Bashir Ahmad; Sotiris Ntouyas

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EXISTENCE OF SOLUTIONS FOR FRACTIONAL DIFFERENTIAL INCLUSIONS WITH NONLOCAL RIEMANN-LIOUVILLE INTEGRAL BOUNDARY CONDITIONS

Bashir Ahmad, Jeddah, Sotiris Ntouyas, Ioánnina

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Abstract. In this paper, we discuss the existence of solutions for a boundary value problem of fractional differential inclusions with nonlocal Riemann-Liouville integral boundary conditions. Our results include the cases when the multivalued map involved in the problem is (i) convex valued, (ii) lower semicontinuous with nonempty closed and decomposable values and (iii) nonconvex valued. In case (i) we apply a nonlinear alternative of Leray-Schauder type, in the second case we combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo, while in the third case we use a fixed point theorem for multivalued contractions due to Covitz and Nadler.

Keywords: differential inclusion; nonlocal condition; integral boundary condition; Leray Schauder alternative; fixed point theorem

MSC 2010: 34A60, 34A08, 34B10

1. Introduction

In this paper, we discuss the existence of solutions for nonlinear fractional differential inclusions of order $q \in (1,2]$ with nonlocal Riemann-Liouville integral boundary conditions given by

$$\tag{1.1} \begin{cases} {}^cD^q x(t) \in F(t, x(t)), \quad 1 < q \leqslant 2,\ t \in [0,1], \\ x(0) = aI^\beta x(\eta) = a \displaystyle\int_0^\eta \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} x(s)\,\mathrm{d}s, \quad 0 < \beta \leqslant 1, \\ x(1) = bI^\alpha x(\sigma) = b \displaystyle\int_0^\sigma \frac{(\sigma - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\,\mathrm{d}s, \quad 0 < \alpha \leqslant 1, \end{cases}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $F\colon [0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$ and a, b, η, σ are real constants with $0<\eta$, $\sigma<1$.

The present work is motivated by a recent article [5], where the authors studied the single-valued case (${}^cD^qx(t)=f(t,x(t))$) of problem (1.1).

Integral boundary conditions have found useful applications in applied fields such as blood flow problems, chemical engineering, thermo-elasticity, underground water flow, population dynamics, etc. For a detailed description of the integral boundary conditions, we refer the reader to the papers [6], [14] and references therein. For the basic theory of fractional differential equations and its applications see [22], [23], [24], [25], whereas the recent developments on the topic can be found in [1], [2], [3], [4], [7], [8], [9], [10], [11], [12] and the references cited therein.

We establish some existence results for the problem (1.1) when the multivalued map involved in the problem is (i) convex valued; (ii) lower semicontinuous with nonempty closed and decomposable values and (iii) nonconvex valued. The first result is shown by applying a nonlinear alternative of Leray-Schauder type. To obtain the second result, we combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo. The third result is based on a fixed point theorem for contraction multivalued maps due to Covitz and Nadler.

The methods used are standard, however their exposition in the framework of problems (1.1) is new.

2. Preliminaries

2.1. Fractional calculus. Let us recall some basic definitions of fractional calculus [22], [25].

Definition 2.1. For $(n-1)$-times absolutely continuous function $g\colon [0,\infty)\to\mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$ {}^cD^qg(t)=\frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}g^{(n)}(s)\,\mathrm{d}s,\quad n-1<q<n,\ n=[q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2. The Riemann-Liouville fractional integral of order q is defined as

$$I^qg(t)=\frac{1}{\Gamma(q)}\int_0^t\frac{g(s)}{(t-s)^{1-q}}\,\mathrm{d}s,\quad q>0,$$

provided the integral exists.

To define the solution for the inclusion problem we need the following lemma.

Lemma 2.3 ([5]). *For any $y \in C[0,1]$, the unique solution of the linear fractional boundary value problem*

$$\begin{cases} {}^cD^q x(t) = y(t), \quad 1 < q \leqslant 2,\ t \in [0,1], \\ x(0) = a \displaystyle\int_0^\eta \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} x(s)\,\mathrm{d}s, \\ x(1) = b \displaystyle\int_0^\sigma \frac{(\sigma - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\,\mathrm{d}s, \quad 0 < \beta,\ \alpha < 1 \end{cases} \tag{2.1}$$

is

$$x(t) = I^q y(t) + (\nu_1 - \nu_4 t) I^{q+\beta} y(\eta) + (\nu_2 + \nu_3 t)(b I^{q+\alpha} y(\sigma) - I^q y(1)), \tag{2.2}$$

where

$$\begin{cases} \nu_1 = \dfrac{a}{\nu}\Big(1 - \dfrac{b\sigma^{\alpha+1}}{\Gamma(\alpha+2)}\Big), \quad \nu_2 = \dfrac{a\eta^{\beta+1}}{\nu\Gamma(\beta+2)}, \\ \nu_3 = \dfrac{1}{\nu}\Big(1 - \dfrac{a\eta^{\beta}}{\Gamma(\beta+1)}\Big), \quad \nu_4 = \dfrac{a}{\nu}\Big(1 - \dfrac{b\sigma^{\alpha}}{\Gamma(\alpha+1)}\Big), \\ \nu = \Big(1 - \dfrac{a\eta^{\beta}}{\Gamma(\beta+1)}\Big)\Big(1 - \dfrac{b\sigma^{\alpha+1}}{\Gamma(\alpha+2)}\Big) + \dfrac{a\eta^{\beta+1}}{\Gamma(\beta+2)}\Big(1 - \dfrac{b\sigma^{\alpha}}{\Gamma(\alpha+1)}\Big). \end{cases} \tag{2.3}$$

2.2. Multivalued analysis. Let us recall some basic definitions on multi-valued maps [17], [19].

For a normed space $(X, \|\cdot\|)$, let $P_{\mathrm{cl}}(X) = \{Y \in \mathcal{P}(X)\colon Y \text{ is closed}\}$, $P_{\mathrm{b}}(X) = \{Y \in \mathcal{P}(X)\colon Y \text{ is bounded}\}$, $P_{\mathrm{cp}}(X) = \{Y \in \mathcal{P}(X)\colon Y \text{ is compact}\}$, and $P_{\mathrm{cp,c}}(X) = \{Y \in \mathcal{P}(X)\colon Y \text{ is compact and convex}\}$. A multi-valued map $G\colon X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \bigcup_{x \in \mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in P_{\mathrm{b}}(X)$ (i.e. $\sup_{x \in \mathbb{B}}\{\sup\{|y|\colon y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. A map G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_{\mathrm{b}}(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. A map G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator

G will be denoted by $\operatorname{Fix} G$. A multivalued map $G\colon [0;1] \to P_{\mathrm{cl}}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \mapsto d(y, G(t)) = \inf\{|y - z|\colon\ z \in G(t)\}$$

is measurable.

Let $C([0,1])$ denote a Banach space of continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|x\| = \sup\limits_{t\in[0,1]} |x(t)|$. Let $L^1([0,1],\mathbb{R})$ be the Banach space of measurable functions $x\colon [0,1] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1 |x(t)|\,\mathrm{d}t$. Also, $AC^1([0,1],\mathbb{R})$ will denote the space of functions $y\colon [0,1] \to \mathbb{R}$ that are absolutely continuous and whose first derivative is absolutely continuous.

Definition 2.4. A multivalued map $F\colon [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \mapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;

(ii) $x \mapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,1]$.

Further, a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0,1],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v|\colon\ v \in F(t,x)\} \leqslant \varphi_\alpha(t)$$

for all $\|x\| \leqslant \alpha$ and for a.e. $t \in [0,1]$.

For each $y \in C([0,1],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,1],\mathbb{R})\colon\ v(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,1]\}.$$

Let X be a nonempty closed subset of a Banach space E and $G\colon X \to \mathcal{P}(E)$ a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X\colon\ G(y) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,1] \times \mathbb{R}$. A is $\mathcal{L} \otimes B$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0,1],\mathbb{R})$ is decomposable if for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,1] = J$, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}}$ belongs to $\mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 2.5. Let Y be a separable metric space and let N be a multivalued operator such that $N\colon Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$. We say N has the property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F\colon [0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F}\colon C([0,1]\times\mathbb{R})\to\mathcal{P}(L^1([0,1],\mathbb{R}))$ associated with F as

$$\mathcal{F}(x)=\{w\in L^1([0,1],\mathbb{R})\colon\ w(t)\in F(t,x(t)) \text{ for a.e. } t\in[0,1]\},$$

which is called the Nemytskii operator associated with F.

Definition 2.6. Let $F\colon [0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Let (X,d) be a metric space induced from the normed space $(X;\|\cdot\|)$. Consider $H_d\colon \mathcal{P}(X)\times\mathcal{P}(X)\to\mathbb{R}\cup\{\infty\}$ given by

$$H_d(A,B)=\max\Big\{\sup_{a\in A} d(a,B),\sup_{b\in B} d(A,b)\Big\},$$

where $d(A,b)=\inf\limits_{a\in A} d(a;b)$ and $d(a,B)=\inf\limits_{b\in B} d(a;b)$. Then $(P_{\mathrm{b,cl}}(X),H_d)$ is a metric space and $(P_{\mathrm{cl}}(X),H_d)$ is a generalized metric space (see [20]).

Definition 2.7. A multivalued operator $N\colon X\to P_{\mathrm{cl}}(X)$ is called:

(a) γ-Lipschitz if and only if there exists $\gamma>0$ such that

$$H_d(N(x),N(y))\leqslant \gamma d(x,y)\quad\text{for each } x,\ y\in X;$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma<1$.

The following lemmas will be used in the sequel.

Lemma 2.8 (Nonlinear alternative for Kakutani maps [18])**.** *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0\in U$. Suppose that $F\colon \overline{U}\to P_{\mathrm{cp,c}}(C)$ is an upper semicontinuous compact map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is a $u\in\partial U$ and $\lambda\in(0,1)$ with $u\in\lambda F(u)$.*

Lemma 2.9 ([21])**.** *Let X be a Banach space. Let $F\colon [0,T]\times\mathbb{R}\to P_{\mathrm{cp,c}}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,1],X)$ to $C([0,1],X)$. Then the operator*

$$\Theta\circ S_F\colon\ C([0,1],X)\to P_{\mathrm{cp,c}}(C([0,1],X)),\quad x\mapsto(\Theta\circ S_F)(x)=\Theta(S_{F,x})$$

is a closed graph operator in $C([0,1],X)\times C([0,1],X)$.

Lemma 2.10 ([13]). *Let Y be a separable metric space and let $N\colon Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator having the property* (BC)*. Then N has a continuous selection, that is, there exists a continuous (single-valued) function $g\colon Y \to L^1([0,1],\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Lemma 2.11 ([16]). *Let (X,d) be a complete metric space. If $N\colon X \to P_{\mathrm{cl}}(X)$ is a contraction, then* $\operatorname{Fix} N \neq \emptyset$.

Definition 2.12. A function $x \in AC^2([0,1],\mathbb{R})$ is said to be a solution of the problem (1.1) if $x(0) = aI^{\beta}x(\eta)$, $x(1) = bI^{\alpha}x(\sigma)$, and there exists a function $f \in L^1([0,1],\mathbb{R})$ such that $f(t) \in F(t,x(t))$ a.e. on $[0,1]$ and

$$\begin{aligned}(2.4)\qquad x(t) = {}&\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,\mathrm{d}s + (\nu_1 - t\nu_4)\int_0^{\eta} \frac{(\eta-s)^{q+\beta-1}}{\Gamma(q+\beta)} f(s)\,\mathrm{d}s \\ &+ (\nu_2+\nu_3 t)\Big\{ b\int_0^{\sigma} \frac{(\sigma-s)^{q+\alpha-1}}{\Gamma(q+\alpha)} f(s)\,\mathrm{d}s - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)\,\mathrm{d}s \Big\},\end{aligned}$$

where ν_1, ν_2, ν_3 and ν_4 are given by (2.3).

3. Existence results

Theorem 3.1 (The Carathéodory case)**.** *Assume that:*

(H$_1$) *$F\colon [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is Carathéodory and has nonempty compact and convex values;*

(H$_2$) *there exist a continuous nondecreasing function $\psi\colon [0,\infty) \to (0,\infty)$ and a function $p \in C([0,1],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y|\colon\ y \in F(t,x)\} \leqslant p(t)\psi(\|x\|) \quad \text{for each } (t,x) \in [0,1]\times\mathbb{R};$$

(H$_3$) *there exists a constant $M > 0$ such that*

$$\frac{M}{\|p\|\psi(M)\Big\{\dfrac{1+|\nu_2|+|\nu_3|}{\Gamma(q+1)} + \dfrac{\eta^{q+\beta}(|\nu_1|+|\nu_4|)}{\Gamma(q+\beta+1)} + \dfrac{|b|\sigma^{q+\alpha}(|\nu_2|+|\nu_3|)}{\Gamma(q+\alpha+1)}\Big\}} > 1.$$

Then the boundary value problem (1.1) *has at least one solution on* $[0,1]$.

P r o o f. Define the operator $\Omega\colon C([0,1],\mathbb{R})\to\mathcal{P}(C([0,1],\mathbb{R}))$ by

$$\begin{aligned}\Omega(x)=\biggl\{&h\in C([0,1],\mathbb{R})\colon\\ &h(t)=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s)\,\mathrm{d}s+(\nu_1-t\nu_4)\int_0^\eta\frac{(\eta-s)^{q+\beta-1}}{\Gamma(q+\beta)}f(s)\,\mathrm{d}s\\ &\quad+(\nu_2+\nu_3t)\biggl(b\int_0^\sigma\frac{(\sigma-s)^{q+\alpha-1}}{\Gamma(q+\alpha)}f(s)\,\mathrm{d}s-\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}f(s)\,\mathrm{d}s\biggr)\biggr\}\end{aligned}$$

for $f\in S_{F,x}$. We will show that Ω satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps. As a first step, we show that Ω *is convex for each* $x\in C([0,1],\mathbb{R})$. This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit the proof.

Next, we show that Ω *maps bounded sets (balls) into bounded sets in* $C([0,1],\mathbb{R})$. For a positive number ϱ, let $B_\varrho=\{x\in C([0,1],\mathbb{R})\colon\ \|x\|\leqslant\varrho\}$ be a bounded ball in $C([0,1],\mathbb{R})$. Then, for each $h\in\Omega(x), x\in B_\varrho$, there exists $f\in S_{F,x}$ such that

$$\begin{aligned}h(t)=&\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s)\,\mathrm{d}s+(\nu_1-t\nu_4)\int_0^\eta\frac{(\eta-s)^{q+\beta-1}}{\Gamma(q+\beta)}f(s)\,\mathrm{d}s\\ &+(\nu_2+\nu_3t)\biggl\{b\int_0^\sigma\frac{(\sigma-s)^{q+\alpha-1}}{\Gamma(q+\alpha)}f(s)\,\mathrm{d}s-\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}f(s)\,\mathrm{d}s\biggr\},\end{aligned}$$

and

$$\begin{aligned}|h(t)|\leqslant&\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}p(s)\psi(\|x\|)\,\mathrm{d}s\\ &+(|\nu_1|+|\nu_4|)|\int_0^\eta\frac{(\eta-s)^{q+\beta-1}}{\Gamma(q+\beta)}p(s)\psi(\|x\|)\,\mathrm{d}s\\ &+(|\nu_2|+|\nu_3|)\biggl(|b|\int_0^\sigma\frac{(\sigma-s)^{q+\alpha-1}}{\Gamma(q+\alpha)}p(s)\psi(\|x\|)\,\mathrm{d}s\\ &+\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}p(s)\psi(\|x\|)\,\mathrm{d}s\biggr)\\ \leqslant&\|p\|\psi(\|x\|)\biggl\{\frac{t^q}{\Gamma(q+1)}+(|\nu_1|+|\nu_4|)\frac{\eta^{q+\beta}}{\Gamma(q+\beta+1)}\\ &+(|\nu_2|+|\nu_3|)\Bigl(\frac{|b|\sigma^{q+\alpha}}{\Gamma(q+\alpha+1)}+\frac{1}{\Gamma(q+1)}\Bigr)\biggr\}\\ \leqslant&\|p\|\psi(\|x\|)\Bigl\{\frac{1+|\nu_2|+|\nu_3|}{\Gamma(q+1)}+\frac{\eta^{q+\beta}(|\nu_1|+|\nu_4|)}{\Gamma(q+\beta+1)}+\frac{|b|\sigma^{q+\alpha}(|\nu_2|+|\nu_3|)}{\Gamma(q+\alpha+1)}\Bigr\}.\end{aligned}$$

Thus,

$$\|h\|\leqslant\|p\|\psi(\varrho)\Bigl\{\frac{1+|\nu_2|+|\nu_3|}{\Gamma(q+1)}+\frac{\eta^{q+\beta}(|\nu_1|+|\nu_4|)}{\Gamma(q+\beta+1)}+\frac{|b|\sigma^{q+\alpha}(|\nu_2|+|\nu_3|)}{\Gamma(q+\alpha+1)}\Bigr\}.$$

Now we show that Ω *maps bounded sets into equicontinuous sets of* $C([0,1],\mathbb{R})$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in B_\varrho$, where B_ϱ is a bounded set of $C([0,1],\mathbb{R})$. For each $h \in \Omega(x)$, we obtain

$$\begin{aligned}
|h(t_2) - h(t_1)| \leqslant \bigg| &\int_0^{t_2} \frac{(t_2 - s)^{q-1}}{\Gamma(q)} p(s)\psi(\varrho)\,\mathrm{d}s - \int_0^{t_1} \frac{(t_1 - s)^{q-1}}{\Gamma(q)} p(s)\psi(\varrho)\,\mathrm{d}s \\
&- \nu_4(t_2 - t_1) \int_0^{\eta} \frac{(\eta - s)^{q+\beta-1}}{\Gamma(q+\beta)} p(s)\psi(\varrho)\,\mathrm{d}s \\
&- \nu_3(t_2 - t_1)\bigg(b\int_0^{\sigma} \frac{(\sigma - s)^{q+\alpha-1}}{\Gamma(q+\alpha)} p(s)\psi(\varrho)\,\mathrm{d}s - \int_0^{1} \frac{(1 - s)^{q-1}}{\Gamma(q)} p(s)\psi(\varrho)\,\mathrm{d}s \bigg)\bigg| \\
\leqslant \psi(\varrho)\|p\| \bigg[&\int_{t_1}^{t_2} \bigg| \frac{(t_2 - s)^{q-1}}{\Gamma(q)} \bigg| \,\mathrm{d}s + \int_0^{t_1} \bigg| \frac{(t_1 - s)^{q-1} - (t_2 - s)^{q-1}}{\Gamma(q)} \bigg| \,\mathrm{d}s \\
&+ |\nu_4|(t_2 - t_1) \int_0^{\eta} \bigg| \frac{(\eta - s)^{q+\beta-1}}{\Gamma(q+\beta)} \bigg| \,\mathrm{d}s \\
&+ |\nu_3|(t_2 - t_1)\bigg(|b| \int_0^{\sigma} \bigg| \frac{(\sigma - s)^{q+\alpha-1}}{\Gamma(q+\alpha)} \bigg| \,\mathrm{d}s + \int_0^{1} \bigg| \frac{(1 - s)^{q-1}}{\Gamma(q)} \,\mathrm{d}s \bigg| \bigg) \bigg].
\end{aligned}$$

Obviously, the right hand side of the above inequality tends to zero independently of $x \in B_\varrho$ as $t_2 - t_1 \to 0$. As Ω satisfies the above three assumptions, it follows by the Arzelà-Ascoli theorem that $\Omega\colon C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ is completely continuous.

In our next step, we show that Ω *has a closed graph.* Let $x_n \to x_*$, $h_n \in \Omega(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \Omega(x_*)$. Associated with $h_n \in \Omega(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$\begin{aligned}
h_n(t) = &\int_0^t \frac{(t - s)^{q-1}}{\Gamma(q)} f_n(s)\,\mathrm{d}s + (\nu_1 - t\nu_4) \int_0^{\eta} \frac{(\eta - s)^{q+\beta-1}}{\Gamma(q+\beta)} f_n(s)\,\mathrm{d}s \\
&+ (\nu_2 + \nu_3 t)\bigg\{ b\int_0^{\sigma} \frac{(\sigma - s)^{q+\alpha-1}}{\Gamma(q+\alpha)} f_n(s)\,\mathrm{d}s - \int_0^{1} \frac{(1 - s)^{q-1}}{\Gamma(q)} f_n(s)\,\mathrm{d}s \bigg\}.
\end{aligned}$$

Thus we have to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0,1]$,

$$\begin{aligned}
h_*(t) = &\int_0^t \frac{(t - s)^{q-1}}{\Gamma(q)} f_*(s)\,\mathrm{d}s + (\nu_1 - t\nu_4) \int_0^{\eta} \frac{(\eta - s)^{q+\beta-1}}{\Gamma(q+\beta)} f_*(s)\,\mathrm{d}s \\
&+ (\nu_2 + \nu_3 t)\bigg\{ b\int_0^{\sigma} \frac{(\sigma - s)^{q+\alpha-1}}{\Gamma(q+\alpha)} f_*(s)\,\mathrm{d}s - \int_0^{1} \frac{(1 - s)^{q-1}}{\Gamma(q)} f_*(s)\,\mathrm{d}s \bigg\}.
\end{aligned}$$

Let us consider the continuous linear operator $\Theta\colon L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by

$$f \mapsto \Theta(f)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,\mathrm{d}s + (\nu_1 - t\nu_4)\int_0^\eta \frac{(\eta-s)^{q+\beta-1}}{\Gamma(q+\beta)} f(s)\,\mathrm{d}s \\ + (\nu_2+\nu_3 t)\biggl\{b\int_0^\sigma \frac{(\sigma-s)^{q+\alpha-1}}{\Gamma(q+\alpha)} f(s)\,\mathrm{d}s - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)\,\mathrm{d}s\biggr\}.$$

Observe that

$$\|h_n(t)-h_*(t)\| = \biggl\|\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (f_n(s)-f_*(s))\,\mathrm{d}s \\ + (\nu_1 - t\nu_4)\int_0^\eta \frac{(\eta-s)^{q+\beta-1}}{\Gamma(q+\beta)} (f_n(s)-f_*(s))\,\mathrm{d}s \\ + (\nu_2+\nu_3 t)\biggl\{|b|\int_0^\sigma \frac{(\sigma-s)^{q+\alpha-1}}{\Gamma(q+\alpha)} (f_n(s)-f_*(s))\,\mathrm{d}s \\ - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} (f_n(s)-f_*(s))\,\mathrm{d}s\biggr\}\biggr\| \to 0,$$

as $n\to\infty$.

Thus, it follows by Lemma 2.9 that $\Theta\circ S_F$ is a closed graph operator. Further, we have $h_n(t)\in\Theta(S_{F,x_n})$. Since $x_n\to x_*$, we have

$$h_*(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)\,\mathrm{d}s + (\nu_1 - t\nu_4)\int_0^\eta \frac{(\eta-s)^{q+\beta-1}}{\Gamma(q+\beta)} f_*(s)\,\mathrm{d}s \\ + (\nu_2+\nu_3 t)\biggl\{b\int_0^\sigma \frac{(\sigma-s)^{q+\alpha-1}}{\Gamma(q+\alpha)} f_*(s)\,\mathrm{d}s - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f_*(s)\,\mathrm{d}s\biggr\}$$

for some $f_*\in S_{F,x_*}$.

Finally, we show there exists an open set $U\subseteq C([0,1],\mathbb{R})$ with $x\notin\Omega(x)$ for $\lambda\in(0,1)$ and $x\in\partial U$. Let $\lambda\in(0,1)$ and $x\in\lambda\Omega(x)$. Then there exists $f\in L^1([0,1],\mathbb{R})$ with $f\in S_{F,x}$ such that, for $t\in[0,1]$, we have

$$h(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,\mathrm{d}s + (\nu_1 - t\nu_4)\int_0^\eta \frac{(\eta-s)^{q+\beta-1}}{\Gamma(q+\beta)} f(s)\,\mathrm{d}s \\ + (\nu_2+\nu_3 t)\biggl\{b\int_0^\sigma \frac{(\sigma-s)^{q+\alpha-1}}{\Gamma(q+\alpha)} f(s)\,\mathrm{d}s - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)\,\mathrm{d}s\biggr\}.$$

Using the computations of the second step above we obtain

$$\|h\| \leqslant \|p\|\psi(\|x\|)\biggl\{\frac{1+|\nu_2|+|\nu_3|}{\Gamma(q+1)} + \frac{\eta^{q+\beta}(|\nu_1|+|\nu_4|)}{\Gamma(q+\beta+1)} + \frac{|b|\sigma^{q+\alpha}(|\nu_2|+|\nu_3|)}{\Gamma(q+\alpha+1)}\biggr\}.$$

Consequently, we have

$$\frac{\|x\|}{\|p\|\psi(\|x\|)\Big\{\dfrac{1+|\nu_2|+|\nu_3|}{\Gamma(q+1)}+\dfrac{\eta^{q+\beta}(|\nu_1|+|\nu_4|)}{\Gamma(q+\beta+1)}+\dfrac{|b|\sigma^{q+\alpha}(|\nu_2|+|\nu_3|)}{\Gamma(q+\alpha+1)}\Big\}}\leqslant 1.$$

In view of (H_3), there exists M such that $\|x\|\neq M$. Let us set

$$U=\{x\in C([0,1],\mathbb{R})\colon\ \|x\|<M\}.$$

Note that the operator $\Omega\colon\ \overline{U}\to\mathcal{P}(C([0,1],\mathbb{R}))$ is upper semicontinuous and completely continuous. Due to the choice of U, there is no $x\in\partial U$ such that $x\in\lambda\Omega(x)$ for some $\lambda\in(0,1)$. Consequently, by the nonlinear alternative of the Leray-Schauder type (Lemma 2.8), we deduce that Ω has a fixed point $x\in\overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

Example 3.1. Consider the strip fractional boundary value problem

$$\begin{cases} {}^cD^{3/2}x(t)\in F(t,x(t)), & 0<t<1,\\ x(0)=I^{1/2}x(1/4), & x(1)=I^{1/3}x(1/2).\end{cases}\tag{3.1}$$

Here $q=3/2$, $a=b=1$, $\beta=1/2$, $\alpha=1/3$, $\eta=1/4$, $\sigma=1/2$, and $F\colon\ [0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$x\to F(t,x)=\Big[\frac{|x|^3}{|x|^3+3}+3t^3+5,\frac{|x|}{|x|+1}+t+1\Big].$$

For $f\in F$ we have

$$|f|\leqslant\max\Big(\frac{|x|^3}{|x|^3+3}+3t^3+5,\frac{|x|}{|x|+1}+t+1\Big)\leqslant 9,\quad x\in\mathbb{R}.$$

Thus,

$$\|F(t,x)\|_{\mathcal{P}}:=\sup\{|y|\colon\ y\in F(t,x)\}\leqslant 9=p(t)\psi(\|x\|),\ x\in\mathbb{R},$$

with $p(t)=1$, $\psi(\|x\|)=9$.

Further, using the condition (H_3) we find that $M>M_1$, where $M_1\approx 15.2244$. Clearly, all the conditions of Theorem 3.1 are satisfied. So there exists at least one solution of the problem (3.1) on $[0,1]$.

Theorem 3.2 (The lower semicontinuous case). *Assume that* (H_2), (H_3) *and the following condition hold:*

(H_4) $F\colon [0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*
  (a) $(t,x)\mapsto F(t,x)$ *is* $\mathcal{L}\otimes B$ *measurable,*
  (b) $x\mapsto F(t,x)$ *is lower semicontinuous for each* $t\in[0,1]$.

Then the boundary value problem (1.1) *has at least one solution on* $[0,1]$.

P r o o f. We make use of the nonlinear alternative of the Leray Schauder type together with the selection theorem of Bressan and Colombo [13] for lower semicontinuous maps with decomposable values to complete the proof. It follows from (H_2) and (H_4) that F is of l.s.c. type. Then by virtue of Lemma 2.10 there exists a continuous function $f\colon C([0,1],\mathbb{R})\to L^1([0,1],\mathbb{R})$ such that $f(x)\in\mathcal{F}(x)$ for all $x\in C([0,1],\mathbb{R})$.

Consider the problem

$$\begin{cases} {}^cD^q x(t)=f(x)(t), \quad t\in[0,1],\ 1<q\leqslant 2,\\ x(0)=aI^\beta x(\eta),\quad x(1)=bI^\alpha x(\sigma).\end{cases}\tag{3.2}$$

Observe that if $x\in AC^2([0,1],\mathbb{R})$ is a solution of (3.2), then x is a solution to the problem (1.1). In order to transform the problem (3.2) into a fixed point problem, we define the operator $\overline{\Omega}$ as

$$\begin{aligned}\overline{\Omega}x(t)=&\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(x)(s)\,\mathrm{d}s+(\nu_1-t\nu_4)\int_0^\eta\frac{(\eta-s)^{q+\beta-1}}{\Gamma(q+\beta)}f(x)(s)\,\mathrm{d}s\\&+(\nu_2+\nu_3t)\Big\{b\int_0^\sigma\frac{(\sigma-s)^{q+\alpha-1}}{\Gamma(q+\alpha)}f(x)(s)\,\mathrm{d}s-\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}f(x)(s)\,\mathrm{d}s\Big\}.\end{aligned}$$

It can be shown easily that $\overline{\Omega}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.1. So we omit it. This completes the proof. □

Our next result dealing with the nonconvex type multivalued map for problem (1.1) relies on a fixed point theorem for multivalued maps due to Covitz and Nadler [16].

Theorem 3.3 (The Lipschitz case). *Assume that the following conditions hold:*

(H_5) $F\colon [0,1]\times\mathbb{R}\to P_{\mathrm{cp}}(\mathbb{R})$ *is such that* $F(\cdot,x)\colon [0,1]\to P_{\mathrm{cp}}(\mathbb{R})$ *is measurable for each* $x\in\mathbb{R}$;

(H_6) $H_d(F(t,x),F(t,\bar{x}))\leqslant m(t)|x-\bar{x}|$ *for almost all* $t\in[0,1]$ *and* $x,\bar{x}\in\mathbb{R}$ *with* $m\in C([0,1],\mathbb{R}^+)$ *and* $d(0,F(t,0))\leqslant m(t)$ *for almost all* $t\in[0,1]$.

If

$$\|m\|\left\{\frac{1+|\nu_2|+|\nu_3|}{\Gamma(q+1)}+\frac{\eta^{q+\beta}(|\nu_1|+|\nu_4|)}{\Gamma(q+\beta+1)}+\frac{|b|\sigma^{q+\alpha}(|\nu_2|+|\nu_3|)}{\Gamma(q+\alpha+1)}\right\}<1$$

then the boundary value problem (1.1) *has at least one solution on* $[0,1]$.

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x\in C([0,1],\mathbb{R})$ by the assumption (H_5), so F has a measurable selection (see [15], Theorem III.6). Now we show that the operator Ω, defined at the beginning of the proof of Theorem 3.1, satisfies the assumptions of Lemma 2.11. To show that $\Omega(x)\in P_{\rm cl}((C[0,1],\mathbb{R}))$ for each $x\in C([0,1],\mathbb{R})$, let $\{u_n\}_{n\geqslant 0}\in\Omega(x)$ be such that $u_n\to u(n\to\infty)$ in $C([0,1],\mathbb{R})$. Then $u\in C([0,1],\mathbb{R})$ and there exists $v_n\in S_{F,u_n}$ such that, for each $t\in[0,1]$,

$$\begin{aligned}u_n(t)=&\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}v_n(s)\,\mathrm{d}s+(\nu_1-t\nu_4)\int_0^\eta\frac{(\eta-s)^{q+\beta-1}}{\Gamma(q+\beta)}v_n(s)\,\mathrm{d}s\\&+(\nu_2+\nu_3t)\left\{b\int_0^\sigma\frac{(\sigma-s)^{q+\alpha-1}}{\Gamma(q+\alpha)}v_n(s)\,\mathrm{d}s-\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}v_n(s)\,\mathrm{d}s\right\}.\end{aligned}$$

As F has compact values, we pass to a subsequence to obtain that v_n converges to v in $L^1([0,1],\mathbb{R})$. Thus, $v\in S_{F,x}$ and for each $t\in[0,1]$,

$$\begin{aligned}u_n(t)\to u(t)=&\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}v(s)\,\mathrm{d}s+(\nu_1-t\nu_4)\int_0^\eta\frac{(\eta-s)^{q+\beta-1}}{\Gamma(q+\beta)}v(s)\,\mathrm{d}s\\&+(\nu_2+\nu_3t)\left\{b\int_0^\sigma\frac{(\sigma-s)^{q+\alpha-1}}{\Gamma(q+\alpha)}v(s)\,\mathrm{d}s-\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}v(s)\,\mathrm{d}s\right\}.\end{aligned}$$

Hence, $u\in\Omega(x)$.

Next we show that there exists $\gamma<1$ such that

$$H_d(\Omega(x),\Omega(\bar{x}))\leqslant\gamma\|x-\bar{x}\|\quad\text{for each } x,\bar{x}\in C([0,1],\mathbb{R}).$$

Let $x,\bar{x}\in C([0,1],\mathbb{R})$ and $h_1\in\Omega(x)$. Then there exists $v_1(t)\in F(t,x(t))$ such that, for each $t\in[0,1]$,

$$\begin{aligned}h_1(t)=&\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}v_1(s)\,\mathrm{d}s+(\nu_1-t\nu_4)\int_0^\eta\frac{(\eta-s)^{q+\beta-1}}{\Gamma(q+\beta)}v_1(s)\,\mathrm{d}s\\&+(\nu_2+\nu_3t)\left\{b\int_0^\sigma\frac{(\sigma-s)^{q+\alpha-1}}{\Gamma(q+\alpha)}v_1(s)\,\mathrm{d}s-\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}v_1(s)\,\mathrm{d}s\right\}.\end{aligned}$$

By (H_6), we have

$$H_d(F(t,x),F(t,\bar{x}))\leqslant m(t)|x(t)-\bar{x}(t)|.$$

So, there exists $w \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \leqslant m(t)|x(t) - \bar{x}(t)|, \quad t \in [0,1].$$

Define $U\colon [0,1] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R}\colon\ |v_1(t) - w| \leqslant m(t)|x(t) - \bar{x}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{x}(t))$ is measurable ([15], Proposition III.4), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [0,1]$, we have $|v_1(t) - v_2(t)| \leqslant m(t)|x(t) - \bar{x}(t)|$.

For each $t \in [0,1]$, let us define

$$\begin{aligned}
h_2(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_2(s)\,\mathrm{d}s + (\nu_1 - t\nu_4) \int_0^\eta \frac{(\eta - s)^{q+\beta-1}}{\Gamma(q+\beta)} v_2(s)\,\mathrm{d}s \\
& + (\nu_2 + \nu_3 t)\Big\{ b \int_0^\sigma \frac{(\sigma - s)^{q+\alpha-1}}{\Gamma(q+\alpha)} v_2(s)\,\mathrm{d}s - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} v_2(s)\,\mathrm{d}s \Big\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
|h_1(t) - h_2(t)| \leqslant {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |v_1(s) - v_2(s)|\,\mathrm{d}s \\
& + (|\nu_1| + |\nu_4|) \int_0^\eta \frac{(\eta - s)^{q+\beta-1}}{\Gamma(q+\beta)} |v_1(s) - v_2(s)|\,\mathrm{d}s \\
& + (|\nu_2| + |\nu_3|)\Big\{ |b| \int_0^\sigma \frac{(\sigma - s)^{q+\alpha-1}}{\Gamma(q+\alpha)} |v_1(s) - v_2(s)|\,\mathrm{d}s \\
& + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |v_1(s) - v_2(s)|\,\mathrm{d}s \Big\} \\
\leqslant {} & \|m\| \Big\{ \frac{1 + |\nu_2| + |\nu_3|}{\Gamma(q+1)} + \frac{\eta^{q+\beta}(|\nu_1| + |\nu_4|)}{\Gamma(q+\beta+1)} + \frac{|b|\sigma^{q+\alpha}(|\nu_2| + |\nu_3|)}{\Gamma(q+\alpha+1)} \Big\} \|x - \bar{x}\|.
\end{aligned}$$

Hence,

$$\|h_1 - h_2\| \leqslant \|m\| \Big\{ \frac{1 + |\nu_2| + |\nu_3|}{\Gamma(q+1)} + \frac{\eta^{q+\beta}(|\nu_1| + |\nu_4|)}{\Gamma(q+\beta+1)} + \frac{|b|\sigma^{q+\alpha}(|\nu_2| + |\nu_3|)}{\Gamma(q+\alpha+1)} \Big\} \|x - \bar{x}\|.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$\begin{aligned}
H_d(\Omega(x), \Omega(\bar{x})) & \leqslant \gamma \|x - \bar{x}\| \\
& \leqslant \|m\| \Big\{ \frac{1 + |\nu_2| + |\nu_3|}{\Gamma(q+1)} + \frac{\eta^{q+\beta}(|\nu_1| + |\nu_4|)}{\Gamma(q+\beta+1)} + \frac{|b|\sigma^{q+\alpha}(|\nu_2| + |\nu_3|)}{\Gamma(q+\alpha+1)} \Big\} \|x - \bar{x}\|.
\end{aligned}$$

Since Ω is a contraction, it follows by Lemma 2.11 that Ω has a fixed point x which is a solution of (1.1). This completes the proof. □

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Authors' addresses: *Bashir Ahmad*, Department of Mathematics, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia, e-mail: `bahmad@kau.edu.sa`; *Sotiris Ntouyas*, Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece, e-mail: `sntouyas@uoi.gr`.