Xinghui Wang; Xiaoqin Li; Shuhe Hu

Complete convergence of weighted sums for arrays of rowwise φ-mixing random variables

Applications of Mathematics, Vol. 59 (2014), No. 5, 589–607

Persistent URL: http://dml.cz/dmlcz/143932

59 (2014) APPLICATIONS OF MATHEMATICS No. 5, 589–607

COMPLETE CONVERGENCE OF WEIGHTED SUMS FOR ARRAYS OF ROWWISE φ-MIXING RANDOM VARIABLES

Xinghui Wang, Xiaoqin Li, Shuhe Hu, Hefei

(Received November 23, 2012)

Abstract. In this paper, we establish the complete convergence and complete moment convergence of weighted sums for arrays of rowwise φ-mixing random variables, and the Baum-Katz-type result for arrays of rowwise φ-mixing random variables. As an application, the Marcinkiewicz-Zygmund type strong law of large numbers for sequences of φ-mixing random variables is obtained. We extend and complement the corresponding results of X. J. Wang, S. H. Hu (2012).

Keywords: complete convergence; φ-mixing sequence; Marcinkiewicz-Zygmund type strong law of large numbers

MSC 2010: 60B10, 60F15

1. Introduction

Assume that $\{X_n, n \geqslant 1\}$ is a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$.

First, we recall the definition of φ-mixing random variables introduced by Dobrushinin [6].

Let m and n be positive integers. Write $\mathcal{F}_n^m = \sigma(X_i, n \leqslant i \leqslant m)$. Given σ-algebras $\mathcal{A}, \mathcal{B}$ in $\mathcal{F}$, let

$$\varphi(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}, B \in \mathcal{B}, P(A) > 0} |P(B|A) - P(B)|.$$

Define the φ-mixing coefficients by

$$\varphi(n) = \sup_{k \geqslant 1} \varphi(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty), \quad n \geqslant 0.$$

The research has been supported by the National Natural Science Foundation of China (11171001, 11201001), Natural Science Foundation of Anhui Province (1208085QA03, 1308085QA03) and Doctoral Research Start-up Funds Projects of Anhui University.

Definition 1.1. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be a φ-mixing sequence if $\varphi(n) \downarrow 0$ as $n \to \infty$.

An array $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ of random variables is called an array of rowwise φ-mixing random variables if for every $n \geqslant 1$, $\{X_{ni}, i \geqslant 1\}$ is a sequence of φ-mixing random variables.

Hsu and Robbins [9] introduced the concept of complete convergence as follows. A sequence $\{U_n, n \geqslant 1\}$ of random variables is said to *converge completely to a constant* C if $\sum_{n=1}^{\infty} P(|U_n - C| > \varepsilon) < \infty$ for all $\varepsilon > 0$. In view of the Borel-Cantelli lemma, this implies that $U_n \to C$ almost surely (a.s.). The converse is true if the $\{U_n, n \geqslant 1\}$ is independent. Hsu and Robbins [9] proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Erdös [7] proved the converse. The result of Hsu-Robbins-Erdös is a fundamental theorem in probability theory and has been generalized in several directions by many authors. One of the most important generalizations is that of Baum and Katz [3] for the strong law of large numbers as follows.

Theorem 1.1. *Let $\alpha > 1/2$ and $\alpha p > 1$. Let $\{X_n, n \geqslant 1\}$ be a sequence of i.i.d. random variables. Assume further $EX_1 = 0$ if $\alpha \leqslant 1$. Then the following statements are equivalent:*

(i) $E|X_1|^p < \infty$;

(ii) $\sum_{n=1}^{\infty} n^{\alpha p - 2} P\Big(\max_{1 \leqslant j \leqslant n} \Big|\sum_{i=1}^{j} X_i\Big| > \varepsilon n^{\alpha}\Big) < \infty$ *for all* $\varepsilon > 0$.

Motivated by the result of Baum and Katz [3] for i.i.d. random variables, many authors further studied the Baum-Katz-type theorem for dependent random variables. One can refer to Jun and Demei [11], Peligrad [15], Peligrad and Gut [16], Qiu et al. [17], Shao [18], Shen et al. [19], Stoica [20], [21], Sung [23], Wang and Hu [26], Wang et al. [29], etc.

Next, we will give the definition of stochastic domination which is used frequently in the paper.

Definition 1.2. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$\sup_{n \geqslant 1} P(|X_n| > x) \leqslant CP(|X| > x)$$

for all $x \geqslant 0$.

An array $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ of rowwise random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$\sup_{i\geqslant 1} P(|X_{ni}| > x) \leqslant CP(|X| > x)$$

for all $x \geqslant 0$ and $n \geqslant 1$.

The complete convergence for arrays of rowwise random variables was studied by many authors. For example, the complete convergence for arrays of rowwise independent random variables was studied by Hu et al. [10], Sung et al. [25], Kruglov et al. [12] and others. Recently, many authors extended the complete convergence for arrays of rowwise independent random variables to the cases of the dependent random variables. Kuczmaszewska [13] obtained the complete convergence for arrays of rowwise ϱ-mixing and $\tilde{\varrho}$-mixing random variables, Chen et al. [4] and Kuczmaszewska [14] established the complete convergence for arrays of rowwise negatively associated random variables, Zhou and Lin [33] obtained the complete convergence for arrays of rowwise ϱ-mixing random variables under some suitable conditions, Sung [24] discussed the complete convergence for arrays of rowwise negatively associated, negatively dependent, φ-mixing and $\tilde{\varrho}$-mixing random variables, and so on. Meanwhile, many authors established the complete convergence of weighted sums for arrays of rowwise dependent random variables. For example, Baek et al. [1] discussed the complete convergence of weighted sums for arrays of rowwise negatively associated random variables, Baek and Park [2] and Wu [31] discussed the convergence of weighted sums for arrays of negatively dependent random variables, Wang et al. [28] discussed the complete convergence for weighted sums of arrays of rowwise asymptotically almost negatively associated random variables, Guo [8] investigated the complete moment convergence of weighted sums of rowwise φ-mixing random variables.

Wang and Hu [26] discussed the complete convergence for φ-mixing random variables and obtained the following results.

Theorem 1.2. *Let $\alpha p > 1$ and $1/2 < \alpha \leqslant 1$. Assume that $\{X_n, n \geqslant 1\}$ is a sequence of φ-mixing random variables which is stochastically dominated by a random variable X with $E|X|^p < \infty$. Assume further that $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ and $EX_k = 0$ for each $k \geqslant 1$. Then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} X_i\right| > \varepsilon n^{\alpha}\right) < \infty.$$

Theorem 1.3. *Let $\{X_n, n \geqslant 1\}$ be a sequence of φ-mixing random variables which is stochastically dominated by a random variable X with $E|X|^p < \infty$ for some $0 < p < 2$. Assume that $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ and $EX_k = 0$ for each $k \geqslant 1$ if $1 \leqslant p < 2$. Then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{-1} P\biggl(\max_{1\leqslant j\leqslant n}\biggl|\sum_{i=1}^{j} X_i\biggr| > \varepsilon n^{1/p}\biggr) < \infty.$$

The main purpose of the paper is to further study the complete convergence and complete moment convergence of weighted sums for arrays of rowwise φ-mixing random variables and the Baum-Katz-type theorem of φ-mixing random variables. As an application, we get the Marcinkiewicz-Zygmund type strong law of large numbers and the necessary and sufficient condition of the complete moment convergence for φ-mixing random variables. We relax the conditions $1/2 < \alpha \leqslant 1$ and $p > 1$ of Theorem 1.2 to the conditions $\alpha > 1/2$ and $p > 0$. Hence, we extend and complement the corresponding results of Wang and Hu [26].

Throughout this paper, the symbols $C, C_1, \ldots$ denote positive constants which may be different at various places. Assume that $I(A)$ is the indicator function of the set A. Let $x^+ = \max(0, x)$ and $\log x = \ln \max(x, \mathrm{e})$, where $\ln x$ denotes the natural logarithm. $a_n = O(b_n)$ stands for $|a_n| \leqslant C|b_n|$.

2. Some lemmas

In this section, we will give some lemmas which are useful to proving our main results.

Lemma 2.1 (cf. [30]). *Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables, which is stochastically dominated by a random variable X. Then for any $a > 0$ and $b > 0$, the following two statements hold:*

$$E|X_n|^a I(|X_n| \leqslant b) \leqslant C_1\{E|X|^a I(|X| \leqslant b) + b^a P(|X| > b)\}$$

and

$$E|X_n|^a I(|X_n| > b) \leqslant C_2 E|X|^a I(|X| > b),$$

where C_1 and C_2 are positive constants.

Lemma 2.2 (cf. [27]). *Let $\{X_n, n \geqslant 1\}$ be a sequence of φ-mixing random variables with $\sum\limits_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$. Assume that $EX_n = 0$ and $E|X_n|^q < \infty$ for some $q \geqslant 2$ and each $n \geqslant 1$. Then there exists a positive constant C depending only on q such that*

$$E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=a+1}^{a+j} X_i\Big|^q\Big) \leqslant C\Big\{\sum_{i=a+1}^{a+n} E|X_i|^q + \Big(\sum_{i=a+1}^{a+n} EX_i^2\Big)^{q/2}\Big\}$$

for every $a \geqslant 0$ and $n \geqslant 1$. In particular, for every $n \geqslant 1$ we have

$$E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} X_i\Big|^q\Big) \leqslant C\Big\{\sum_{i=1}^{n} E|X_i|^q + \Big(\sum_{i=1}^{n} EX_i^2\Big)^{q/2}\Big\}.$$

Lemma 2.3 (cf. [22]). *Let $\{Y_n, n \geqslant 1\}$ and $\{Z_n, n \geqslant 1\}$ be sequences of random variables. Then for any $q > 1$, $\varepsilon > 0$ and $a > 0$,*

$$E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j}(Y_i + Z_i)\Big| - \varepsilon a\Big)^+ \leqslant \Big(\frac{1}{\varepsilon^q} + \frac{1}{q-1}\Big)\frac{1}{a^{q-1}} E\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} Y_i\Big|^q + E\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} Z_i\Big|.$$

Lemma 2.4 (cf. [32]). *Assume that events A_1, A_2, …, A_n satisfy*

$$\operatorname{Var}\Big(\sum_{i=1}^{n} I(A_i)\Big) \leqslant C\sum_{i=1}^{n} P(A_i),$$

then

$$\Big(1 - P\Big(\bigcup_{i=1}^{n} A_i\Big)\Big)^2 \sum_{i=1}^{n} P(A_i) \leqslant CP\Big(\bigcup_{i=1}^{n} A_i\Big).$$

3. Main results and their proofs

In this section, let $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ be an array of rowwise φ-mixing random variables, precisely, $\{X_{ni}, i \geqslant 1\}$ is a sequence of φ-mixing random variables with the common mixing coefficients $\{\varphi(i), i \geqslant 1\}$ for every $n \geqslant 1$. Assume that $\{a_{ni}, i \geqslant 1, n \geqslant 1\}$ is an array of real numbers. Let $\{X_n, n \geqslant 1\}$ be a sequence of φ-mixing random variables with the mixing coefficients $\{\varphi(n), n \geqslant 1\}$.

In the following, let $\psi(x) = 1$ or $\psi(x) = \log x$. Note that the function $\psi(x)$ has the following properties (see Chen and Volodin [5]):

(a) for all $m \geqslant k \geqslant 1$,

$$\sum_{n=k}^{m} n^{r-1}\psi(n) \leqslant Cm^r\psi(m) \quad \text{if } r > 0 \tag{3.1}$$

and

$$\sum_{n=m}^{\infty} n^{r-1}\psi(n) \leqslant Cm^r\psi(m) \quad \text{if } r < 0; \tag{3.2}$$

(b) for all $p > 0$, $x \in \mathbb{R}$,

$$\psi(|x|^p) \leqslant C(p)\psi(|x|) \leqslant C(p)\psi(1 + |x|), \tag{3.3}$$

where $C(p)$ is a constant depending only on p.

Theorem 3.1. *Let $\alpha > 1/2$ and $\alpha p \geqslant 1$. Assume that $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ is an array of rowwise φ-mixing random variables which is stochastically dominated by a random variable X. Assume that $\{a_{ni}, i \geqslant 1, n \geqslant 1\}$ is an array of real numbers with*

$$\sum_{i=1}^{n} |a_{ni}|^q = O(n) \tag{3.4}$$

for some $q > \max\{(p\alpha - 1)/(\alpha - 1/2), 2\}$. Assume further that $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ and $EX_{ni} = 0$ for $i \geqslant 1$ and $n \geqslant 1$ if $p \geqslant 1$. If

$$E|X|^p\psi(|X|) < \infty, \tag{3.5}$$

then

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\psi(n)P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon n^{\alpha}\right) < \infty \quad \textit{for all } \varepsilon > 0. \tag{3.6}$$

P r o o f. i) Let $p > 1$. For fixed $n \geqslant 1$, let $X'_{ni} = X_{ni} I(|X_{ni}| \leqslant n^{\alpha})$ and $X''_{ni} = X_{ni} - X'_{ni}$, $i \geqslant 1$. Then it is easy to check that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} \psi(n) P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni} X_{ni}\Big| > \varepsilon n^{\alpha}\Big) \\
&\quad \leqslant \sum_{n=1}^{\infty} n^{\alpha p-2} \psi(n) P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}(X'_{ni} - EX'_{ni})\Big| > \varepsilon n^{\alpha}/2\Big) \\
&\qquad + \sum_{n=1}^{\infty} n^{\alpha p-2} \psi(n) P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}(X''_{ni} - EX''_{ni})\Big| > \varepsilon n^{\alpha}/2\Big) := I^* + J^*.
\end{aligned}$$

By C_r's inequality and $\sum_{i=1}^{n} |a_{ni}|^q = O(n)$, it is easy to check that for all $0 < \gamma \leqslant q$,

$$\frac{1}{n}\sum_{i=1}^{n} |a_{ni}|^{\gamma} \leqslant \Big(\frac{1}{n}\sum_{i=1}^{n} |a_{ni}|^{q}\Big)^{\gamma/q} = O(1). \tag{3.7}$$

For J^*, we have by Markov's inequality, Lemma 2.1, (3.7), and (3.3) that

$$\begin{aligned}
J^* &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\psi(n)\sum_{i=1}^{n} |a_{ni}| E|X''_{ni}| \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha}\psi(n) E|X| I(|X| > n^{\alpha}) \\
&= C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha}\psi(n)\sum_{j=n}^{\infty} E|X| I(j < |X|^{1/\alpha} \leqslant j+1) \\
&= C\sum_{j=1}^{\infty} E|X| I(j < |X|^{1/\alpha} \leqslant j+1)\sum_{n=1}^{j} n^{\alpha p-1-\alpha}\psi(n) \\
&\leqslant C\sum_{j=1}^{\infty} j^{\alpha p-\alpha}\psi(j) E|X| I(j < |X|^{1/\alpha} \leqslant j+1) \\
&\leqslant CE|X|^p \psi(|X|^{1/\alpha}) \\
&\leqslant CE|X|^p \psi(|X|) < \infty.
\end{aligned} \tag{3.8}$$

For I^*, by Markov's inequality, Lemma 2.2 and Jensen's inequality we have that for any $r \geqslant 2$,

$$I^* \leqslant C_r \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}\psi(n) E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j}(a_{ni}X'_{ni} - Ea_{ni}X'_{ni})\Big|^{r}\Big) \tag{3.9}$$

$$\leqslant C_r \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}\psi(n)\sum_{i=1}^{n}|a_{ni}|^r E|X'_{ni}|^r$$
$$+ C_r \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}\psi(n)\Bigl(\sum_{i=1}^{n} a_{ni}^2 E(X'_{ni})^2\Bigr)^{r/2} := I_1^* + I_2^*.$$

We consider the following three cases:

Case 1. $\alpha > 1/2$, $\alpha p > 1$ and $p \geqslant 2$.

Take $r = q$. By $q > \max\{(\alpha p-1)/(\alpha-1/2), 2\}$, it follows that $q > p$ and $\alpha p - 2 - \alpha q + q/2 < -1$.

For I_1^*, we have by C_r's inequality that

$$\begin{aligned}
(3.10)\quad I_1^* &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}\psi(n)\\
&\quad\times \sum_{i=1}^{n}|a_{ni}|^q\bigl(E|X_{ni}|^q I(|X_{ni}|\leqslant n^\alpha) + n^{\alpha q}P(|X_{ni}|>n^\alpha)\bigr)\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}\psi(n)\sum_{i=1}^{n}|a_{ni}|^q(E|X|^q I(|X|\leqslant n^\alpha) + n^{\alpha q}P(|X|>n^\alpha))\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha q}\psi(n)E|X|^q I(|X|\leqslant n^\alpha)\\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha}\psi(n)E|X|I(|X|>n^\alpha)\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha(p-q)-1}\sum_{j=1}^{n} j^{\alpha q}P(j-1<|X|^{1/\alpha}\leqslant j) + C\\
&\leqslant C\sum_{j=1}^{\infty} j^{\alpha q}P(j-1<|X|^{1/\alpha}\leqslant j)\sum_{n=j}^{\infty} n^{\alpha(p-q)-1}\psi(n) + C\\
&\leqslant C\sum_{j=1}^{\infty} j^{\alpha p}\psi(j)P(j-1<|X|^{1/\alpha}\leqslant j) + C\\
&\leqslant CE|X|^p\psi(|X|) + C < \infty.
\end{aligned}$$

For I_2^*, note that $EX^2<\infty$ if $E|X|^p\psi(|X|)<\infty$ for $p\geqslant 2$. We have by (3.7) that

$$\begin{aligned}
I_2^* &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}\psi(n)\Bigl(\sum_{i=1}^{n} a_{ni}^2 EX_{ni}^2\Bigr)^{q/2}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}\psi(n)\Bigl(\sum_{i=1}^{n} a_{ni}^2 EX^2\Bigr)^{q/2} \leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q+q/2}\psi(n) < \infty.
\end{aligned}$$

Case 2. $\alpha > 1/2$, $\alpha p > 1$ and $1 < p < 2$.

Take $r = 2$. Similarly to the proofs of (3.8)–(3.10) , we have that

$$(3.11)\quad \begin{aligned} I^* &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha}\psi(n)\sum_{i=1}^{n} a_{ni}^2 (EX_{ni}^2 I(|X_{ni}| \leqslant n^\alpha) + n^{2\alpha}P(|X_{ni}| > n^\alpha)) \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha}\psi(n)EX^2 I(|X| \leqslant n^\alpha) \\ &\quad + C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha}\psi(n)E|X|I(|X| > n^\alpha) < \infty. \end{aligned}$$

Case 3. $\alpha > 1/2$, $\alpha p = 1$ and $p > 1$

Take $r = 2$. Note that $1/2 < \alpha < 1$ if $\alpha p = 1$. Similarly to the proof of (3.11), it follows that $I^* < \infty$.

ii) Let $p = 1$. Note that $\alpha \geqslant 1$ due to $\alpha p \geqslant 1$. By $EX_{ni} = 0$ for $i \geqslant 1$ and $n \geqslant 1$, Lemma 2.1, (3.7) and (3.5), we have that

$$\begin{aligned} n^{-\alpha}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}EX'_{ni}\right| &= n^{-\alpha}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}EX_{ni}I(|X_{ni}| > n^\alpha)\right| \\ &\leqslant n^{-\alpha}\sum_{i=1}^{n} |a_{ni}|E|X_{ni}|I(|X_{ni}| > n^\alpha) \\ &\leqslant n^{1-\alpha}E|X|I(|X| > n^\alpha) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence, for n large enough, we have

$$(3.12)\qquad n^{-\alpha}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}EX'_{ni}\right| < \frac{\varepsilon}{2}.$$

It follows that

$$(3.13)\quad \begin{aligned} &\sum_{n=1}^{\infty} n^{\alpha-2}\psi(n)P\Big(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}\right| > \varepsilon n^\alpha\Big) \\ &\leqslant \sum_{n=1}^{\infty} n^{\alpha-2}\psi(n)\sum_{i=1}^{n} P(|X_{ni}| > n^\alpha) \\ &\quad + \sum_{n=1}^{\infty} n^{\alpha-2}\psi(n)P\Big(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X'_{ni}\right| > \varepsilon n^\alpha\Big) \end{aligned}$$

$$\begin{aligned}
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha-1}\psi(n)P(|X|>n^{\alpha})\\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha-2}\psi(n)P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}(X'_{ni}-EX'_{ni})\right|>\frac{\varepsilon n^{\alpha}}{2}\right)\\
&:= CI_1+CI_2.
\end{aligned}$$

For I_1, we have by (3.1) and (3.5) that

$$\begin{aligned}
I_1 &= \sum_{n=1}^{\infty} n^{\alpha-1}\psi(n)\sum_{i=n}^{\infty} P(i^{\alpha}<|X|\leqslant (i+1)^{\alpha})\\
&= \sum_{i=1}^{\infty} P(i^{\alpha}<|X|\leqslant (i+1)^{\alpha})\sum_{n=1}^{i} n^{\alpha-1}\psi(n)\\
&\leqslant C\sum_{i=1}^{\infty} P(i^{\alpha}<|X|\leqslant (i+1)^{\alpha})i^{\alpha}\psi(i)\leqslant CE|X|\psi(|X|)<\infty.
\end{aligned} \tag{3.14}$$

For I_2, we have by Markov's inequality, Lemma 2.2, Lemma 2.1, (3.2) and (3.3) that

(3.15)

$$\begin{aligned}
I_2 &\leqslant C\sum_{n=1}^{\infty} n^{-\alpha-2}\psi(n)E\max_{1\leqslant j\leqslant n}\left(\sum_{i=1}^{j} a_{ni}(X'_{ni}-EX'_{ni})\right)^2\\
&\leqslant C\sum_{n=1}^{\infty} n^{-\alpha-2}\psi(n)\sum_{i=1}^{n} a_{ni}^2E(X'_{ni})^2\\
&= C\sum_{n=1}^{\infty} n^{-\alpha-2}\psi(n)\left\{\sum_{i=1}^{n} a_{ni}^2EX_{ni}^2I(|X_{ni}|\leqslant n^{\alpha})+n^{2\alpha}\sum_{i=1}^{n} a_{ni}^2P(|X_{ni}|>n^{\alpha})\right\}\\
&\leqslant C\sum_{n=1}^{\infty} n^{-\alpha-1}\psi(n)EX^2I(|X|\leqslant n^{\alpha})+C\sum_{n=1}^{\infty} n^{\alpha-1}\psi(n)P(|X|>n^{\alpha})\\
&= C\sum_{n=1}^{\infty} n^{-\alpha-1}\psi(n)\sum_{k=1}^{n} EX^2I((k-1)^{\alpha}<|X|\leqslant k^{\alpha})+C\\
&= C\sum_{k=1}^{\infty} EX^2I((k-1)^{\alpha}<|X|\leqslant k^{\alpha})\sum_{n=k}^{\infty} n^{-\alpha-1}\psi(n)+C\\
&\leqslant C\sum_{k=1}^{\infty} k^{-\alpha}\psi(k)EX^2I((k-1)^{\alpha}<|X|\leqslant k^{\alpha})+C\\
&\leqslant CE|X|\psi(|X|)+C<\infty.
\end{aligned}$$

By (3.13)–(3.15), (3.6) holds for the case $p=1$.

iii) Let $0 < p < 1$. Denote

$$\sum_{i=1}^{j} a_{ni}X_{ni} = \sum_{i=1}^{j} a_{ni}X_{ni}I(|X_{ni}| \leqslant n^{\alpha}) + \sum_{i=1}^{j} a_{ni}X_{ni}I(|X_{ni}| > n^{\alpha}) \tag{3.16}$$
$$=: S'_{nj} + S''_{nj}.$$

Noting that $E|X|^p\psi(|X|) < \infty$, we have by Markov's inequality, Lemma 2.1, (3.2), (3.3), and (3.7) that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2}\psi(n)P\Big(\max_{1\leqslant j\leqslant n}|S'_{nj}| > \varepsilon n^{\alpha}\Big) \qquad (3.17)\\
&\leqslant \varepsilon^{-1}\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\psi(n)E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}X_{ni}I(|X_{ni}| \leqslant n^{\alpha})\Big|\Big)\\
&\leqslant \varepsilon^{-1}\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\psi(n)\sum_{i=1}^{n}|a_{ni}|E|X_{ni}|I(|X_{ni}| \leqslant n^{\alpha})\\
&\leqslant C\varepsilon^{-1}\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha}\psi(n)E|X|I(|X| \leqslant n^{\alpha})\\
&\quad + C\varepsilon^{-1}\sum_{n=1}^{\infty} n^{\alpha p-1}\psi(n)P(|X| > n^{\alpha})\\
&= C\varepsilon^{-1}\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha}\psi(n)\sum_{j=1}^{n}E|X|I(j-1 < |X|^{1/\alpha} \leqslant j)\\
&\quad + C\varepsilon^{-1}\sum_{n=1}^{\infty} n^{\alpha p-1}\psi(n)\sum_{j=n}^{\infty}P(j < |X|^{1/\alpha} \leqslant j+1)\\
&= C\varepsilon^{-1}\sum_{j=1}^{\infty} j^{\alpha}P(j-1 < |X|^{1/\alpha} \leqslant j)\sum_{n=j}^{\infty} n^{\alpha p-1-\alpha}\psi(n)\\
&\quad + C\varepsilon^{-1}\sum_{j=1}^{\infty} P(j < |X|^{1/\alpha} \leqslant j+1)\sum_{n=1}^{j} n^{\alpha p-1}\psi(n)\\
&\leqslant C\varepsilon^{-1}\sum_{j=1}^{\infty} j^{\alpha p}\psi(j)P(j-1 < |X|^{1/\alpha} \leqslant j)\\
&\quad + C\varepsilon^{-1}\sum_{j=1}^{\infty} j^{\alpha p}\psi(j)P(j < |X|^{1/\alpha} \leqslant j+1)\\
&\leqslant CE|X|^p\psi(|X|) < \infty
\end{aligned}$$

and

$$
\begin{aligned}
(3.18)\qquad \sum_{n=1}^{\infty} n^{\alpha p-2}\psi(n)P\Big(\max_{1\leqslant j\leqslant n}|S''_{nj}|>\varepsilon n^{\alpha}\Big)&\\
&\leqslant \varepsilon^{-p/2}\sum_{n=1}^{\infty} n^{\alpha p/2-2}\psi(n)E\bigg(\max_{1\leqslant j\leqslant n}\bigg|\sum_{i=1}^{j} a_{ni}X_{ni}I(|X_{ni}|>n^{\alpha})\bigg|\bigg)^{p/2}\\
&\leqslant \varepsilon^{-p/2}\sum_{n=1}^{\infty} n^{\alpha p/2-2}\psi(n)\sum_{i=1}^{n}|a_{ni}|^{p/2}E|X_{ni}|^{p/2}I(|X_{ni}|>n^{\alpha})\\
&\leqslant C\varepsilon^{-p/2}\sum_{n=1}^{\infty} n^{\alpha p/2-1}\psi(n)E|X|^{p/2}I(|X|>n^{\alpha})\\
&= C\varepsilon^{-p/2}\sum_{n=1}^{\infty} n^{\alpha p/2-1}\psi(n)\sum_{j=n}^{\infty}E|X|^{p/2}I(j<|X|^{1/\alpha}\leqslant j+1)\\
&= C\varepsilon^{-p/2}\sum_{j=1}^{\infty} j^{\alpha p/2}P(j<|X|^{1/\alpha}\leqslant j+1)\sum_{n=1}^{j}n^{\alpha p/2-1}\psi(n)\\
&\leqslant C\varepsilon^{-p/2}\sum_{j=1}^{\infty} j^{\alpha p}\psi(j)P(j-1<|X|^{1/\alpha}\leqslant j)\\
&\leqslant CE|X|^{p}\psi(|X|)<\infty.
\end{aligned}
$$

Hence, (3.16)–(3.18) implies (3.6). The proof of the theorem is completed. □

R e m a r k 3.1. Under the conditions of Theorem 3.1, we have that for $p>1$

$$
(3.19)\qquad \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\psi(n)E\bigg(\max_{1\leqslant j\leqslant n}\bigg|\sum_{i=1}^{j} a_{ni}X_{ni}\bigg|-\varepsilon n^{\alpha}\bigg)^{+}<\infty \text{ for all } \varepsilon>0.
$$

In fact, by Lemma 2.3 with $r\geqslant 2$

$$
\begin{aligned}
\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\psi(n)E\bigg(\max_{1\leqslant j\leqslant n}\bigg|\sum_{i=1}^{j} a_{ni}X_{ni}\bigg|-\varepsilon n^{\alpha}\bigg)^{+}&\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r}\psi(n)E\bigg(\max_{1\leqslant j\leqslant n}\bigg|\sum_{i=1}^{j}(a_{ni}X'_{ni}-Ea_{ni}X'_{ni})\bigg|\bigg)^{r}\\
&\quad+\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\psi(n)E\bigg(\max_{1\leqslant j\leqslant n}\bigg|\sum_{i=1}^{j}(a_{ni}X''_{ni}-Ea_{ni}X''_{ni})\bigg|\bigg).
\end{aligned}
$$

By applying the process of the proof of Theorem 3.1 for the case $p>1$, it follows that (3.19) holds.

R e m a r k 3.2. Zhou and Lin [33] and Guo [8] established the complete convergence for arrays of dependent random variables. Theorem 3.2 of Zhou and Lin [33] yields the complete convergence of weighted sums for arrays of rowwise ϱ-mixing random variables stochastically dominated by a random variables X with $E|X|^p < \infty$ for $1 \leqslant p \leqslant 2$. The weighted condition in Theorem 3.2 of Zhou and Lin [33] guarantees that for some $r \geqslant 2$

$$\max_{1\leqslant i\leqslant n} |a_{ni}|^p = O(n^{\nu-1}) \text{ for some } 0 < \nu < 2/r. \tag{3.20}$$

Note that in the case of non-weight (take $a_{ni} \equiv 1$) (3.20), cannot be satisfied but (3.4), can; hence (3.20) is stronger than (3.4). Actually, by (3.20), it follows that

$$\sum_{i=1}^{n} |a_{ni}|^q \leqslant Cn^{1+\frac{\nu-1}{p}q} \leqslant Cn.$$

We discuss the complete convergence of weighted sums for arrays of rowwise φ-mixing random variables stochastically dominated by a random variable X with $E|X|^p < \infty$ for $p > 0$ under the condition (3.4) which is satisfied for the case of non-weight (take $a_{ni} \equiv 1$). The Corollary 2.5 of Guo [8] establishes the complete moment convergence for arrays of rowwise φ-mixing random variables stochastically dominated by a random variable X with $E|X|^p l(|X|^{1/\alpha})$ for $\alpha p > 1$ and $1/2 < \alpha < 1$, where $l(x) > 0$ is a slowly varying function. In Remark 3.1, we consider the complete moment convergence of weighted sums for arrays of rowwise φ-mixing random variables to two special slowly varying functions $\psi(x) = 1$ or $\psi(x) = \log x$, and relax the conditions $\alpha p > 1$ and $1/2 < \alpha < 1$ to the case $\alpha p \geqslant 1$, $p > 1$, and $\alpha > 1/2$.

Similarly to the proof of Theorem 3.1, we can get easily the following result.

Theorem 3.2. *Let $\alpha > 1/2$ and $\alpha p \geqslant 1$. Suppose that $\{X_n, n \geqslant 1\}$ is a sequence of φ-mixing random variables which is stochastically dominated by a random variable X. Assume that $\{a_{ni}, i \geqslant 1, n \geqslant 1\}$ is an array of real numbers with $\sum_{i=1}^{n} |a_{ni}|^q = O(n)$ for some $q > \max\{(\alpha p - 1)/(\alpha - 1/2), 2\}$. Assume further that $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ and $EX_n = 0$ for $n \geqslant 1$ if $p \geqslant 1$. If (3.5) holds, then*

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\psi(n)P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| > \varepsilon n^{\alpha}\Big) < \infty \text{ for all } \varepsilon > 0.$$

Corollary 3.1. *Let $\alpha > 1/2$ and $\alpha p \geqslant 1$. Suppose that $\{X_n, n \geqslant 1\}$ is a sequence of φ-mixing random variables which is stochastically dominated by a random variable X. Assume that $\{a_n, n \geqslant 1\}$ is a sequence of real numbers with $\sum_{i=1}^{n} |a_i|^q = O(n)$ for some $q > \max\{(\alpha p - 1)/(\alpha - 1/2), 2\}$. Assume further that $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ and $EX_n = 0$ for $n \geqslant 1$ if $p \geqslant 1$. If $E|X|^p < \infty$, then*

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P\Big(\max_{1 \leqslant j \leqslant n} \Big|\sum_{i=1}^{j} a_i X_i\Big| > \varepsilon n^{\alpha}\Big) < \infty \text{ for all } \varepsilon > 0 \tag{3.21}$$

and

$$n^{-\alpha} \sum_{i=1}^{n} a_i X_i \to 0 \text{ a.s. } n \to \infty. \tag{3.22}$$

Proof. Taking $\psi(x) = 1$, and $a_{ni} = a_i$ for $1 \leqslant i \leqslant n$ and $a_{ni} = 0$ otherwise in Theorem 3.2, we obtain (3.21) immediately. We will prove (3.22).

By (3.21), it follows that for all $\varepsilon > 0$,

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{\alpha p - 2} P\Big(\max_{1 \leqslant j \leqslant n} \Big|\sum_{i=1}^{j} a_i X_i\Big| > \varepsilon n^{\alpha}\Big) \\
&= \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} n^{\alpha p - 2} P\Big(\max_{1 \leqslant j \leqslant n} \Big|\sum_{i=1}^{j} a_i X_i\Big| > \varepsilon n^{\alpha}\Big) \\
&\geqslant \begin{cases} \sum_{k=0}^{\infty} (2^k)^{\alpha p - 2} 2^k P\Big(\max_{1 \leqslant j \leqslant 2^k} \Big|\sum_{i=1}^{j} a_i X_i\Big| > \varepsilon 2^{(k+1)\alpha}\Big), & \text{if } \alpha p \geqslant 2, \\ \sum_{k=0}^{\infty} (2^{k+1})^{\alpha p - 2} 2^k P\Big(\max_{1 \leqslant j \leqslant 2^k} \Big|\sum_{i=1}^{j} a_i X_i\Big| > \varepsilon 2^{(k+1)\alpha}\Big), & \text{if } 1 \leqslant \alpha p < 2. \end{cases} \\
&\geqslant \begin{cases} \sum_{k=0}^{\infty} P\Big(\max_{1 \leqslant j \leqslant 2^k} \Big|\sum_{i=1}^{j} a_i X_i\Big| > \varepsilon 2^{(k+1)\alpha}\Big), & \text{if } \alpha p \geqslant 2, \\ 1/2 \sum_{k=0}^{\infty} P\Big(\max_{1 \leqslant j \leqslant 2^k} \Big|\sum_{i=1}^{j} a_i X_i\Big| > \varepsilon 2^{(k+1)\alpha}\Big), & \text{if } 1 \leqslant \alpha p < 2. \end{cases}
\end{aligned}$$

By the Borel-Cantelli lemma, we obtain that

$$\frac{\max_{1 \leqslant j \leqslant 2^k} \Big|\sum_{i=1}^{j} a_i X_i\Big|}{2^{(k+1)\alpha}} \to 0 \quad \text{a.s. } k \to \infty. \tag{3.23}$$

For every positive integer n there exists a positive integer k such that $2^{k-1} \leqslant n \leqslant 2^k$. We have by (3.23) that

$$n^{-\alpha}\Big|\sum_{i=1}^{n} a_i X_i\Big| \leqslant \max_{2^{k-1}\leqslant n\leqslant 2^k} n^{-\alpha}\Big|\sum_{i=1}^{n} a_i X_i\Big| \leqslant \frac{2^{\alpha}\max\limits_{1\leqslant j\leqslant 2^k}\Big|\sum\limits_{i=1}^{j} a_i X_i\Big|}{2^{(k+1)\alpha}} \to 0 \quad \text{a.s. } k\to\infty,$$

which implies that

$$n^{-\alpha}\sum_{i=1}^{n} a_i X_i \to 0 \quad \text{a.s. } n\to\infty.$$

This completes the proof of the corollary. □

R e m a r k 3.3. Take $a_n \equiv 1$ in Corollary 3.1. Compared with Theorem 1.2, we relax the conditions $1/2 < \alpha \leqslant 1$ and $p > 1$ to the conditions $\alpha > 1/2$ and $p > 0$, and also consider the case $\alpha p = 1$. Taking $\alpha p = 1$ in Corollary 3.1, we can get Theorem 1.3 immediately, i.e., Theorem 1.3 is a special case of Corollary 3.1. Taking $\alpha = 1$ and $p = 2$ in Corollary 3.1, we can get the Hsu-Robbins-type theorem (see Hsu and Robbins [9]) for φ-mixing random variables. Hence, we extend and complement the corresponding results of Wang and Hu [26].

If the condition of stochastic domination in Theorem 3.1 is replaced by the stronger condition that (3.24) below is satisfied, we get the following result.

Theorem 3.3. *Let $\alpha > 1/2$ and $\alpha p \geqslant 1$. Let $\{X_{ni}, i \geqslant, n \geqslant 1\}$ be an array of rowwise φ-mixing random variables. Assume that $\{a_{ni}, i \geqslant 1, n \geqslant 1\}$ is an array of real numbers with $\sum_{i=1}^{n} |a_{ni}|^q = O(n)$ for some $q > \max\{(\alpha p - 1)/(\alpha - 1/2), 2\}$. Assume further that $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ and $EX_{ni} = 0$ for $i \geqslant 1$ and $n \geqslant 1$ if $p \geqslant 1$. If there exist a random variable X and positive numbers C_1 and C_2 such that for all $x \geqslant 0$, $n \geqslant 1$,*

$$C_1 P(|X| \geqslant x) \leqslant \inf_{i\geqslant 1} P(|X_{ni}| \geqslant x) \leqslant \sup_{i\geqslant 1} P(|X_{ni}| \geqslant x) \leqslant C_2 P(|X| \geqslant x), \tag{3.24}$$

then (3.5) *is equivalent to* (3.6).

P r o o f. By Theorem 3.1, we can see that (3.5) implies (3.6) under the conditions of Theorem 3.3. So we only need to prove that (3.6) implies (3.5).

By (3.6), taking $a_{ni} \equiv 1$ for all $i \geqslant 1$ and $n \geqslant 1$, it follows that for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{\alpha p - 2}\psi(n) P\Big(\max_{1\leqslant j\leqslant n} |X_{nj}| > \varepsilon n^{\alpha}\Big) < \infty. \tag{3.25}$$

We have by (3.25) that

$$P\Big(\max_{1\leqslant j\leqslant n}|X_{nj}|>\varepsilon n^{\alpha}\Big)\to 0 \quad \text{as } n\to\infty. \tag{3.26}$$

By Lemma 2.2, one has that

$$\begin{aligned}\operatorname{Var}\Big(\sum_{i=1}^{n}I(|X_{ni}|>\varepsilon n^{\alpha})\Big) &= E\Big(\sum_{i=1}^{n}(I(|X_{ni}|>\varepsilon n^{\alpha})-EI(|X_{ni}|>\varepsilon n^{\alpha}))\Big)^2\\ &\leqslant C\sum_{i=1}^{n}P(|X_{ni}|>\varepsilon n^{\alpha}),\end{aligned}$$

which implies that by Lemma 2.4

$$\begin{aligned}\Big(1-P\Big(\max_{1\leqslant j\leqslant n}|X_{nj}|>\varepsilon n^{\alpha}\Big)\Big)^2 &\sum_{i=1}^{n}P(|X_{ni}|>\varepsilon n^{\alpha})\\ &\leqslant CP\Big(\max_{1\leqslant j\leqslant n}|X_{nj}|>\varepsilon n^{\alpha}\Big).\end{aligned} \tag{3.27}$$

Combining (3.24) with (3.26) and (3.27), we have that for all $\varepsilon>0$

$$nP(|X|>\varepsilon n^{\alpha})\leqslant C\sum_{j=1}^{n}P(|X_{nj}|>\varepsilon n^{\alpha})\leqslant CP\Big(\max_{1\leqslant j\leqslant n}|X_{nj}|>\varepsilon n^{\alpha}\Big). \tag{3.28}$$

Take $\varepsilon=1$. It follows from (3.25) and (3.28) and (3.1) that

$$\begin{aligned}\infty &> \sum_{n=1}^{\infty}n^{\alpha p-2}\psi(n)P\Big(\max_{1\leqslant j\leqslant n}|X_{nj}|>n^{\alpha}\Big)\\ &\geqslant C\sum_{n=1}^{\infty}n^{\alpha p-1}\psi(n)P(|X|>n^{\alpha})\\ &= C\sum_{n=1}^{\infty}n^{\alpha p-1}\psi(n)\sum_{j=n}^{\infty}P(j<|X|^{1/\alpha}\leqslant j+1)\\ &= C\sum_{j=1}^{\infty}P(j<|X|^{1/\alpha}\leqslant j+1)\sum_{n=1}^{j}n^{\alpha p-1}\psi(n)\\ &\geqslant C\sum_{j=1}^{\infty}P(j<|X|^{1/\alpha}\leqslant j+1)j^{\alpha p}\psi(j)\\ &\geqslant CE|X|^p\psi(|X|),\end{aligned}$$

i.e. (3.5) holds. The proof of the theorem is completed. □

Similarly to the proof of Theorem 3.3, we obtain the following result easily.

Theorem 3.4. *Let $\alpha > 1/2$ and $\alpha p \geqslant 1$. Assume that $\{X_n, n \geqslant 1\}$ is a sequence of φ-mixing random variables. Assume further that $\sum\limits_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ and $EX_n = 0$ for $n \geqslant 1$ if $p \geqslant 1$. If there exist a random variable X and positive numbers C_1 and C_2 such that for all $x \geqslant 0$*

$$C_1 P(|X| \geqslant x) \leqslant \inf_{i \geqslant 1} P(|X_i| \geqslant x) \leqslant \sup_{i \geqslant 1} P(|X_i| \geqslant x) \leqslant C_2 P(|X| \geqslant x), \tag{3.29}$$

then (3.5) *is equivalent to* (3.6).

If $\{X_n, n \geqslant 1\}$ is a sequence of identically distributed random variables, then (3.29) is satisfied. So we can get the following corollary from Theorem 3.4.

Corollary 3.2. *Let $\alpha > 1/2$ and $\alpha p \geqslant 1$. Let $\{X_n, n \geqslant 1\}$ be a sequence of identically distributed φ-mixing random variables. Assume that $\sum\limits_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ and $EX_n = 0$ for $n \geqslant 1$ if $p \geqslant 1$. Then the following two statements are equivalent:*

(i) $E|X_1|^p < \infty$;

(ii) $\sum\limits_{n=1}^{\infty} n^{\alpha p - 2} P\Big(\max\limits_{1 \leqslant j \leqslant n} \Big|\sum\limits_{i=1}^{j} X_i\Big| > \varepsilon n^{\alpha}\Big) < \infty$ *for all* $\varepsilon > 0$.

Remark 3.4. Corollary 3.2 extends the Baum-Katz Theorem (i.e. Theorem 1.1) for i.i.d. random variables to the case of φ-mixing random variables. In addition, we complement the case $\alpha p = 1$ and $\alpha > 1/2$.

Corollary 3.3. *Let $\alpha > 1/2$, $\alpha p \geqslant 1$ and $p > 1$. Let $\{X_n, n \geqslant 1\}$ be a sequence of zero mean and identically distributed φ-mixing random variables. Assume that $\sum\limits_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$. Then the following two statements are equivalent:*

(i) $E|X_1|^p < \infty$;

(ii) $\sum\limits_{n=1}^{\infty} n^{\alpha p - 2 - \alpha} E\Big(\max\limits_{1 \leqslant j \leqslant n} \Big|\sum\limits_{i=1}^{j} X_i\Big| - \varepsilon n^{\alpha}\Big)^+ < \infty$ *for all* $\varepsilon > 0$.

Proof. Taking $\psi(x) = 1$ in Theorem 3.1, (i) implies (ii) from Theorem 3.1 and Remark 3.1 immediately. We only need to prove (ii) implies (i).

It is easy to check that

$$\varepsilon n^{\alpha} P\Big(\max_{1 \leqslant j \leqslant n} \Big|\sum_{i=1}^{j} X_i\Big| > 2\varepsilon n^{\alpha}\Big) \leqslant E\Big(\max_{1 \leqslant j \leqslant n} \Big|\sum_{i=1}^{j} X_i\Big| - \varepsilon n^{\alpha}\Big)^+.$$

Hence, (ii) implies that

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P\Big(\max_{1 \leqslant j \leqslant n} \Big|\sum_{i=1}^{j} X_i\Big| > \varepsilon n^{\alpha}\Big) < \infty \quad \text{for all } \varepsilon > 0.$$

The rest of the proof is similar to that of Theorem 3.3 and is omitted. This completes the proof of the corollary. □

A c k n o w l e d g e m e n t. The authors are most grateful to the Editor Viktor Beneš and an anonymous referee for their careful reading of the manuscript and valuable suggestions which helped in significantly improving an earlier version of this paper.

References

[1] *J.-I. Baek, I.-B. Choi, S.-L. Niu*: On the complete convergence of weighted sums for arrays of negatively associated variables. J. Korean Stat. Soc. *37* (2008), 73–80.

[2] *J.-I. Baek, S.-T. Park*: Convergence of weighted sums for arrays of negatively dependent random variables and its applications. RETRACTED. J. Theor. Probab. *23* (2010), 362–377; retraction ibid. *26* (2013), 899–900.

[3] *L. E. Baum, M. Katz*: Convergence rates in the law of large numbers. Trans. Am. Math. Soc. *120* (1965), 108–123.

[4] *P. Chen, T.-C. Hu, X. Liu, A. Volodin*: On complete convergence for arrays of row-wise negatively associated random variables. Theory Probab. Appl. *52* (2008), 323–328; and Teor. Veroyatn. Primen. *52* (2007), 393–397.

[5] *P. Chen, T.-C. Hu, A. Volodin*: Limiting behaviour of moving average processes under negative association assumption. Theory Probab. Math. Stat. *77* (2008), 165–176; and Teor. Jmovirn. Mat. Stat. *77* (2007), 149–160.

[6] *R. L. Dobrushin*: Central limit theorem for non-stationary Markov chains. I, II. Teor. Veroyatn. Primen. *1* (1956), 72–89; Berichtigung. Ibid. *3* (1958), 477.

[7] *P. Erdős*: On a theorem of Hsu and Robbins. Ann. Math. Stat. *20* (1949), 286–291.

[8] *M. L. Guo*: Complete moment convergence of weighted sums for arrays of rowwise φ-mixing random variables. Int. J. Math. Math. Sci. *2012* (2012), Article ID 730962, 13 pp.

[9] *P. L. Hsu, H. Robbins*: Complete convergence and the law of large numbers. Proc. Natl. Acad. Sci. USA *33* (1947), 25–31.

[10] *T.-C. Hu, M. Ordóñez Cabrera, S. H. Sung, A. Volodin*: Complete convergence for arrays of rowwise independent random variables. Commun. Korean Math. Soc. *18* (2003), 375–383.

[11] *A. Jun, Y. Demei*: Complete convergence of weighted sums for $\varrho *$-mixing sequence of random variables. Stat. Probab. Lett. *78* (2008), 1466–1472.

[12] *V. M. Kruglov, A. I. Volodin, T.-C. Hu*: On complete convergence for arrays. Stat. Probab. Lett. *76* (2006), 1631–1640.

[13] *A. Kuczmaszewska*: On complete convergence for arrays of rowwise dependent random variables. Stat. Probab. Lett. *77* (2007), 1050–1060.

[14] *A. Kuczmaszewska*: On complete convergence for arrays of rowwise negatively associated random variables. Stat. Probab. Lett. *79* (2009), 116–124.

[15] *M. Peligrad*: Convergence rates of the strong law for stationary mixing sequences. Z. Wahrscheinlichkeitstheor. Verw. Geb. *70* (1985), 307–314.

[16] *M. Peligrad, A. Gut*: Almost-sure results for a class of dependent random variables. J. Theor. Probab. *12* (1999), 87–104.

[17] *D. H. Qiu, T.-C. Hu, M. O. Cabrera, A. Volodin*: Complete convergence for weighted sums of arrays of Banach-space-valued random elements. Lith. Math. J. *52* (2012), 316–325.

[18] *Q. M. Shao*: A moment inequality and its applications. Acta Math. Sin. *31* (1988), 736–747. (In Chinese.)

[19] *A. T. Shen, X. H. Wang, J. M. Ling*: On complete convergence for non-stationary φ-mixing random variables. Commun. Stat. Theory Methods, DOI:10.1080/03610926.2012.725501.

[20] *G. Stoica*: Baum-Katz-Nagaev type results for martingales. J. Math. Anal. Appl. *336* (2007), 1489–1492.

[21] *G. Stoica*: A note on the rate of convergence in the strong law of large numbers for martingales. J. Math. Anal. Appl. *381* (2011), 910–913.

[22] *S. H. Sung*: Moment inequalities and complete moment convergence. J. Inequal. Appl. *2009* (2009), Article ID 271265, 14 pp.

[23] *S. H. Sung*: Complete convergence for weighted sums of ϱ^*-mixing random variables. Discrete Dyn. Nat. Soc. *2010* (2010), Article ID 630608, 13 pp.

[24] *S. H. Sung*: On complete convergence for weighted sums of arrays of dependent random variables. Abstr. Appl. Anal. *2011* (2011), Article ID 630583, 11 pp.

[25] *S. H. Sung, A. I. Volodin, T.-C. Hu*: More on complete convergence for arrays. Stat. Probab. Lett. *71* (2005), 303–311.

[26] *X. J. Wang, S. H. Hu*: Some Baum-Katz type results for φ-mixing random variables with different distributions. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM *106* (2012), 321–331.

[27] *X. J. Wang, S. H. Hu, W. Z. Yang, Y. Shen*: On complete convergence for weighted sums of φ-mixing random variables. J. Inequal. Appl. *2010* (2010), Article ID 372390, 13 pp.

[28] *X. J. Wang, S. H. Hu, W. Z. Yang, X. H. Wang*: Convergence rates in the strong law of large numbers for martingale difference sequences. Abstr. Appl. Anal. *2012* (2012), Article ID 572493, 13 pp.

[29] *X. J. Wang, S. H. Hu, W. Z. Yang, X. H. Wang*: On complete convergence of weighted sums for arrays of rowwise asymptotically almost negatively associated random variables. Abstr. Appl. Anal. *2012* (2012), Article ID 315138, 15 pp.

[30] *Q. Y. Wu*: Probability Limit Theory for Mixed Sequence. China Science Press, Beijing, 2006. (In Chinese.)

[31] *Q. Y. Wu*: A complete convergence theorem for weighted sums of arrays of rowwise negatively dependent random variables. J. Inequal. Appl. *2012* (2012), Article ID 50, 10 pp. (electronic only).

[32] *L. X. Zhang, J. W. Wen*: The strong law of large numbers for B-valued random fields. Chin. Ann. Math., Ser. A *22* (2001), 205–216. (In Chinese.)

[33] *X. C. Zhou, J. G. Lin*: On complete convergence for arrays of rowwise ϱ-mixing random variables and its applications. J. Inequal. Appl. *2010* (2010), Article ID 769201, 12 pp.

Authors' address: *Xinghui Wang*, *Xiaoqin Li* (corresponding author), *Shuhe Hu*, Department of Statistics, Anhui University, Hefei 230039, P. R. China, e-mail: `wangxinghuial@163.com`, `lixiaoqin1983@163.com`, `hushuhe@263.net`.