$$

where $k \geqslant 0$, $\gamma \in [-1,1)$ and is said to be in the class of k-uniformly convex functions of order γ, denoted by k-UCV(γ) [3], if

$$\Re\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > k\Big|\frac{zf''(z)}{f'(z)}\Big| + \gamma \tag{1.3}$$

where $k \geqslant 0$, $\gamma \in [-1,1)$.

The classes 1-UST(0) = UST and 1-UCV(0) = UCV were defined by Goodman [8] and Rønning [28], respectively. Furthermore, the classes k-UST(0) and k-UCV(0) are the well-known classes of k-starlike and k-uniformly convex functions, respectively, introduced by Kanas and Wiśniowska [16], [15]. Using the Alexander type relation, we can get the following relation:

$$f(z) \in k\text{-UCV}(\gamma) \Longleftrightarrow zf'(z) \in k\text{-UST}(\gamma). \tag{1.4}$$

Geometric interpretation: $f \in k$-UST(γ) and $f \in k$-UCV(γ) if and only if $zf''(z)/f(z)$ and $1 + zf''(z)/f'(z)$, respectively, take all the values in the conic domain $\mathcal{R}_{k,\gamma}$ which is included in the right half plane such that

$$\mathcal{R}_{k,\gamma} = \big\{w = u + \mathrm{i}v \in \mathbb{C}\colon\ u > k\sqrt{(u-1)^2 + v^2} + \gamma,\ k \geqslant 0 \text{ and } \gamma \in [-1,1)\big\}. \tag{1.5}$$

Denote by $\mathcal{P}(P_{k,\gamma})$ ($k \geqslant 0$, $-1 \leqslant \gamma < 1$) the family of functions p such that $p \in \mathcal{P}$, where $\mathcal{P}$ denotes the well-known class of Caratheodory functions and $p \prec P_{k,\gamma}$ in $\mathcal{U}$. The function $P_{k,\gamma}$ maps the unit disk conformally onto the domain $\mathcal{R}_{k,\gamma}$ such that $1 \in \mathcal{R}_{k,\gamma}$ and $\partial\mathcal{R}_{k,\gamma}$ is the curve defined by the equality

$$\begin{aligned} \partial\mathcal{R}_{k,\gamma} := \{&w = u + \mathrm{i}v \in \mathbb{C}\colon\ u^2 = (k\sqrt{(u-1)^2 + v^2} + \gamma)^2,\\ &k \geqslant 0 \text{ and } \gamma \in [-1,1)\}. \end{aligned} \tag{1.6}$$

From elementary computations we see that (1.6) represents conic sections symmetric about the real axis. Thus $\mathcal{R}_{k,\gamma}$ is an elliptic domain for $k > 1$, a parabolic domain for $k = 1$, a hyperbolic domain for $0 < k < 1$ and the right half plane $u > \gamma$ for $k = 0$.

The functions $P_{k,\gamma}$, which play the role of extremal functions of the class $\mathcal{P}(P_{k,\gamma})$, were obtained in [2], and for some unique $\theta \in (0,1)$, every positive number k can be expressed as

$$k = \cosh \frac{\pi \mathcal{K}'(\theta)}{4\mathcal{K}(\theta)} \tag{1.7}$$

where $\mathcal{K}$ is Legendre's complete elliptic integral of the first kind and $\mathcal{K}'$ is the complementary integral of $\mathcal{K}$ (for details see [2], [16], [15], [14], [19] and [20]).

For $f(z)$ belonging to $\mathcal{A}$, the *multiplier differential operator* $D^n_{\lambda,\mu} f$ was defined by the authors in [6] as follows:

$$\begin{aligned}
&D^0_{\lambda,\mu} f(z) = f(z),\\
&D^1_{\lambda,\mu} f(z) = D_{\lambda,\mu} f(z) = \lambda\mu z^2 (f(z))'' + (\lambda-\mu) z (f(z))' + (1-\lambda+\mu) f(z),\\
&D^2_{\lambda,\mu} f(z) = D_{\lambda,\mu}(D^1_{\lambda,\mu} f(z)),\\
&\qquad\qquad \vdots\\
&D^n_{\lambda,\mu} f(z) = D_{\lambda,\mu}(D^{n-1}_{\lambda,\mu} f(z)),
\end{aligned}$$

where $\lambda \geqslant \mu \geqslant 0$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

If f is given by (1.1) then from the definition of the operator $D^n_{\lambda,\mu} f(z)$ it is easy to see that

$$D^n_{\lambda,\mu} f(z) = z + \sum_{j=2}^{\infty} A^n_j a_j z^j, \tag{1.8}$$

where $A^n_j = [1 + (\lambda\mu j + \lambda - \mu)(j-1)]^n$.

It should be remarked that $D^n_{\lambda,\mu}$ is a generalization of many other linear operators considered earlier by different authors. In particular, for $f \in \mathcal{A}$ we have the following:

- ▷ $D^n_{1,0} f(z) \equiv D^n f(z)$, the operator investigated by Sălăgean (see [30]).
- ▷ $D^n_{\lambda,0} f(z) \equiv D^n_\lambda f(z)$, the operator studied by Al-Oboudi (see [1]).
- ▷ $D^n_{\lambda,\mu} f(z)$ the operator for $0 \leqslant \mu \leqslant \lambda \leqslant 1$, called the Răducanu-Orhan operator (see [27]).

Now we define new subclasses of $\mathcal{A}$.

Definition 1.1. Let $k \geqslant 0$, $-1 \leqslant \gamma < 1$, and let s, t be real numbers with $s \neq t$. We denote by $k\text{-UST}^n_{\lambda,\mu}(s,t,\gamma)$ the class of functions $f \in \mathcal{A}$ which satisfy the following condition:

$$\Re\left\{\frac{(s-t)z(D^n_{\lambda,\mu}f(z))'}{D^n_{\lambda,\mu}f(sz)-D^n_{\lambda,\mu}f(tz)}\right\} > k\left|\frac{(s-t)z(D^n_{\lambda,\mu}f(z))'}{D^n_{\lambda,\mu}f(sz)-D^n_{\lambda,\mu}f(tz)}-1\right|+\gamma, \quad z \in \mathcal{U}. \tag{1.9}$$

Using the Alexander type relation, we define the class $k\text{-UCV}^n_{\lambda,\mu}(s,t,\gamma)$ as follows:

$$f \in k\text{-UCV}^n_{\lambda,\mu}(s,t,\gamma) \quad \text{if and only if} \quad zf' \in k\text{-UST}^n_{\lambda,\mu}(s,t,\gamma),$$

and also

$$k\text{-UCV}^n_{\lambda,\mu}(s,t,\gamma) \subseteq k\text{-UST}^n_{\lambda,\mu}(s,t,\gamma).$$

Geometric interpretation: From (1.9), $f \in k\text{-UST}^n_{\lambda,\mu}(s,t,\gamma)$ if and only if $q(z) = (s-t)z(D^n_{\lambda,\mu}f(z))'/(D^n_{\lambda,\mu}f(sz)-D^n_{\lambda,\mu}f(tz))$ take all the values in the conic domain $\mathcal{R}_{k,\gamma}$ given in (1.5) which is included in the right half plane.

Detailed information about uniformly convex functions can be found in Gangadharan, Shanmugam, Srivastava [7], Kanas, Srivastava [13] and Orhan, Deniz, Raducanu [20].

We note that by specializing the parameters $s, t, \gamma, \lambda, \mu, n$ and k we obtain the subclasses studied by various authors, some of them are the following classes:

1. The class $0\text{-UST}^0_{\lambda,\mu}(s,t,0)$ was introduced by Sakaguchi [29]. Therefore, a function $0\text{-UST}^0_{\lambda,\mu}(1,-1,\gamma)$ is called a Sakaguchi function of order γ (see [4] and [24]).
2. The class $0\text{-UST}^0_{\lambda,\mu}(1,t,\gamma)$ was introduced and studied by Owa et al. [25].
3. The class $0\text{-UST}^n_{\lambda,\mu}(1,t,\gamma)$ was introduced and studied by Orhan et al. [23].
4. The class $k\text{-UST}^0_{\lambda,\mu}(1,t,\gamma)$ was studied by Goyal et al. [9].

To prove our results, we will need the following lemmas.

Lemma 1.1 (see [2] also [20]). *Let $0 \leqslant k < \infty$ and $-1 \leqslant \gamma < 1$ be fixed and let $P_{k,\gamma}$ be the Riemann map of $\mathcal{U}$ onto $\mathcal{R}_{k,\gamma}$, satisfying $P_{k,\gamma}(0) = 1$ and $P'_{k,\gamma}(0) > 0$. If*

$$P_{k,\gamma}(z) = 1 + P_1 z + P_2 z^2 + \ldots, \quad z \in \mathcal{U}, \tag{1.10}$$

then

$$P_1 = \begin{cases} \dfrac{2(1-\gamma)\mathcal{B}^2}{1-k^2}, & 0 \leqslant k < 1; \\[2mm] \dfrac{8(1-\gamma)}{\pi^2}, & k = 1; \\[2mm] \dfrac{\pi^2(1-\gamma)}{4(k^2-1)\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)}, & k > 1; \end{cases}$$

and

$$P_2 = \begin{cases} \dfrac{\mathcal{B}^2+2}{3}P_1, & 0 \leqslant k < 1; \\ \dfrac{2}{3}P_1, & k = 1; \\ \dfrac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{24\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)}P_1, & k > 1; \end{cases}$$

where

$$\mathcal{B} = \frac{2}{\pi}\arccos k \tag{1.11}$$

and $\mathcal{K}(\theta)$ *is the complete elliptic integral of the first kind.*

Lemma 1.2 ([17])**.** *Let* $h \in \mathcal{P}$ *be given by*

$$h(z) = 1 + c_1 z + c_2 z^2 + \ldots, \quad z \in \mathcal{U}. \tag{1.12}$$

Then

$$|vc_1^2 - c_2| \leqslant \begin{cases} -4v+2, & v \leqslant 0; \\ 2, & 0 \leqslant v \leqslant 1; \\ 4v-2, & v \geqslant 1. \end{cases} \tag{1.13}$$

When $v < 0$ or $v > 1$, the equality holds if and only if $h_1(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0 < v < 1$, then the equality holds if and only if $h_1(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $v = 0$, the equality holds if and only if

$$h_1(z) = \left(\frac{1}{2} + \frac{1}{2}\eta\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\eta\right)\frac{1-z}{1+z}, \quad 0 \leqslant \eta \leqslant 1, \tag{1.14}$$

or one of its rotations. If $v = 1$, the equality holds if and only if $h_1(z)$ is the reciprocal of one of the functions such that the equality holds in the case $v = 0$.

The above upper bound is sharp. When $0 < v < 1$, it can be improved as follows:

$$|vc_1^2 - c_2| + v|c_1|^2 \leqslant 2, \quad 0 < v \leqslant \tfrac{1}{2}, \tag{1.15}$$

and

$$|vc_1^2 - c_2| + (1-v)|c_1|^2 \leqslant 2, \quad \tfrac{1}{2} < v < 1. \tag{1.16}$$

2. Main results

In this section, we will give some upper bounds for the Fekete-Szegő functional $|\mu a_2^2 - a_3|$.

In order to prove our main results we have to recall the following facts.

First, the following calculations will be used in the proofs of each of Theorems 2.1–2.11. By geometric interpretation there exists a function w satisfying the conditions of the Schwarz lemma such that

$$\frac{(s-t)z(D^n_{\lambda,\mu}f(z))'}{D^n_{\lambda,\mu}f(sz) - D^n_{\lambda,\mu}f(tz)} = P_{k,\gamma}(w(z)), \quad z \in \mathcal{U}, \tag{2.1}$$

where $P_{k,\gamma}$ is the function defined in Lemma 1.1.

Define the function h in $\mathcal{P}$ by

$$h(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \ldots, \quad z \in \mathcal{U}.$$

It follows that

$$w(z) = \frac{c_1}{2}z + \frac{1}{2}\Big(c_2 - \frac{c_1^2}{2}\Big)z^2 + \ldots$$

and

$$\begin{aligned} P_{k,\gamma}(w(z)) &= 1 + P_1\Big\{\frac{c_1}{2}z + \frac{1}{2}\Big(c_2 - \frac{c_1^2}{2}\Big)z^2 + \ldots\Big\} \\ &\quad + P_2\Big\{\frac{c_1}{2}z + \frac{1}{2}\Big(c_2 - \frac{c_1^2}{2}\Big)z^2 + \ldots\Big\}^2 + \ldots \\ &= 1 + \frac{P_1 c_1}{2}z + \Big\{\frac{1}{2}\Big(c_2 - \frac{c_1^2}{2}\Big)P_1 + \frac{1}{4}c_1^2 P_2\Big\}z^2. \end{aligned} \tag{2.2}$$

Thus, by using (2.1) and (2.2), we obtain

$$a_2 = \frac{P_1}{2(2-s-t)A_2^n}c_1, \tag{2.3}$$

and

$$a_3 = \frac{1}{2(3-s^2-st-t^2)A_3^n}\Big\{P_1\Big(c_2 - \frac{c_1^2}{2}\Big) + \frac{P_2 c_1^2}{2} - \frac{(s+t)P_1^2 c_1^2}{2(s+t-2)}\Big\}. \tag{2.4}$$

In all our theorems and corollaries we suppose that $s \neq t$ and $s + t \neq 2$.

Theorem 2.1. *Let the function f given by (1.1) be in the class k-$\mathrm{UST}^n_{\lambda,\mu}(s,t,\gamma)$ $(-1 \leqslant \gamma < 1;\ 0 \leqslant k < 1)$ and let $s^2+st+t^2>3$. Then*

$$|\mu a_2^2 - a_3| \leqslant \begin{cases} \frac{2(1-\gamma)\mathcal{B}^2}{(1-k^2)(s^2+st+t^2-3)A_3^n}\Big(\frac{2(1-\gamma)\mathcal{B}^2(s^2+st+t^2-3)A_3^n}{(1-k^2)(s+t-2)^2A_2^{2n}}\mu \\ \quad +\frac{\mathcal{B}^2+2}{3} - \frac{2(s+t)\mathcal{B}^2}{(1-k^2)(s+t-2)}\Big), & \mu \geqslant \sigma_1; \\ \frac{2(1-\gamma)\mathcal{B}^2}{(1-k^2)(s^2+st+t^2-3)A_3^n}, & \sigma_2 \leqslant \mu \leqslant \sigma_1; \\ \frac{2(1-\gamma)\mathcal{B}^2}{(1-k^2)(s^2+st+t^2-3)A_3^n}\Big(\frac{2(s+t)\mathcal{B}^2}{(1-k^2)(s+t-2)} \\ \quad -\frac{\mathcal{B}^2+2}{3} - \frac{2(1-\gamma)\mathcal{B}^2(s^2+st+t^2-3)A_3^n}{(1-k^2)(s+t-2)^2A_2^{2n}}\mu\Big), & \mu \leqslant \sigma_2; \end{cases} \tag{2.5}$$

where $\mathcal{B}$ is given by (1.11), and

$$\sigma_1 = \frac{(s+t-2)A_2^{2n}}{(s^2+st+t^2-3)A_3^n}\Big(s+t-\frac{(s+t-2)(1-k^2)(\mathcal{B}^2-1)}{6(1-\gamma)\mathcal{B}^2}\Big), \tag{2.6}$$

$$\sigma_2 = \frac{(s+t-2)A_2^{2n}}{(s^2+st+t^2-3)A_3^n}\Big(s+t-\frac{(s+t-2)(1-k^2)(\mathcal{B}^2+5)}{6(1-\gamma)\mathcal{B}^2}\Big). \tag{2.7}$$

Each of the estimates in (2.5) is sharp.

Proof. From (2.3) and (2.4) we have

$$\mu a_2^2 - a_3 = \mu\frac{P_1^2}{4(2-s-t)^2A_2^{2n}}c_1^2 - \frac{1}{2(3-s^2-st-t^2)A_3^n} \times \Big\{P_1\Big(c_2-\frac{c_1^2}{2}\Big)+\frac{P_2c_1^2}{2}-\frac{(s+t)P_1^2c_1^2}{2(s+t-2)}\Big\}, \tag{2.8}$$

from Lemma 1.1 for $0 \leqslant k < 1$ if we put $P_2 = (\mathcal{B}^2+2)P_1/3$, we get

$$\mu a_2^2 - a_3 = \frac{P_1}{2}\Big\{\Big[\mu\frac{P_1}{2(2-s-t)^2A_2^{2n}} - \frac{1}{(3-s^2-st-t^2)A_3^n} \times \Big(-\frac{s+t}{2(2-s-t)}P_1+\frac{\mathcal{B}^2-1}{6}\Big)\Big]c_1^2 - \frac{1}{2(3-s^2-st-t^2)A_3^n}c_2\Big\}, \tag{2.9}$$

again from Lemma 1.1 for $0 \leqslant k < 1$ if we take $P_1 = 2(1-\gamma)\mathcal{B}^2/(1-k^2)$, we obtain

$$\begin{aligned} \mu a_2^2 - a_3 &= \frac{(1-\gamma)\mathcal{B}^2}{(1-k^2)}\Big\{\Big[\mu\frac{(1-\gamma)\mathcal{B}^2}{(1-k^2)(s+t-2)^2A_2^{2n}} - \frac{(1-\gamma)(s+t)\mathcal{B}^2}{(1-k^2)(s+t-2)(s^2+st+t^2-3)A_3^n} \\ &\quad + \frac{\mathcal{B}^2-1}{6(s^2+st+t^2-3)A_3^n}\Big]c_1^2 + \frac{1}{(s^2+st+t^2-3)A_3^n}c_2\Big\} \\ &= \frac{-(1-\gamma)\mathcal{B}^2}{(1-k^2)(s^2+st+t^2-3)A_3^n}\{vc_1^2 - c_2\} \end{aligned} \tag{2.10}$$

where

$$v:=\Big[\mu\frac{-(1-\gamma)\mathcal{B}^2(s^2+st+t^2-3)A_3^n}{(1-k^2)(s+t-2)^2A_2^{2n}}+\frac{(1-\gamma)(s+t)\mathcal{B}^2}{(1-k^2)(s+t-2)}-\frac{\mathcal{B}^2-1}{6}\Big]. \tag{2.11}$$

We have

$$|\mu a_2^2-a_3|=\frac{(1-\gamma)\mathcal{B}^2}{(1-k^2)(s^2+st+t^2-3)A_3^n}|vc_1^2-c_2|. \tag{2.12}$$

If $\mu\geqslant\sigma_1$, then, according to Lemma 1.2, we get

$$\begin{aligned}|\mu a_2^2-a_3|\leqslant{}&\frac{2(1-\gamma)\mathcal{B}^2}{(1-k^2)(s^2+st+t^2-3)A_3^n}\\&\times\Big(\frac{2(1-\gamma)\mathcal{B}^2(s^2+st+t^2-3)A_3^n}{(1-k^2)(s+t-2)^2A_2^{2n}}\mu+\frac{\mathcal{B}^2+2}{3}-\frac{2(s+t)\mathcal{B}^2}{(1-k^2)(s+t-2)}\Big),\end{aligned} \tag{2.13}$$

which is the first assertion of (2.5).

Next, if $\mu\leqslant\sigma_2$, by applying Lemma 1.2 we get

$$\begin{aligned}|\mu a_2^2-a_3|\leqslant{}&\frac{2(1-\gamma)\mathcal{B}^2}{(1-k^2)(s^2+st+t^2-3)A_3^n}\\&\times\Big(\frac{2(s+t)\mathcal{B}^2}{(1-k^2)(s+t-2)}-\frac{\mathcal{B}^2+2}{3}-\frac{2(1-\gamma)\mathcal{B}^2(s^2+st+t^2-3)A_3^n}{(1-k^2)(s+t-2)^2A_2^{2n}}\mu\Big),\end{aligned} \tag{2.14}$$

which is the third assertion of (2.5)

If $\sigma_2\leqslant\mu\leqslant\sigma_1$, by using again Lemma 1.2 we obtain

$$|\mu a_2^2-a_3|\leqslant\frac{2(1-\gamma)\mathcal{B}^2}{(1-k^2)(s^2+st+t^2-3)A_3^n}, \tag{2.15}$$

which is the second part of the assertion (2.5).

To show that these bounds are sharp, we define functions K_{φ_m}, $m=2,3,4,\dots$, by

$$\frac{(s-t)z(D^n_{\lambda,\mu}K_{\varphi_m}(z))'}{D^n_{\lambda,\mu}K_{\varphi_m}(sz)-D^n_{\lambda,\mu}K_{\varphi_m}(tz)}=P_{k,\gamma}(z^{m-1}),\quad K_{\varphi_m}(0)=0=K'_{\varphi_m}(0)-1 \tag{2.16}$$

and functions F_α and G_α, $0\leqslant\alpha<1$, by

$$\frac{(s-t)z(D^n_{\lambda,\mu}F_\alpha(z))'}{D^n_{\lambda,\mu}F_\lambda(sz)-D^n_{\lambda,\mu}F_\alpha(tz)}=P_{k,\gamma}\Big(\frac{z(z+\alpha)}{1+\alpha z}\Big),\quad F_\lambda(0)=0=F'_\lambda(0)-1 \tag{2.17}$$

and

$$(2.18)\quad \frac{(s-t)z(D^n_{\lambda,\mu}G_\alpha(z))'}{D^n_{\lambda,\mu}G_\alpha(sz)-D^n_{\lambda,\mu}G_\alpha(tz)} = P_{k,\gamma}\Big(\frac{-z(z+\alpha)}{1+\alpha z}\Big),\quad G_\lambda(0)=0=G'_\lambda(0)-1.$$

Clearly, these functions satisfy $K_{\varphi_m}, F_\alpha, G_\alpha \in k\text{-UST}^n_{\lambda,\mu}(s,t,\gamma)$.

Also, we write $K_\varphi = K_{\varphi_2}$. If $\mu > \sigma_1$, then equality holds in (2.5) if and only if equality holds in (2.13). This happens if and only if $|c_1| = 2$ and $|c_1^2 - c_2| = 2$. Thus $w(z) = z$, i.e., equality holds if and only if f is K_φ or one of its rotations. If $\mu < \sigma_2$, then equality holds in (2.5) if and only if equality holds in (2.14). This happens if and only if $|c_1| = 2$ and $|c_2| = 2$. Thus $w(z) = z$, i.e., equality holds if and only if f is K_φ or one of its rotations. If $\mu = \sigma_2$, then equality holds if and only if $|c_2| = 2$. In this case, equality holds if and only if f is F_α or one of its rotations.

Similarly, for $\mu = \sigma_1$, equality holds if and only if f is G_α or one of its rotations. Finally, if $\sigma_2 < \mu < \sigma_1$, then equality holds if and only if f is K_{φ_3} or one of its rotations.

The proof of Theorem 2.1 is now completed. □

Corollary 2.2. *Let the function f given by* (1.1) *be in the class k-$\text{UST}^n_{\lambda,\mu}(s,t,\gamma)$; $-1 \leqslant \gamma < 1$; $0 \leqslant k < 1$, and $s^2+st+t^2 < 3$. Then*

$$(2.19)\qquad |\mu a_2^2 - a_3| \leqslant \begin{cases} \frac{2(1-\gamma)\mathcal{B}^2}{(1-k^2)(3-s^2-st-t^2)A_3^n}\Big(\frac{2(1-\gamma)\mathcal{B}^2(s^2+st+t^2-3)A_3^n}{(1-k^2)(s+t-2)^2A_2^{2n}}\mu \\ \qquad +\frac{\mathcal{B}^2+2}{3} - \frac{2(s+t)\mathcal{B}^2}{(1-k^2)(s+t-2)}\Big), & \mu \leqslant \sigma_1; \\ \frac{2(1-\gamma)\mathcal{B}^2}{(1-k^2)(3-s^2-st-t^2)A_3^n}, & \sigma_1 \leqslant \mu \leqslant \sigma_2; \\ \frac{2(1-\gamma)\mathcal{B}^2}{(1-k^2)(3-s^2-st-t^2)A_3^n}\Big(\frac{2(s+t)\mathcal{B}^2}{(1-k^2)(s+t-2)} \\ \qquad -\frac{\mathcal{B}^2+2}{3} - \frac{2(1-\gamma)\mathcal{B}^2(s^2+st+t^2-3)A_3^n}{(1-k^2)(s+t-2)^2A_2^{2n}}\mu\Big), & \mu \geqslant \sigma_2; \end{cases}$$

where $\mathcal{B}$, σ_1 and σ_2 are given by (1.11), (2.6), (2.7), *respectively.*

Theorem 2.3. *Let the function f given by* (1.1) *be in the class k-$\text{UST}^n_{\lambda,\mu}(s,t,\gamma)$; $-1 \leqslant \gamma < 1$; $k = 1$, and $s^2+st+t^2 > 3$. Then*

$$(2.20)\qquad |\mu a_2^2 - a_3| \leqslant \begin{cases} \frac{16(1-\gamma)}{\pi^2(s^2+st+t^2-3)A_3^n}\Big(\frac{4(1-\gamma)(s^2+st+t^2-3)A_3^n}{\pi^2(s+t-2)^2A_2^{2n}}\mu \\ \qquad +\frac{1}{3} - \frac{4(s+t)(1-\gamma)}{\pi^2(s+t-2)}\Big), & \mu \geqslant \delta_1; \\ \frac{8(1-\gamma)}{\pi^2(s^2+st+t^2-3)A_3^n}, & \delta_2 \leqslant \mu \leqslant \delta_1; \\ \frac{16(1-\gamma)}{\pi^2(s^2+st+t^2-3)A_3^n}\Big(\frac{4(s+t)(1-\gamma)}{\pi^2(s+t-2)} \\ \qquad -\frac{1}{3} - \frac{4(1-\gamma)(s^2+st+t^2-3)A_3^n}{\pi^2(s+t-2)^2A_2^{2n}}\mu\Big), & \mu \leqslant \delta_2; \end{cases}$$

where

$$\delta_1 = \frac{(s+t-2)A_2^{2n}}{(s^2+st+t^2-3)A_3^n}\Big(s+t+\frac{\pi^2(s+t-2)}{24(1-\gamma)}\Big), \tag{2.21}$$

$$\delta_2 = \frac{(s+t-2)A_2^{2n}}{(s^2+st+t^2-3)A_3^n}\Big(s+t-\frac{5\pi^2(s+t-2)}{24(1-\gamma)}\Big). \tag{2.22}$$

The results are sharp.

P r o o f. From Lemma 1.1 for $k=1$ if we put $P_2 = 2P_1/3$ in (2.8), we get

$$\begin{aligned}\mu a_2^2 - a_3 = \frac{P_1}{2}\Big\{\Big[\mu\frac{P_1}{2(s+t-2)^2A_2^{2n}} - \frac{1}{(s^2+st+t^2-3)A_3^n}\\ \times\Big(\frac{s+t}{2(s+t-2)}P_1+\frac{1}{6}\Big)\Big]c_1^2 + \frac{1}{(s^2+st+t^2-3)A_3^n}c_2\Big\},\end{aligned} \tag{2.23}$$

again from Lemma 1.1 for $k=1$ if we take $P_1 = 8(1-\gamma)/\pi^2$, we obtain

$$\begin{aligned}\mu a_2^2 - a_3 &= \frac{4(1-\gamma)}{\pi^2}\Big\{\Big[\mu\frac{4(1-\gamma)}{\pi^2(s+t-2)^2A_2^{2n}} - \frac{1}{(s^2+st+t^2-3)A_3^n}\\ &\quad\times\Big(\frac{4(s+t)(1-\gamma)}{\pi^2(s+t-2)}+\frac{1}{6}\Big)\Big]c_1^2 + \frac{1}{(s^2+st+t^2-3)A_3^n}c_2\Big\}\\ &= \frac{-4(1-\gamma)}{\pi^2(s^2+st+t^2-3)A_3^n}\{vc_1^2 - c_2\}\end{aligned} \tag{2.24}$$

where

$$v := \Big[\mu\frac{-4(1-\gamma)(s^2+st+t^2-3)A_3^n}{\pi^2(s+t-2)^2A_2^{2n}} + \frac{4(s+t)(1-\gamma)}{\pi^2(s+t-2)} + \frac{1}{6}\Big]. \tag{2.25}$$

We have

$$|\mu a_2^2 - a_3| = \frac{4(1-\gamma)}{\pi^2(s^2+st+t^2-3)A_3^n}|vc_1^2 - c_2|. \tag{2.26}$$

If $\mu \geqslant \delta_1$, then, according to Lemma 1.2, we get

$$\begin{aligned}|\mu a_2^2 - a_3| \leqslant \frac{16(1-\gamma)}{\pi^2(s^2+st+t^2-3)A_3^n}\\ \times\Big(\frac{4(1-\gamma)(s^2+st+t^2-3)A_3^n}{\pi^2(s+t-2)^2A_2^{2n}}\mu + \frac{1}{3} - \frac{4(s+t)(1-\gamma)}{\pi^2(s+t-2)}\Big)\end{aligned} \tag{2.27}$$

which is the first assertion of (2.20).

Next, if $\mu \leqslant \delta_2$, by applying Lemma 1.2 we get

$$(2.28)\qquad |\mu a_2^2 - a_3| \leqslant \frac{16(1-\gamma)}{\pi^2(s^2+st+t^2-3)A_3^n} \times \Big(\frac{4(s+t)(1-\gamma)}{\pi^2(s+t-2)} - \frac{1}{3} - \frac{4(1-\gamma)(s^2+st+t^2-3)A_3^n}{\pi^2(s+t-2)^2A_2^{2n}}\mu\Big),$$

which is the third assertion of (2.20)

If $\delta_2 \leqslant \mu \leqslant \delta_1$, by using again Lemma 1.2 we obtain

$$(2.29)\qquad |\mu a_2^2 - a_3| \leqslant \frac{8(1-\gamma)}{\pi^2(s^2+st+t^2-3)A_3^n},$$

which is the second part of the assertion (2.20).

The sharpness of the estimates in (2.20) can be proved as in Theorem 2.1, using K_{φ_m}, F_α, G_α. □

Corollary 2.4. *Let the function f given by* (1.1) *be in the class k-$\mathrm{UST}^n_{\lambda,\mu}(s,t,\gamma)$; $-1 \leqslant \gamma < 1$; $k = 1$, and $s^2+st+t^2 < 3$. Then*

$$(2.30)\qquad |\mu a_2^2 - a_3| \leqslant \begin{cases} \frac{16(1-\gamma)}{\pi^2(3-s^2-st-t^2)A_3^n}\Big(\frac{4(1-\gamma)(s^2+st+t^2-3)A_3^n}{\pi^2(s+t-2)^2A_2^{2n}}\mu \\ \qquad + \frac{1}{3} - \frac{4(s+t)(1-\gamma)}{\pi^2(s+t-2)}\Big), & \mu \leqslant \delta_1; \\ \frac{8(1-\gamma)}{\pi^2(3-s^2-st-t^2)A_3^n}, & \delta_1 \leqslant \mu \leqslant \delta_2; \\ \frac{16(1-\gamma)}{\pi^2(3-s^2-st-t^2)A_3^n}\Big(\frac{4(s+t)(1-\gamma)}{\pi^2(s+t-2)} \\ \qquad - \frac{1}{3} - \frac{4(1-\gamma)(s^2+st+t^2-3)A_3^n}{\pi^2(s+t-2)^2A_2^{2n}}\mu\Big), & \mu \geqslant \delta_2; \end{cases}$$

where δ_1 and δ_2 are given by (2.21) *and* (2.22), *respectively.*

Theorem 2.5. *Let the function f given by* (1.1) *be in the class k-$\mathrm{UST}^n_{\lambda,\mu}(s,t,\gamma)$; $-1 \leqslant \gamma < 1$; $1 < k < \infty$, $s^2+st+t^2 > 3$ and let θ be the unique positive number in the open interval* $(0,1)$ *defined by* (1.7). *Then*

$$(2.31)\qquad |\mu a_2^2 - a_3| \leqslant \begin{cases} \frac{P_1}{2(s^2+st+t^2-3)A_3^n}\Big(\frac{2P_1(s^2+st+t^2-3)A_3^n}{(s+t-2)^2A_2^{2n}}\mu \\ \qquad + \frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{12\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)} - \frac{2(s+t)}{(s+t-2)}P_1\Big), & \mu \geqslant \varrho_1; \\ \frac{P_1}{(s^2+st+t^2-3)A_3^n}, & \varrho_2 \leqslant \mu \leqslant \varrho_1; \\ \frac{P_1}{2(s^2+st+t^2-3)A_3^n}\Big(\frac{2(s+t)}{(s+t-2)}P_1 - \frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{12\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)} \\ \qquad - \frac{2P_1(s^2+st+t^2-3)A_3^n}{(s+t-2)^2A_2^{2n}}\mu\Big), & \mu \leqslant \varrho_2; \end{cases}$$

where $\mathcal{K}(t)$ is the complete elliptic integral of the first kind, P_1 is given by Lemma 1.1 *and*

$$\varrho_1 = \frac{2(s+t-2)^2 A_2^{2n}}{P_1(s^2+st+t^2-3)A_3^n} \times \Big(\frac{(s+t)}{2(s+t-2)}P_1 + \frac{1}{2} - \frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{48\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)}\Big), \tag{2.32}$$

$$\varrho_2 = \frac{2(s+t-2)^2 A_2^{2n}}{P_1(s^2+st+t^2-3)A_3^n} \times \Big(\frac{(s+t)}{2(s+t-2)}P_1 - \frac{1}{2} - \frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{48\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)}\Big). \tag{2.33}$$

Proof. From Lemma 1.1 for $1 < k < \infty$ if we put $P_2 = [4\mathcal{K}^2(\theta)(\theta^2+6\theta+1) - \pi^2]P_1/(24\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta))$ in (2.8), we obtain

$$\begin{aligned}\mu a_2^2 - a_3 &= \frac{P_1}{2}\Big\{\Big[\mu\frac{P_1}{2(s+t-2)^2A_2^{2n}} - \frac{1}{(s^2+st+t^2-3)A_3^n}\Big(\frac{s+t}{2(s+t-2)}P_1 \\ &\quad - \frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{48\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)} + \frac{1}{2}\Big)\Big]c_1^2 + \frac{1}{(s^2+st+t^2-3)A_3^n}c_2\Big\} \\ &= \frac{-P_1}{2(s^2+st+t^2-3)A_3^n}\{vc_1^2 - c_2\}\end{aligned} \tag{2.34}$$

where

$$v := \Big[\mu\frac{-P_1(s^2+st+t^2-3)A_3^n}{2(s+t-2)^2A_2^{2n}} - \frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{48\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)} + \frac{s+t}{2(s+t-2)}P_1 + \frac{1}{2}\Big]. \tag{2.35}$$

We have

$$|\mu a_2^2 - a_3| = \frac{P_1}{2(s^2+st+t^2-3)A_3^n}|vc_1^2 - c_2|. \tag{2.36}$$

If $\mu \geqslant \varrho_1$, then, according to Lemma 1.2, we get

$$|\mu a_2^2 - a_3| \leqslant \frac{P_1}{2(s^2+st+t^2-3)A_3^n}\Big(\frac{2P_1(s^2+st+t^2-3)A_3^n}{(s+t-2)^2A_2^{2n}}\mu + \frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{12\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)} - \frac{2(s+t)}{(s+t-2)}P_1\Big), \tag{2.37}$$

which is the first assertion of (2.31).

Next, if $\mu \leqslant \varrho_2$, by applying Lemma 1.2 we get

$$(2.38)\qquad |\mu a_2^2 - a_3| \leqslant \frac{P_1}{2(s^2+st+t^2-3)A_3^n}\Big(\frac{2(s+t)}{(s+t-2)}P_1 - \frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{12\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)} - \frac{2P_1(s^2+st+t^2-3)A_3^n}{(s+t-2)^2A_2^{2n}}\mu\Big),$$

which is the third assertion of (2.31).

If $\varrho_2 \leqslant \mu \leqslant \varrho_1$, by using again Lemma 1.2 we obtain

$$(2.39)\qquad |\mu a_2^2 - a_3| \leqslant \frac{P_1}{(s^2+st+t^2-3)A_3^n},$$

which is the second part of the assertion (2.31).

The sharpness of the estimates in (2.31) can be proved as in Theorem 2.1. □

Corollary 2.6. *Let the function f given by* (1.1) *be in the class k-$\mathrm{UST}^n_{\lambda,\mu}(s,t,\gamma)$; $-1 \leqslant \gamma < 1$; $1 < k < \infty$, $s^2+st+t^2 < 3$ and let θ be the unique positive number in the open interval* $(0,1)$ *defined by* (1.7)*. Then*

$$(2.40)\qquad |\mu a_2^2 - a_3| \leqslant \begin{cases} \frac{P_1}{2(3-s^2-st-t^2)A_3^n}\Big(\frac{2P_1(s^2+st+t^2-3)A_3^n}{(s+t-2)^2A_2^{2n}}\mu + \frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{12\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)} - \frac{2(s+t)}{(s+t-2)}P_1\Big), & \mu \leqslant \varrho_1; \\ \frac{P_1}{(3-s^2-st-t^2)A_3^n}, & \varrho_1 \leqslant \mu \leqslant \varrho_2; \\ \frac{P_1}{2(3-s^2-st-t^2)A_3^n}\Big(\frac{2(s+t)}{(s+t-2)}P_1 - \frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{12\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)} - \frac{2P_1(s^2+st+t^2-3)A_3^n}{(s+t-2)^2A_2^{2n}}\mu\Big), & \mu \geqslant \varrho_2; \end{cases}$$

where $\mathcal{K}(t)$ is the complete elliptic integral of the first kind, P_1, ϱ_1, ϱ_2 are given by Lemma 1.1, (2.32) *and* (2.33)*, respectively.*

Theorem 2.7. *Let the function f given by* (1.1) *be in the class k-$\mathrm{UST}^n_{\lambda,\mu}(s,t,\gamma)$; $-1 \leqslant \gamma < 1$; $0 \leqslant k < 1$, and $s^2+st+t^2 > 3$. Then*

$$(2.41)\qquad |\mu a_2^2 - a_3| + (\mu - \sigma_2)|a_2^2| \leqslant \frac{2(1-\gamma)\mathcal{B}^2}{(1-k^2)(s^2+st+t^2-3)A_3^n}, \qquad \sigma_2 \leqslant \mu \leqslant \sigma_3$$

and

$$(2.42)\qquad |\mu a_2^2 - a_3| + (\sigma_1 - \mu)|a_2^2| \leqslant \frac{2(1-\gamma)\mathcal{B}^2}{(1-k^2)(s^2+st+t^2-3)A_3^n}, \qquad \sigma_3 \leqslant \mu \leqslant \sigma_1$$

where $\mathcal{B}$, σ_1 and σ_2 are given by (1.11), (2.6) and (2.7), respectively, and

$$\sigma_3 = \frac{(s+t-2)A_2^{2n}}{(s^2+st+t^2-3)A_3^n}\Big(s+t-\frac{(s+t-2)(1-k^2)(\mathcal{B}^2+2)}{6(1-\gamma)\mathcal{B}^2}\Big). \tag{2.43}$$

P r o o f. Suppose that $0 \leqslant k < 1$ and $\sigma_2 \leqslant \mu \leqslant \sigma_3$. Using (2.3) for $|\mu a_2^2 - a_3|$ and (2.12) for $|a_2|$, we have

$$\begin{aligned} |\mu a_2^2 - a_3| + (\mu - \sigma_2)|a_2^2| &= \frac{(1-\gamma)\mathcal{B}^2}{(1-k^2)(s^2+st+t^2-3)A_3^n}|vc_1^2 - c_2| \\ &\quad + (\mu-\sigma_2)\frac{(1-\gamma)^2\mathcal{B}^4}{(1-k^2)^2(s+t-2)^2A_2^{2n}}|c_1^2|. \end{aligned} \tag{2.44}$$

After some computations we get

$$\begin{aligned} |\mu a_2^2 - a_3| + (\mu - \sigma_2)|a_2^2| &= \frac{(1-\gamma)\mathcal{B}^2}{(1-k^2)(s^2+st+t^2-3)A_3^n} \\ &\quad \times \{|vc_1^2 - c_2| + (1-v)|c_1^2|\} \end{aligned} \tag{2.45}$$

where

$$v := \Big[\mu\frac{-(1-\gamma)\mathcal{B}^2(s^2+st+t^2-3)A_3^n}{(1-k^2)(s+t-2)^2A_2^{2n}} + \frac{(1-\gamma)(s+t)\mathcal{B}^2}{(1-k^2)(s+t-2)} - \frac{\mathcal{B}^2-1}{6}\Big].$$

From (1.16) we obtain

$$|\mu a_2^2 - a_3| + (\mu - \sigma_2)|a_2^2| \leqslant \frac{2(1-\gamma)\mathcal{B}^2}{(1-k^2)(s^2+st+t^2-3)A_3^n}, \tag{2.46}$$

which proves (2.41).

Similarly, for the values $\sigma_3 \leqslant \mu \leqslant \sigma_1$, we write

$$\begin{aligned} |\mu a_2^2 - a_3| + (\sigma_1 - \mu)|a_2^2| &= \frac{(1-\gamma)\mathcal{B}^2}{(1-k^2)(s^2+st+t^2-3)A_3^n}|vc_1^2 - c_2| \\ &\quad + (\sigma_1-\mu)\frac{(1-\gamma)^2\mathcal{B}^4}{(1-k^2)^2(s+t-2)^2A_2^{2n}}|c_1^2| \\ &= \frac{(1-\gamma)\mathcal{B}^2}{(1-k^2)(s^2+st+t^2-3)A_3^n}\{|vc_1^2 - c_2| + v|c_1^2|\}. \end{aligned} \tag{2.47}$$

From (1.15) we get

$$|\mu a_2^2 - a_3| + (\sigma_1 - \mu)|a_2^2| \leqslant \frac{2(1-\gamma)\mathcal{B}^2}{(1-k^2)(s^2+st+t^2-3)A_3^n}, \tag{2.48}$$

which proves (2.42).

This completes the proof of Theorem 2.7. □

Corollary 2.8. *Let the function f given by (1.1) be in the class k-$\mathrm{UST}^n_{\lambda,\mu}(s,t,\gamma)$; $-1 \leqslant \gamma < 1$; $0 \leqslant k < 1$, and let $s^2+st+t^2 < 3$. Then*

$$|\mu a_2^2 - a_3| + (\mu - \sigma_1)|a_2^2| \leqslant \frac{2(1-\gamma)\mathcal{B}^2}{(1-k^2)(3-s^2-st-t^2)A_3^n}, \qquad \sigma_1 \leqslant \mu \leqslant \sigma_3 \tag{2.49}$$

and

$$|\mu a_2^2 - a_3| + (\sigma_2 - \mu)|a_2^2| \leqslant \frac{2(1-\gamma)\mathcal{B}^2}{(1-k^2)(3-s^2-st-t^2)A_3^n}, \qquad \sigma_3 \leqslant \mu \leqslant \sigma_2 \tag{2.50}$$

where $\mathcal{B}$, σ_1, σ_2 and σ_3 are given by (1.11), (2.6), (2.7) and (2.43), respectively.

Theorem 2.9. *Let the function f given by (1.1) be in the class k-$\mathrm{UST}^n_{\lambda,\mu}(s,t,\gamma)$; $-1 \leqslant \gamma < 1$; $k = 1$, and let $s^2+st+t^2 > 3$. Then*

$$|\mu a_2^2 - a_3| + (\mu - \delta_2)|a_2^2| \leqslant \frac{8(1-\gamma)}{\pi^2(s^2+st+t^2-3)A_3^n}, \qquad \delta_2 \leqslant \mu \leqslant \delta_3 \tag{2.51}$$

and

$$|\mu a_2^2 - a_3| + (\delta_1 - \mu)|a_2^2| \leqslant \frac{8(1-\gamma)}{\pi^2(s^2+st+t^2-3)A_3^n}, \qquad \delta_3 \leqslant \mu \leqslant \delta_1 \tag{2.52}$$

where δ_1 and δ_2 are given by (2.21) and (2.22), respectively, and

$$\delta_3 = \frac{(s+t-2)A_2^{2n}}{(s^2+st+t^2-3)A_3^n}\left(s+t-\frac{2\pi^2(s+t-2)}{24(1-\gamma)}\right). \tag{2.53}$$

Corollary 2.10. *Let the function f given by (1.1) be in the class k-$\mathrm{UST}^n_{\lambda,\mu}(s,t,\gamma)$; $-1 \leqslant \gamma < 1$; $k = 1$, and let $s^2+st+t^2 < 3$. Then*

$$|\mu a_2^2 - a_3| + (\mu - \delta_1)|a_2^2| \leqslant \frac{8(1-\gamma)}{\pi^2(s^2+st+t^2-3)A_3^n}, \qquad \delta_1 \leqslant \mu \leqslant \delta_3 \tag{2.54}$$

and

$$|\mu a_2^2 - a_3| + (\delta_2 - \mu)|a_2^2| \leqslant \frac{8(1-\gamma)}{\pi^2(s^2+st+t^2-3)A_3^n}, \qquad \delta_3 \leqslant \mu \leqslant \delta_2 \tag{2.55}$$

where δ_1, δ_2 and δ_3 are given by (2.21), (2.22) and (2.53), respectively.

Theorem 2.11. *Let the function f given by* (1.1) *be in the class* k-$\mathrm{UST}^n_{\lambda,\mu}(s,t,\gamma)$*;* $-1 \leqslant \gamma < 1$*;* $1 < k < \infty$*,* $s^2+st+t^2 > 3$ *and let θ be the unique positive number in the open interval* $(0,1)$ *defined by* (1.7)*. Then*

$$|\mu a_2^2 - a_3| + (\mu - \varrho_2)|a_2^2| \leqslant \frac{P_1}{(s^2+st+t^2-3)A_3^n}, \qquad \varrho_2 \leqslant \mu \leqslant \varrho_3 \tag{2.56}$$

and

$$|\mu a_2^2 - a_3| + (\varrho_1 - \mu)|a_2^2| \leqslant \frac{P_1}{(s^2+st+t^2-3)A_3^n}, \qquad \varrho_3 \leqslant \mu \leqslant \varrho_1 \tag{2.57}$$

where ϱ_1, ϱ_2 *and* P_1 *are given by* (2.32), (2.33), *Lemma* 1.1, *respectively, and*

$$\varrho_3 = \frac{2(s+t-2)^2 A_2^{2n}}{P_1(s^2+st+t^2-3)A_3^n}\Big(\frac{s+t}{2(s+t-2)}P_1 - \frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{48\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)}\Big). \tag{2.58}$$

P r o o f s o f t h e T h e o r e m 2 . 9 a n d T h e o r e m 2 . 1 1 . The proofs of Theorem 2.9 and Theorem 2.11 are similar to the proof of Theorem 2.7, except for some obvious changes. Therefore, we omit the details.

Corollary 2.12. *Let the function f given by* (1.1) *be in the class* k-$\mathrm{UST}^n_{\lambda,\mu}(s,t,\gamma)$*;* $-1 \leqslant \gamma < 1$*;* $1 < k < \infty$*,* $s^2+st+t^2 < 3$ *and let θ be the unique positive number in the open interval* $(0,1)$ *defined by* (1.7)*. Then*

$$|\mu a_2^2 - a_3| + (\mu - \varrho_1)|a_2^2| \leqslant \frac{P_1}{(s^2+st+t^2-3)A_3^n}, \qquad \varrho_1 \leqslant \mu \leqslant \varrho_3 \tag{2.59}$$

and

$$|\mu a_2^2 - a_3| + (\varrho_2 - \mu)|a_2^2| \leqslant \frac{P_1}{(s^2+st+t^2-3)A_3^n}, \qquad \varrho_3 \leqslant \mu \leqslant \varrho_2 \tag{2.60}$$

where ϱ_1, ϱ_2, ϱ_3 *and* P_1 *are given by* (2.32), (2.33), (2.58) *and Lemma* 1.1, *respectively.*

3. Applications to functions defined by fractional derivatives

For functions $f, g \in \mathcal{A}$, given by $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ and $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$, we define the *Hadamard product* (or *convolution*) of $f(z)$ and $g(z)$ by

$$(f * g)(z) := z + \sum_{j=2}^{\infty} a_j b_j z^j =: (g * f)(z), \qquad z \in \mathcal{U}. \tag{3.1}$$

For fixed $g \in \mathcal{A}$, let $k\text{-UST}_{\lambda,\mu}^{n,g}(s,t,\gamma)$ be the class of functions $f \in \mathcal{A}$ for which $(f * g) \in k\text{-UST}_{\lambda,\mu}^{n}(s,t,\gamma)$.

In order to introduce the class $k\text{-UST}_{\lambda,\mu}^{n}(s,t,\gamma)$, we need

Definition 3.1 ([26]). Let $f(z)$ be an analytic function in a simply connected region of the z-plane containing the origin. The fractional derivative of f of order τ is defined by

$$D_z^{\tau} f(z) = \frac{1}{\Gamma(1-\tau)} \frac{\mathrm{d}}{\mathrm{d}z} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\tau}} \,\mathrm{d}\zeta, \qquad 0 \leqslant \tau < 1,$$

where the multiplicity of $(z-\zeta)^{\eta}$ is removed by requiring that $\log(z-\zeta)$ is real for $z - \zeta > 0$.

Using Definition 3.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [26] introduced the operator $\Omega^{\tau}\colon \mathcal{A} \to \mathcal{A}$ defined by

$$\Omega^{\tau} f(z) = \Gamma(2-\tau) z^{\tau} D_z^{\tau} f(z), \qquad \tau \neq 2, 3, 4, \ldots .$$

The class $k\text{-UST}_{\lambda,\mu}^{n,\tau}(s,t,\gamma)$ consists of functions $f \in \mathcal{A}$ for which

$$\Omega^{\tau} f \in k\text{-UST}_{\lambda,\mu}^{n}(s,t,\gamma).$$

The class $k\text{-UST}_{\lambda,\mu}^{n,\tau}(s,t,\gamma)$ is the special case of the class $k\text{-UST}_{\lambda,\mu}^{n,g}(s,t,\gamma)$ when

$$g(z) = z + \sum_{j=2}^{\infty} \frac{\Gamma(j+1)\Gamma(2-\gamma)}{\Gamma(j+1-\gamma)} z^j. \tag{3.2}$$

Now for the function $(f * g)(z) = z + a_2 g_2 z^2 + a_3 g_3 z^3 + \ldots$, we get the following theorem after an obvious change of the parameter μ:

Theorem 3.1. *Let $g(z) = z + \sum_{j=2}^{\infty} g_j z^j$ $(g_j > 0)$ and $s^2 + st + t^2 > 3$. If the function f given by* (1.1) *is in the class $k\text{-UST}_{\lambda,\mu}^{n,g}(s,t,\gamma)$; $-1 \leqslant \gamma < 1$; $0 \leqslant k < 1$,* *then*

$$|\mu a_2^2 - a_3| \leqslant \begin{cases} \frac{2(1-\gamma)\mathcal{B}^2}{(1-k^2)(s^2+st+t^2-3)g_3A_3^n}\Big(\frac{2(1-\gamma)\mathcal{B}^2(s^2+st+t^2-3)g_3A_3^n}{(1-k^2)(s+t-2)^2g_2^2A_2^{2n}}\mu \\ \qquad +\frac{\mathcal{B}^2+2}{3} - \frac{2(s+t)\mathcal{B}^2}{(1-k^2)(s+t-2)}\Big), & \mu \geqslant \sigma_1^*; \\ \frac{2(1-\gamma)\mathcal{B}^2}{(1-k^2)(s^2+st+t^2-3)g_3A_3^n}, & \sigma_2^* \leqslant \mu \leqslant \sigma_1^*; \\ \frac{2(1-\gamma)\mathcal{B}^2}{(1-k^2)(s^2+st+t^2-3)g_3A_3^n}\Big(\frac{2(s+t)\mathcal{B}^2}{(1-k^2)(s+t-2)} \\ \qquad -\frac{\mathcal{B}^2+2}{3} - \frac{2(1-\gamma)\mathcal{B}^2(s^2+st+t^2-3)g_3A_3^n}{(1-k^2)(s+t-2)^2g_2^2A_2^{2n}}\mu\Big), & \mu \leqslant \sigma_2^*; \end{cases} \tag{3.3}$$

where $\mathcal{B}$ *is given by* (1.11), *and*

$$\sigma_1^* = \frac{(s+t-2)g_2^2A_2^{2n}}{(s^2+st+t^2-3)g_3A_3^n}\Big(s+t-\frac{(s+t-2)(1-k^2)(\mathcal{B}^2-1)}{6(1-\gamma)\mathcal{B}^2}\Big), \tag{3.4}$$

$$\sigma_2^* = \frac{(s+t-2)g_2^2A_2^{2n}}{(s^2+st+t^2-3)g_3A_3^n}\Big(s+t-\frac{(s+t-2)(1-k^2)(\mathcal{B}^2+5)}{6(1-\gamma)\mathcal{B}^2}\Big). \tag{3.5}$$

Each of the estimates in (3.3) *is sharp.*

Corollary 3.2. *Let* $g(z) = z + \sum_{j=2}^{\infty} g_j z^j$; $g_j > 0$ *and* $s^2+st+t^2 < 3$. *If the function* f *given by* (1.1) *is in the class* k-$\mathrm{UST}_{\lambda,\mu}^{n,g}(s,t,\gamma)$; $-1 \leqslant \gamma < 1$; $0 \leqslant k < 1$, *then*

$$|\mu a_2^2 - a_3| \leqslant \begin{cases} \frac{2(1-\gamma)\mathcal{B}^2}{(1-k^2)(3-s^2-st-t^2)g_3A_3^n}\Big(\frac{2(1-\gamma)\mathcal{B}^2(s^2+st+t^2-3)g_3A_3^n}{(1-k^2)(s+t-2)^2g_2^2A_2^{2n}}\mu \\ \qquad +\frac{\mathcal{B}^2+2}{3} - \frac{2(s+t)\mathcal{B}^2}{(1-k^2)(s+t-2)}\Big), & \mu \leqslant \sigma_1^*; \\ \frac{2(1-\gamma)\mathcal{B}^2}{(1-k^2)(3-s^2-st-t^2)g_3A_3^n}, & \sigma_1^* \leqslant \mu \leqslant \sigma_2^*; \\ \frac{2(1-\gamma)\mathcal{B}^2}{(1-k^2)(3-s^2-st-t^2)g_3A_3^n}\Big(\frac{2(s+t)\mathcal{B}^2}{(1-k^2)(s+t-2)} \\ \qquad -\frac{\mathcal{B}^2+2}{3} - \frac{2(1-\gamma)\mathcal{B}^2(s^2+st+t^2-3)g_3A_3^n}{(1-k^2)(s+t-2)^2g_2^2A_2^{2n}}\mu\Big), & \mu \geqslant \sigma_2^*; \end{cases} \tag{3.6}$$

where $\mathcal{B}$, σ_1^*, σ_2^* *are given by* (1.11), (3.4) *and* (3.5), *respectively.*

Theorem 3.3. *Let* $g(z) = z + \sum_{j=2}^{\infty} g_j z^j$; $g_j > 0$ *and* $s^2+st+t^2 > 3$. *If the function* f *given by* (1.1) *is in the class* k-$\mathrm{UST}_{\lambda,\mu}^{n,g}(s,t,\gamma)$; $-1 \leqslant \gamma < 1$; $k = 1$, *then*

$$|\mu a_2^2 - a_3| \leqslant \begin{cases} \frac{16(1-\gamma)}{\pi^2(s^2+st+t^2-3)g_3A_3^n}\Big(\frac{4(1-\gamma)(s^2+st+t^2-3)g_3A_3^n}{\pi^2(s+t-2)^2g_2^2A_2^{2n}}\mu \\ \qquad +\frac{1}{3} - \frac{4(s+t)(1-\gamma)}{\pi^2(s+t-2)}\Big), & \mu \geqslant \delta_1^*; \\ \frac{8(1-\gamma)}{\pi^2(s^2+st+t^2-3)g_3A_3^n}, & \delta_2^* \leqslant \mu \leqslant \delta_1^*; \\ \frac{16(1-\gamma)}{\pi^2(s^2+st+t^2-3)g_3A_3^n}\Big(\frac{4(s+t)(1-\gamma)}{\pi^2(s+t-2)} \\ \qquad -\frac{1}{3} - \frac{4(1-\gamma)(s^2+st+t^2-3)g_3A_3^n}{\pi^2(s+t-2)^2g_2^2A_2^{2n}}\mu\Big), & \mu \leqslant \delta_2^*; \end{cases} \tag{3.7}$$

where

$$\delta_1^* = \frac{(s+t-2)g_2^2A_2^{2n}}{(s^2+st+t^2-3)g_3A_3^n}\Big(s+t+\frac{\pi^2(s+t-2)}{24(1-\gamma)}\Big), \tag{3.8}$$

$$\delta_2^* = \frac{(s+t-2)g_2^2A_2^{2n}}{(s^2+st+t^2-3)g_3A_3^n}\Big(s+t-\frac{5\pi^2(s+t-2)}{24(1-\gamma)}\Big). \tag{3.9}$$

The results are sharp.

Corollary 3.4. *Let* $g(z) = z+\sum\limits_{j=2}^{\infty} g_jz^j$; $g_j > 0$ *and* $s^2+st+t^2 < 3$. *If the function* f *given by* (1.1) *is in the class* k-$\mathrm{UST}_{\lambda,\mu}^{n,g}(s,t,\gamma)$; $-1 \leqslant \gamma < 1$; $k = 1$, *then*

$$|\mu a_2^2 - a_3| \leqslant \begin{cases} \frac{16(1-\gamma)}{\pi^2(3-s^2-st-t^2)g_3A_3^n}\Big(\frac{4(1-\gamma)(s^2+st+t^2-3)g_3A_3^n}{\pi^2(s+t-2)^2g_2^2A_2^{2n}}\mu \\ \qquad +\frac{1}{3}-\frac{4(s+t)(1-\gamma)}{\pi^2(s+t-2)}\Big), & \mu \leqslant \delta_1^*; \\ \frac{8(1-\gamma)}{\pi^2(3-s^2-st-t^2)g_3A_3^n}, & \delta_1^* \leqslant \mu \leqslant \delta_2^*; \\ \frac{16(1-\gamma)}{\pi^2(3-s^2-st-t^2)g_3A_3^n}\Big(\frac{4(s+t)(1-\gamma)}{\pi^2(s+t-2)} \\ \qquad -\frac{1}{3}-\frac{4(1-\gamma)(s^2+st+t^2-3)g_3A_3^n}{\pi^2(s+t-2)^2g_2^2A_2^{2n}}\mu\Big), & \mu \geqslant \delta_2^*; \end{cases} \tag{3.10}$$

where δ_1^*, δ_2^* *are given by* (3.8) *and* (3.9), *respectively.*

Theorem 3.5. *Let* $g(z) = z+\sum\limits_{j=2}^{\infty} g_jz^j$; $g_j > 0$ *and* $s^2+st+t^2 > 3$. *Let the function* f *given by* (1.1) *be in the class* k-$\mathrm{UST}_{\lambda,\mu}^{n,g}(s,t,\gamma)$; $-1 \leqslant \gamma < 1$; $1 < k < \infty$, *and let* θ *be the unique positive number in the open interval* $(0,1)$ *defined by* (1.7). *Then*

$$|\mu a_2^2 - a_3| \leqslant \begin{cases} \frac{P_1}{2(s^2+st+t^2-3)g_3A_3^n}\Big(\frac{2P_1(s^2+st+t^2-3)g_3A_3^n}{(s+t-2)^2g_2^2A_2^{2n}}\mu \\ \qquad +\frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{12\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)}-\frac{2(s+t)}{(s+t-2)}P_1\Big), & \mu \geqslant \varrho_1^*; \\ \frac{P_1}{(s^2+st+t^2-3)g_3A_3^n}, & \varrho_2^* \leqslant \mu \leqslant \varrho_1^*; \\ \frac{P_1}{2(s^2+st+t^2-3)g_3A_3^n}\Big(\frac{2(s+t)}{(s+t-2)}P_1-\frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{12\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)} \\ \qquad -\frac{2P_1(s^2+st+t^2-3)g_3A_3^n}{(s+t-2)^2g_2^2A_2^{2n}}\mu\Big); & \mu \leqslant \varrho_2^*, \end{cases} \tag{3.11}$$

where $\mathcal{K}(t)$ *is the complete elliptic integral of the first kind,* P_1 *is given by Lemma* 1.1,

$$\varrho_1^* = \frac{2(s+t-2)^2g_2^2A_2^{2n}}{P_1(s^2+st+t^2-3)g_3A_3^n} \times\Big(\frac{(s+t)}{2(s+t-2)}P_1+\frac{1}{2}-\frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{48\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)}\Big), \tag{3.12}$$

$$(3.13)\qquad \varrho_2^* = \frac{2(s+t-2)^2 g_2^2 A_2^{2n}}{P_1(s^2+st+t^2-3)g_3A_3^n} \times \Big(\frac{(s+t)}{2(s+t-2)}P_1 - \frac{1}{2} - \frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{48\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)}\Big).$$

Corollary 3.6. *Let $g(z) = z + \sum\limits_{j=2}^{\infty} g_j z^j$; $g_j > 0$ and let $s^2+st+t^2 < 3$. Let the function f given by* (1.1) *be in the class k-$\mathrm{UST}_{\lambda,\mu}^{n,g}(s,t,\gamma)$; $-1 \leqslant \gamma < 1$; $1 < k < \infty$, and let θ be the unique positive number in the open interval* $(0,1)$ *defined by* (1.7). *Then*

$$(3.14)\qquad |\mu a_2^2 - a_3| \leqslant \begin{cases} \frac{P_1}{2(3-s^2-st-t^2)g_3A_3^n}\Big(\frac{2P_1(s^2+st+t^2-3)g_3A_3^n}{(s+t-2)^2g_2^2A_2^{2n}}\mu \\ \qquad +\frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{12\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)} - \frac{2(s+t)}{(s+t-2)}P_1\Big), & \mu \leqslant \varrho_1^*; \\ \frac{P_1}{(3-s^2-st-t^2)g_3A_3^n}, & \varrho_1^* \leqslant \mu \leqslant \varrho_2^*; \\ \frac{P_1}{2(3-s^2-st-t^2)g_3A_3^n}\Big(\frac{2(s+t)}{(s+t-2)}P_1 - \frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{12\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)} \\ \qquad -\frac{2P_1(s^2+st+t^2-3)g_3A_3^n}{(s+t-2)^2g_2^2A_2^{2n}}\mu\Big), & \mu \geqslant \varrho_2^*; \end{cases}$$

where $\mathcal{K}(t)$ is the complete elliptic integral of the first kind, P_1, ϱ_1^*, ϱ_2^* are given by Lemma 1.1, (3.12) *and* (3.13), *respectively.*

Since

$$(3.15)\qquad \Omega^\tau(f)(z) = z + \sum_{j=2}^{\infty} \frac{\Gamma(j+1)\Gamma(2-\tau)}{\Gamma(j+1-\tau)} a_j z^j,$$

we have

$$(3.16)\qquad g_2 := \frac{\Gamma(3)\Gamma(2-\tau)}{\Gamma(3-\tau)} = \frac{2}{2-\tau}$$

and

$$(3.17)\qquad g_3 := \frac{\Gamma(4)\Gamma(2-\tau)}{\Gamma(4-\tau)} = \frac{6}{(2-\tau)(3-\tau)}.$$

For g_2 and g_3 given by the above equalities, Theorem 3.1, Theorem 3.3 and Theorem 3.5 reduce to the following statements:

Theorem 3.7. *Let $\tau < 2$ and $s^2+st+t^2 > 3$. If the function f given by (1.1) is in the class k-$\mathrm{UST}^{n,\tau}_{\lambda,\mu}(s,t,\gamma)$; $-1 \leqslant \gamma < 1$; $0 \leqslant k < 1$, then*

$$(3.18)\qquad |\mu a_2^2 - a_3| \leqslant \begin{cases} \frac{(1-\gamma)(2-\tau)(3-\tau)\mathcal{B}^2}{3(1-k^2)(s^2+st+t^2-3)A_3^n}\Big(\frac{3(1-\gamma)(2-\tau)\mathcal{B}^2(s^2+st+t^2-3)}{(1-k^2)(3-\tau)(s+t-2)^2} \\ \qquad \times\frac{A_3^n}{A_2^{2n}}\mu + \frac{\mathcal{B}^2+2}{3} - \frac{2(s+t)\mathcal{B}^2}{(1-k^2)(s+t-2)}\Big), \quad \mu \geqslant \sigma_1^{**}; \\ \frac{(1-\gamma)(2-\tau)(3-\tau)\mathcal{B}^2}{3(1-k^2)(s^2+st+t^2-3)A_3^n}, \qquad \sigma_2^{**} \leqslant \mu \leqslant \sigma_1^{**}; \\ \frac{(1-\gamma)(2-\tau)(3-\tau)\mathcal{B}^2}{3(1-k^2)(s^2+st+t^2-3)A_3^n}\Big(\frac{2(s+t)\mathcal{B}^2}{(1-k^2)(s+t-2)} - \frac{\mathcal{B}^2+2}{3} \\ \qquad -\frac{3(1-\gamma)(2-\tau)\mathcal{B}^2(s^2+st+t^2-3)A_3^n}{(1-k^2)(3-\tau)(s+t-2)^2A_2^{2n}}\mu\Big), \quad \mu \leqslant \sigma_2^{**}; \end{cases}$$

where $\mathcal{B}$ is given by (1.11), and

$$(3.19)\ \sigma_1^{**} = \frac{2(3-\tau)(s+t-2)g_2^2A_2^{2n}}{3(2-\tau)(s^2+st+t^2-3)g_3A_3^n}\Big(s+t-\frac{(s+t-2)(1-k^2)(\mathcal{B}^2-1)}{6(1-\gamma)\mathcal{B}^2}\Big),$$

$$(3.20)\ \sigma_2^{**} = \frac{2(3-\tau)(s+t-2)g_2^2A_2^{2n}}{3(2-\tau)(s^2+st+t^2-3)g_3A_3^n}\Big(s+t-\frac{(s+t-2)(1-k^2)(\mathcal{B}^2+5)}{6(1-\gamma)\mathcal{B}^2}\Big).$$

Each of the estimates in (3.18) is sharp.

Corollary 3.8. *Let $\tau < 2$ and $s^2+st+t^2 < 3$. If the function f given by (1.1) is in the class k-$\mathrm{UST}^{n,\tau}_{\lambda,\mu}(s,t,\gamma)$; $-1 \leqslant \gamma < 1$; $0 \leqslant k < 1$, then*

$$(3.21)\qquad |\mu a_2^2 - a_3| \leqslant \begin{cases} \frac{(1-\gamma)(2-\tau)(3-\tau)\mathcal{B}^2}{3(1-k^2)(3-s^2-st-t^2)A_3^n}\Big(\frac{3(1-\gamma)(2-\tau)\mathcal{B}^2(s^2+st+t^2-3)}{(1-k^2)(3-\tau)(s+t-2)^2} \\ \qquad \times\frac{A_3^n}{A_2^{2n}}\mu + \frac{\mathcal{B}^2+2}{3} - \frac{2(s+t)\mathcal{B}^2}{(1-k^2)(s+t-2)}\Big), \quad \mu \leqslant \sigma_1^{**}; \\ \frac{(1-\gamma)(2-\tau)(3-\tau)\mathcal{B}^2}{3(1-k^2)(3-s^2-st-t^2)A_3^n}, \qquad \sigma_1^{**} \leqslant \mu \leqslant \sigma_2^{**}; \\ \frac{(1-\gamma)(2-\tau)(3-\tau)\mathcal{B}^2}{3(1-k^2)(3-s^2-st-t^2)A_3^n}\Big(\frac{2(s+t)\mathcal{B}^2}{(1-k^2)(s+t-2)} - \frac{\mathcal{B}^2+2}{3} \\ \qquad -\frac{3(1-\gamma)(2-\tau)\mathcal{B}^2(s^2+st+t^2-3)A_3^n}{(1-k^2)(3-\tau)(s+t-2)^2A_2^{2n}}\mu\Big), \quad \mu \geqslant \sigma_2^{**}; \end{cases}$$

where $\mathcal{B}$, σ_1^{**}, σ_2^{**} are given by (1.11), (3.19) and (3.20), respectively.

Theorem 3.9. *Let $\tau < 2$ and $s^2+st+t^2 > 3$. If the function f given by (1.1) is in the class k-$\mathrm{UST}^{n,\tau}_{\lambda,\mu}(s,t,\gamma)$; $-1 \leqslant \gamma < 1$; $k = 1$, then*

$$(3.22)\qquad |\mu a_2^2 - a_3| \leqslant \begin{cases} \frac{8(1-\gamma)(2-\tau)(3-\tau)}{3\pi^2(s^2+st+t^2-3)A_3^n}\Big(\frac{12(1-\gamma)(2-\tau)(s^2+st+t^2-3)A_3^n}{\pi^2(s+t-2)^2 2(3-\tau)A_2^{2n}}\mu \\ \qquad +\frac{1}{3} - \frac{4(s+t)(1-\gamma)}{\pi^2(s+t-2)}\Big), \quad \mu \geqslant \delta_1^{**}; \\ \frac{4(1-\gamma)(2-\tau)(3-\tau)}{3\pi^2(s^2+st+t^2-3)A_3^n}, \qquad \delta_2^{**} \leqslant \mu \leqslant \delta_1^{**}; \\ \frac{8(1-\gamma)(2-\tau)(3-\tau)}{3\pi^2(s^2+st+t^2-3)A_3^n}\Big(\frac{4(s+t)(1-\gamma)}{\pi^2(s+t-2)} \\ \qquad -\frac{1}{3} - \frac{12(1-\gamma)(2-\tau)(s^2+st+t^2-3)A_3^n}{\pi^2(s+t-2)^2 2(3-\tau)A_2^{2n}}\mu\Big), \quad \mu \leqslant \delta_2^{**}; \end{cases}$$

where

$$\delta_1^{**} = \frac{2(s+t-2)(3-\tau)A_2^{2n}}{3(s^2+st+t^2-3)(2-\tau)A_3^n}\Big(s+t+\frac{\pi^2(s+t-2)}{24(1-\gamma)}\Big), \tag{3.23}$$

$$\delta_2^{**} = \frac{2(s+t-2)(3-\tau)A_2^{2n}}{3(s^2+st+t^2-3)(2-\tau)A_3^n}\Big(s+t-\frac{5\pi^2(s+t-2)}{24(1-\gamma)}\Big). \tag{3.24}$$

The results are sharp.

Corollary 3.10. *Let $\tau < 2$ and $s^2+st+t^2 < 3$. If the function f given by (1.1) is in the class k-$\mathrm{UST}_{\lambda,\mu}^{n,\tau}(s,t,\gamma)$; $-1 \leqslant \gamma < 1$; $k=1$, then*

$$|\mu a_2^2 - a_3| \leqslant \begin{cases} \frac{8(1-\gamma)(2-\tau)(3-\tau)}{3\pi^2(3-s^2-st-t^2)A_3^n}\Big(\frac{12(1-\gamma)(2-\tau)(s^2+st+t^2-3)A_3^n}{\pi^2(s+t-2)^2 2(3-\tau)A_2^{2n}}\mu \\ \qquad +\frac{1}{3}-\frac{4(s+t)(1-\gamma)}{\pi^2(s+t-2)}\Big), & \mu \leqslant \delta_1^{**}; \\ \frac{4(1-\gamma)(2-\tau)(3-\tau)}{3\pi^2(3-s^2-st-t^2)A_3^n}, & \delta_1^{**} \leqslant \mu \leqslant \delta_2^{**}; \\ \frac{8(1-\gamma)(2-\tau)(3-\tau)}{3\pi^2(3-s^2-st-t^2)A_3^n}\Big(\frac{4(s+t)(1-\gamma)}{\pi^2(s+t-2)} \\ \qquad -\frac{1}{3}-\frac{12(1-\gamma)(2-\tau)(s^2+st+t^2-3)A_3^n}{\pi^2(s+t-2)^2 2(3-\tau)A_2^{2n}}\mu\Big), & \mu \geqslant \delta_2^{**}; \end{cases} \tag{3.25}$$

where δ_1^{**}, δ_2^{**} are given by (3.23) and (3.24), respectively.

Theorem 3.11. *Let $s^2+st+t^2 > 3$. Let the function f given by (1.1) be in the class k-$\mathrm{UST}_{\lambda,\mu}^{n,\tau}(s,t,\gamma)$; $-1 \leqslant \gamma < 1$; $1 < k < \infty$, and let θ be the unique positive number in the open interval $(0,1)$ defined by (1.7). Then*

$$|\mu a_2^2 - a_3| \leqslant \begin{cases} \frac{(2-\tau)(3-\tau)P_1}{12(s^2+st+t^2-3)A_3^n}\Big(\frac{6(2-\tau)(s^2+st+t^2-3)A_3^n P_1}{(s+t-2)^2 2(3-\tau)A_2^{2n}}\mu \\ \qquad +\frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{12\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)}-\frac{2(s+t)}{(s+t-2)}P_1\Big), & \mu \geqslant \varrho_1^{**}; \\ \frac{(2-\tau)(3-\tau)P_1}{6(s^2+st+t^2-3)A_3^n}, & \varrho_2^{**} \leqslant \mu \leqslant \varrho_1^{**}; \\ \frac{(2-\tau)(3-\tau)P_1}{12(s^2+st+t^2-3)A_3^n}\Big(\frac{2(s+t)}{(s+t-2)}P_1-\frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{12\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)} \\ \qquad -\frac{6(2-\tau)(s^2+st+t^2-3)A_3^n P_1}{(s+t-2)^2 2(3-\tau)A_2^{2n}}\mu\Big), & \mu \leqslant \varrho_2^{**}; \end{cases} \tag{3.26}$$

where $\mathcal{K}(t)$ is the complete elliptic integral of the first kind, P_1 is given by Lemma 1.1 *and*

$$\varrho_1^{**} = \frac{4(s+t-2)^2(3-\tau)A_2^{2n}}{3(s^2+st+t^2-3)(2-\tau)A_3^n P_1} \times\Big(\frac{(s+t)}{2(s+t-2)}P_1+\frac{1}{2}-\frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{48\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)}\Big), \tag{3.27}$$

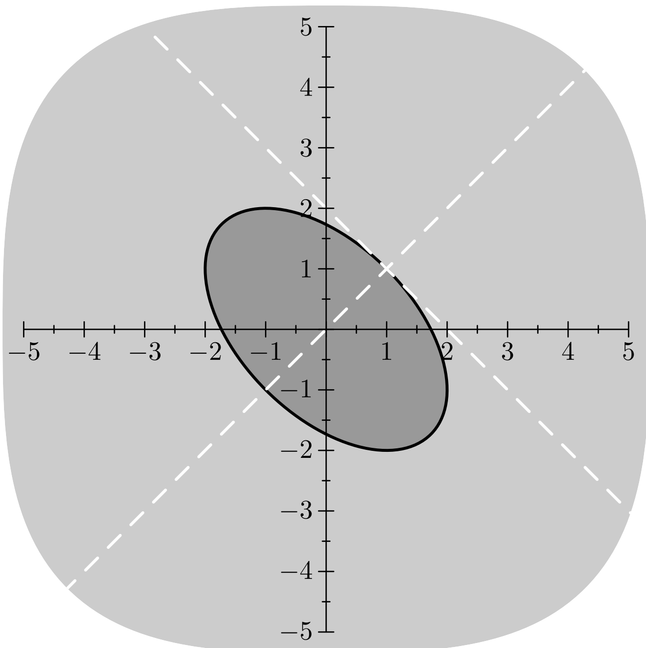
$$\varrho_2^{**} = \frac{4(s+t-2)^2(3-\tau)A_2^{2n}}{(s^2+st+t^2-3)3(2-\tau)A_3^n P_1} \times\Big(\frac{(s+t)}{2(s+t-2)}P_1-\frac{1}{2}-\frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{48\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)}\Big). \tag{3.28}$$

Corollary 3.12. *Let $s^2+st+t^2<3$. Let the function f given by (1.1) be in the class k-$\mathrm{UST}^{n,\tau}_{\lambda,\mu}(s,t,\gamma)$; $-1\leqslant\gamma<1$; $1<k<\infty$, and let θ be the unique positive number in the open interval $(0,1)$ defined by (1.7). Then*

$$
(3.29)\qquad |\mu a_2^2-a_3|\leqslant\begin{cases}\frac{(2-\tau)(3-\tau)P_1}{12(3-s^2-st-t^2)A_3^n}\Big(\frac{6(2-\tau)(s^2+st+t^2-3)A_3^nP_1}{(s+t-2)^2 2(3-\tau)A_2^{2n}}\mu \\ \qquad +\frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{12\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)}-\frac{2(s+t)}{(s+t-2)}P_1\Big), & \mu\leqslant\varrho_1^{**};\\ \frac{(2-\tau)(3-\tau)P_1}{6(3-s^2-st-t^2)A_3^n}, & \varrho_1^{**}\leqslant\mu\leqslant\varrho_2^{**};\\ \frac{(2-\tau)(3-\tau)P_1}{12(3-s^2-st-t^2)A_3^n}\Big(\frac{2(s+t)}{(s+t-2)}P_1-\frac{4\mathcal{K}^2(\theta)(\theta^2+6\theta+1)-\pi^2}{12\sqrt{\theta}(1+\theta)\mathcal{K}^2(\theta)} \\ \qquad -\frac{6(2-\tau)(s^2+st+t^2-3)A_3^nP_1}{(s+t-2)^2 2(3-\tau)A_2^{2n}}\mu\Big), & \mu\geqslant\varrho_2^{**};\end{cases}
$$

where $\mathcal{K}(t)$ is the complete elliptic integral of the first kind, P_1, ϱ_1^{**}, ϱ_2^{**} are given by Lemma 1.1, (3.23) *and* (3.24), *respectively.*

R e m a r k 3.1. If we take $s=1$, $t=-1$ and $n=0$ in corollaries 2.2–3.12, we get all theorems in [9].



The region of $s^2+st+t^2<3$, $s\neq t$ and $s+t\neq 2$.
The region of $s^2+st+t^2>3$, $s\neq t$ and $s+t\neq 2$.

For a brief history of the Fekete-Szegő problem for the class of starlike, convex and close to convex functions, see the recent papers by Deniz and Orhan [6], Deniz et al. [5], Kanas and Lecko [12], Kanas and Darwish [11], Kanas [10], Mishra and Gochhayat [18], Orhan and Gunes [21], Orhan and Răducanu [22], Orhan et al. [20], Srivastava and Mishra [31], Srivastava et al. [32].

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Authors' addresses: *Nihat Yagmur*, Department of Mathematics, Faculty of Science and Art, Erzincan University, 24000 Erzincan, Turkey, e-mail: `nhtyagmur@gmail.com`; *Halit Orhan* (corresponding author), Department of Mathematics, Faculty of Science, Atatürk University, 25240 Erzurum, Turkey, e-mail: `horhan@atauni.edu.tr`.