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Decaying Regularly Varying Solutions of Third-order Differential Equations with a Singular Nonlinearity

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Abstract

This paper is concerned with asymptotic analysis of strongly decaying solutions of the third-order singular differential equation $x'''+q(t)x^{-\gamma}=0$, by means of regularly varying functions, where γ is a positive constant and q is a positive continuous function on $[a,\infty)$. It is shown that if q is a regularly varying function, then it is possible to establish necessary and sufficient conditions for the existence of slowly varying solutions and regularly varying solutions of (A) which decrease to 0 as $t\to\infty$ and to acquire precise information about the asymptotic behavior at infinity of these solutions. The main tool is the Schauder–Tychonoff fixed point theorem combined with the basic theory of regular variation.

Key words: third order nonlinear differential equation, singular nonlinearity, positive solution, decaying solution, asymptotic behavior, regularly varying functions

2010 Mathematics Subject Classification: 34C11, 26A12

1 Introduction

In this paper we are interested in the existence and accurate asymptotic behavior near infinity of positive solutions of the third order nonlinear differential equation

$$x''' + q(t)x^{-\gamma} = 0, \tag{A}$$

where

(a) γ is a positive constant;
(b) $q\colon [a,\infty) \to (0,\infty)$ is a continuous function, $a > 0$.

By a proper solution of equation (A) we mean three times continuously differentiable function $x\colon [t_0,\infty) \to (0,\infty)$, $t_0 \ge a$, which satisfies (A) on $[t_0,\infty)$. To examine these solutions it may be useful to make an "apriori" classification of proper solutions of (A) according to their asymptotic behavior at infinity.

Let x be a positive solution of (A) defined on some $[t_0,\infty)$. Then for $t \in [t_0,\infty)$ $x'''(t) < 0$ holds, which means that x'' decreases. This implies that $x''(t) > 0$ (else it would lead to the contradiction with the assumption of positivity of the solution x), and so there exists a finite limit $x''(\infty) := \lim_{t\to\infty} x''(t) \ge 0$ (using L'Hospital rule with unbounded denominator we know that also $\lim_{t\to\infty} \frac{x(t)}{t^2} = \frac{1}{2}\lim_{t\to\infty} x''(t) \ge 0$) and that x' increases. It may increase to zero, positive constant or to infinity (negative constant is excluded again because of the assumption of positivity of the solution), i.e. $\lim_{t\to\infty} \frac{x(t)}{t} = \lim_{t\to\infty} x'(t) := x'(\infty) \ge 0$. In case x' increases to positive constant or to infinity, $x'(t) > 0$ holds in the neighborhood of infinity and that means that x increases to infinity. The case x' increases to zero implies that $x'(t) < 0$ for $t \in [t_0,\infty)$, so there are two possibilities: x decreases to zero or to positive constant (let us put $x(\infty) := \lim_{t\to\infty} x(t)$).

Summarizing the above observations, we see that if x is a positive increasing solution of (A) then its asymptotic behavior falls into one of the three types

$$\lim_{t\to\infty} \frac{x(t)}{t^2} = \text{const} > 0, \tag{I}$$

$$\lim_{t\to\infty} \frac{x(t)}{t^2} = 0, \quad \lim_{t\to\infty} \frac{x(t)}{t} = \infty, \tag{II}$$

$$\lim_{t\to\infty} \frac{x(t)}{t} = \text{const} > 0, \tag{III}$$

and that if x is a positive decreasing solution of (A), then it satisfies either

$$\lim_{t\to\infty} x(t) = \text{const} > 0, \tag{IV}$$

or

$$\lim_{t\to\infty} x(t) = 0. \tag{V}$$

Let us introduce a relation $\sim$ defined by

$$f(t) \sim g(t) \text{ for } t \to \infty \iff \lim_{t\to\infty} \frac{f(t)}{g(t)} = 1,$$

The expression $f(t) \sim g(t)$ is read "f is asymptotically equivalent to g as $t \to \infty$" or "f behaves as g for $t \to \infty$".

Using terms of this symbol we can distinguish solutions $x(t) \sim ct^2$, $x(t) \sim ct$, $x(t) \sim c$ for some constant $c > 0$ (solutions of type (I), (III) and (IV) respectively). These solutions are called also *primitive* solutions of Eq. (A), while solutions of types (V) and (II) are *nonprimitive* ones.

In this paper, we focus our attention on solutions of type (V), which are often referred to as *strongly decaying solutions.* We try to get as precise information about their asymptotic behavior as possible. To that end some more assumptions on Eq. (A) are put and only a certain subset of strongly decaying solutions is considered.

The first step in our investigation is to find a corresponding integral equation for this solution type. Let x be a strongly decaying solution of (A). Integrating (A) three times on $[t,\infty)$, where $t \in [t_0,\infty)$, and using the fact that in this case $x''(\infty) = x'(\infty) = x(\infty) = 0$, we have for x the integral equation

$$x(t) = \int_t^\infty \int_s^\infty \int_r^\infty q(u)x(u)^{-\gamma} du dr ds, \quad t \geq t_0. \tag{1.1}$$

Conversly, a positive continuous function x satisfying (1.1) for $t \geq t_0$ gives a strongly decaying solution of (A) on $[t_0,\infty)$. The integral equation (1.1) can be approximated at infinity by the asymptotic integral relation

$$x(t) \sim \int_t^\infty \int_s^\infty \int_r^\infty q(u)x(u)^{-\gamma} du dr ds, \quad t \to \infty. \tag{AR}$$

To solve this asymptotical integral relation is a very difficult problem for the case where q is a general continuous function. However, the recent development of asymptotic analysis of differential equations by means of regular variation suggests investigating the problem in the framework of regularly varying functions (or Karamata functions). For the reader's benefit we recall here the definition and the most important properties of regularly varying functions.

Definition 1.1 [9] A measurable function $f\colon (a,\infty) \to (0,\infty)$ is called *regularly varying of index* ρ (with $\rho \in \mathbb{R}$) if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho,$$

for all $\lambda \in \mathbb{R}^+$ or equivalently if it can be expressed in the form

$$f(t) = c(t) \exp\left(\int_a^t \frac{\varrho(s)}{s} ds \right), \ t \geq a,$$

for some $a > 0$ and some measurable functions $c(t)$ and $\varrho(t)$ satisfying $\lim_{t\to\infty} c(t) = c_0 > 0$ and $\lim_{t\to\infty} \varrho(t) = \rho$.

Totality of regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. If in particular $\rho = 0$, we use SV instead of $\mathrm{RV}(0)$ and refer to members of SV as *slowly varying* functions. It is clear that $\mathrm{RV}(\rho)$-functions can be expressed as

$$f(t) = t^\rho L(t), \quad L \in \mathrm{SV}.$$

Some other basic properties used in the paper can be seen easily:

Proposition 1.1 [9] *If L, L_1 and L_2 are slowly varying, then also L^α for every real α is slowly varying and $L_1.L_2$ is slowly varying, too.*

Proposition 1.2 [9] *If some positive measurable function f behaves as a regularly varying function of index ρ, i.e. if $f(t) \sim t^\rho L(t)$, as $t \to \infty$, then f is a regularly varying function of index ρ, i.e. $f(t) = t^\rho L^*(t)$, where $L^* \in \mathrm{SV}$ and in general $L^* \neq L$ but $L^*(t) \sim L(t)$, as $t \to \infty$.*

Definition 1.2 Regularly varying function f will be called a *trivial regularly varying function* (*of index ρ*) if it is possible to write it in the form $f(t) = t^\rho L(t)$, where $L(t)$ is slowly varying fulfilling

$$\lim_{t\to\infty} L(t) = \text{const} > 0.$$

Otherwise f is called a *nontrivial regularly varying function* (*of index ρ*). In the case $\rho = 0$, terminology *trivial*, resp. *nontrivial slowly varying function* is used.

According to this definition a primitive solution x of (A) such that $x(t) \sim ct^j$, for some $c > 0$ and $j \in \{0, 1, 2\}$ is a trivial regularly varying solution of index j. As we will see later, nonprimitive solutions of (A) of type (V) can be trivial or nontrivial.

The most crucial tool used in our analysis is the following proposition known as Karamata's integration theorem:

Proposition 1.3 [1] *Let $L \in \mathrm{SV}$. Then for $t \to \infty$*

(i) if $\alpha > -1$,

$$\int_a^t s^\alpha L(s)ds \sim \frac{1}{\alpha+1} t^{\alpha+1} L(t),$$

(ii) if $\alpha < -1$,

$$\int_t^\infty s^\alpha L(s)ds \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t),$$

(iii) if $\alpha = -1$,

$$l(t) := \int_a^t \frac{L(s)}{s} ds \in \mathrm{SV} \quad \text{and} \quad \lim_{t\to\infty} \frac{L(t)}{l(t)} = 0$$

and if $\frac{L(t)}{t}$ is integrable on $[a, \infty)$, then

$$m(t) := \int_t^\infty \frac{L(s)}{s} ds \in \mathrm{SV} \quad \text{and} \quad \lim_{t\to\infty} \frac{L(t)}{m(t)} = 0.$$

For more about regular variation see Bingham et al [1] and for results up to 2000 of its applications to second order ordinary differential equations see Marić [9]. For the literature devoted to differential equations (of the second order) with singular nonlinearities more or less related to present work, we refer to Tanigawa and Kusano [8], [10] and Kamo and Usami [5]. The Emden-Fowler-type equation of the second and higher order with a positive exponent in context of the regular variation was studied e.g. in [3], [4], [6] and [7]. We partially extend these results to the singular equation (A). Similar type of results obtained by another approach of examination of Emden-Fowler like equations under various and quite general settings can be found in papers by Evtukhov et al, see e.g. [2], where unbounded solutions are studied.

2 Asymptotic analysis of the integral asymptotic relation

In this section we find necessary and sufficient conditions for the existence of regularly varying strongly decaying solutions of integral asymptotic relation

$$x(t) \sim \int_t^\infty \int_s^\infty \int_r^\infty q(u)x(u)^{-\gamma} du dr ds, \tag{AR}$$

where q is regularly varying. Precise information about asymptotic behavior near infinity of these solutions is found.

Theorem 2.1 *Let q be a regularly varying function of index σ. The integral asymptotic relation* (AR) *has slowly varying solutions if and only if $\sigma = -3$ and*

$$\int_a^\infty t^2 q(t)\, dt < \infty, \tag{2.1}$$

in which case any such solution x is nontrivial slowly varying and has the asymptotic behavior

$$x(t) \sim \left[\frac{1+\gamma}{2} \int_t^\infty s^2 q(s)\, ds\right]^{\frac{1}{1+\gamma}}, \quad t \to \infty. \tag{2.2}$$

Theorem 2.2 *Let q be a regularly varying function of index σ. The integral asymptotic relation* (AR) *has regularly varying solutions of index $\rho < 0$ if and only if $\sigma < -3$, in which case the index ρ is given by*

$$\rho = \frac{\sigma + 3}{1+\gamma} \tag{2.3}$$

and any such solution x has the asymptotic behavior

$$x(t) \sim \left[\frac{t^3 q(t)}{(-\rho)(1-\rho)(2-\rho)}\right]^{\frac{1}{1+\gamma}}, \quad t \to \infty. \tag{2.4}$$

Proof of Theorems 2.1, 2.2. Let us put $q(t) = t^\sigma l(t)$, $l \in \mathrm{SV}$.

(The "only if" part) Suppose that (AR) has a regularly varying solution $x \in \mathrm{RV}(\rho)$ on $[t_0, \infty)$. It is clear that $\rho \leq 0$ and that x satisfies (V). Let

$x(t) = t^{\rho}\xi(t)$, $\xi \in \mathrm{SV}$. Since the expression $q(t)x(t)^{-\gamma} = t^{\sigma-\rho\gamma}l(t)\xi(t)$ in (AR) has to be three times integrable on $[t_0, \infty)$, it implies that $\sigma - \rho\gamma \le -3$ holds.

(a) First we consider the case $\sigma - \rho\gamma = -3$. Then the right-hand side of (AR) becomes

$$\int_t^{\infty}\int_s^{\infty}\int_r^{\infty} u^{-3}l(u)\xi(u)^{-\gamma}dudrds. \tag{2.5}$$

Proposition 1.1 implies $l\xi^{-\gamma} \in \mathrm{SV}$ so we can use Proposition 1.3 (ii) twice and we have that (2.5) is asymptotically equivalent to

$$\frac{1}{2}\int_t^{\infty} s^{-1}l(s)\xi(s)^{-\gamma}ds. \tag{2.6}$$

From Proposition 1.3 (iii) it follows that (2.6) is from SV, so we have

$$x(t) \sim \frac{1}{2}\int_t^{\infty} s^{-1}l(s)\xi(s)^{-\gamma}ds \in \mathrm{SV}.$$

We see that x is asymptotically equivalent to slowly varying function, according to Proposition 1.2 x is a slowly varying function, i.e. $\rho = 0$ which implies that $\sigma = -3$ and $x(t) = \xi(t)$ on $[t_0, \infty)$. So we have

$$x(t) \sim \frac{1}{2}\int_t^{\infty} s^{-1}l(s)x(s)^{-\gamma}ds, \quad t \to \infty. \tag{2.7}$$

Let us denote the right-hand side of (2.7) by $y(t)$ ($\Rightarrow x(t) \sim y(t)$) and rewrite it into the following differential asymptotic relation:

$$y'(t) = -\frac{1}{2}t^{-1}l(t)x(t)^{-\gamma} \sim -\frac{1}{2}t^{-1}l(t)y(t)^{-\gamma},$$

that is

$$-y(t)^{\gamma}y'(t) \sim \frac{1}{2}t^{-1}l(t) = \frac{1}{2}t^2q(t). \tag{2.8}$$

The left-hand side of (2.8) is integrable on $[t, \infty)$ for every $t \ge t_0$ (since $y(\infty) := \lim_{t\to\infty} y(t) = \lim_{t\to\infty} x(t) = 0$), so is $t^2q(t)$ (and the condition (2.1) is fulfilled) and we have for $t \to \infty$:

$$-\frac{y(\infty)^{1+\gamma}}{1+\gamma} + \frac{y(t)^{1+\gamma}}{1+\gamma} \sim \frac{1}{2}\int_t^{\infty} s^2q(s)\,ds,$$

$$x(t) \sim y(t) \sim \left[\frac{1+\gamma}{2}\int_t^{\infty} s^2q(s)\,ds\right]^{\frac{1}{1+\gamma}},$$

what shows that (2.2) holds and that x is nontrivial slowly varying (since $x \in \mathrm{SV}$ and $\lim_{t\to\infty} x(t) = 0$).

(b) Next we consider the case $\sigma - \rho\gamma < -3$. Then we can use Proposition 1.3 (ii) three times to get

$$x(t) \sim \int_t^\infty \int_s^\infty \int_r^\infty u^{\sigma-\rho\gamma} l(u)\xi(u)^{-\gamma} du dr ds$$
$$\sim \frac{t^{\sigma-\rho\gamma+3} l(t)\xi(t)^{-\gamma}}{[-(\sigma-\rho\gamma+1)][-(\sigma-\rho\gamma+2)][-(\sigma-\rho\gamma+3)]}, \quad t \to \infty. \tag{2.9}$$

This together with Proposition 1.2 implies that $x \in \mathrm{RV}(\sigma - \rho\gamma + 3)$, i.e. $\rho = \sigma - \rho\gamma + 3 < 0$, from what follows that (2.3) and $\sigma < -3$ hold. Returning to the functions q and x and using the equality $\rho = \sigma - \rho\gamma + 3$ in denominator of (2.9), we get for $t \to \infty$

$$x(t) \sim \frac{t^3 q(t) x(t)^{-\gamma}}{(2-\rho)(1-\rho)(-\rho)}$$

and finally

$$x(t) \sim \left[\frac{t^3 q(t)}{(-\rho)(1-\rho)(2-\rho)}\right]^{\frac{1}{1+\gamma}}, \quad t \to \infty.$$

This completes the proof of the "only if" parts of Theorems 2.1 and 2.2.

(The "if" part) Let us define the function

$$X(t) = \begin{cases} \left[\frac{1+\gamma}{2}\int_t^\infty s^2 q(s)ds\right]^{\frac{1}{1+\gamma}} & \text{if } \sigma = -3 \text{ and (2.1) holds;} \\ \left[\frac{t^3 q(t)}{(-\rho)(1-\rho)(2-\rho)}\right]^{\frac{1}{1+\gamma}} & \text{if } \sigma < -3, \text{ where } \rho = \dfrac{\sigma+3}{1+\gamma} \end{cases} \tag{2.10}$$

and show that it satisfies asymptotic relation

$$X(t) \sim \int_t^\infty \int_s^\infty \int_r^\infty q(u) X(u)^{-\gamma} du dr ds, \quad t \to \infty. \tag{2.11}$$

If $\sigma = -3$ and (2.1) holds, then the right-hand side of (2.11) becomes

$$\int_t^\infty \int_s^\infty \int_r^\infty u^{-3} l(u) \left[\frac{1+\gamma}{2}\int_u^\infty v^2 q(v) dv\right]^{\frac{-\gamma}{1+\gamma}} du dr ds. \tag{2.12}$$

Since $l(t)\left[\dfrac{1+\gamma}{2}\int_t^\infty s^2 q(s)ds\right]^{\frac{-\gamma}{1+\gamma}} \in \mathrm{SV}$, we can use Proposition 1.3 (ii) twice to see that (2.12) is asymptotically equivalent to

$$\frac{1}{2}\int_t^\infty s^{-1} l(s) \left[\frac{1+\gamma}{2}\int_s^\infty r^{-1} l(r)\, dr\right]^{\frac{-\gamma}{1+\gamma}} ds = -\int_{X(t)}^0 dX(s) = X(t),$$

as $t \to \infty$. It is evident that the solution X is nontrivial slowly varying in this case.

If $\sigma < -3$, then we have

$$\int_t^\infty \int_s^\infty \int_r^\infty q(u)X(u)^{-\gamma} du dr ds = \int_t^\infty \int_s^\infty \int_r^\infty u^\sigma l(u) \left[\frac{t^3 q(t)}{(-\rho)(1-\rho)(2-\rho)}\right]^{\frac{-\gamma}{1+\gamma}} du dr ds$$

$$= \frac{\int_t^\infty \int_s^\infty \int_r^\infty u^{\sigma-\rho\gamma} l(u)^{\frac{1}{1+\gamma}} du dr ds}{[(-\rho)(1-\rho)(2-\rho)]^{\frac{-\gamma}{1+\gamma}}} = \frac{\int_t^\infty \int_s^\infty \int_r^\infty u^{\rho-3} l(u)^{\frac{1}{1+\gamma}} du dr ds}{[(-\rho)(1-\rho)(2-\rho)]^{\frac{-\gamma}{1+\gamma}}}, \quad (2.13)$$

where $\rho = (\sigma+3)/(1+\gamma)$. Using (ii) of Proposition 1.3 three times we see that (2.13) is asymptotically equivalent to

$$\frac{t^\rho l(t)^{\frac{1}{1+\gamma}}}{(-\rho)(1-\rho)(2-\rho)\,[(-\rho)(1-\rho)(2-\rho)]^{\frac{-\gamma}{1+\gamma}}} = \left[\frac{t^3 q(t)}{(-\rho)(1-\rho)(2-\rho)}\right]^{\frac{1}{1+\gamma}} = X(t),$$

as $t \to \infty$.

This proves the "if" parts of Theorems 2.1 and 2.2. □

3 Existence of strongly decaying solutions of the equation (A)

Now we will relax the assumption on the coefficient q. For this purpose let us introduce the following relation. Let f and g be two positive functions defined on some $[T, \infty)$. We use the notation

$$f(t) \asymp g(t), \quad t \to \infty$$

to denote that there exist positive constants m and M such that

$$mg(t) \le f(t) \le Mg(t), \quad \text{for } t \in [T, \infty).$$

Definition 3.1 Let f be a positive function defined on $[0, \infty)$. If there exists $g \in \mathrm{RV}(\rho)$ such that

$$f(t) \asymp g(t), \quad t \to \infty,$$

then we call f *nearly regularly varying of index* ρ. If $\rho = 0$, f is said to be *nearly slowly varying.*

Theorem 3.1 *Let $\gamma < 1$ and let q be nearly regularly varying of index σ, i.e. $q \asymp q_\sigma$ at infinity for some $q_\sigma \in \mathrm{RV}(\sigma)$. Suppose that $\sigma = -3$ and* (2.1) *holds. Then, Eq.* (A) *possesses a strongly decaying solution x which is nearly slowly varying and satisfies*

$$x(t) \asymp \left[\frac{1+\gamma}{2} \int_t^\infty s^2 q_\sigma(s)\, ds\right]^{\frac{1}{1+\gamma}}, \quad t \to \infty. \quad (3.1)$$

Theorem 3.2 *Let $\gamma < 1$ and let q be nearly regularly varying of index σ, i.e. $q \asymp q_\sigma$ at infinity, for some $q_\sigma \in \mathrm{RV}(\sigma)$. Suppose that $\sigma < -3$. Then, Eq.* (A) *possesses a strongly decaying solution x which is nearly regularly varying of index $\rho < 0$ and satisfies*

$$x(t) \asymp \left[\frac{t^3 q_\sigma(t)}{(-\rho)(1-\rho)(2-\rho)}\right]^{\frac{1}{1+\gamma}}, \quad t \to \infty, \tag{3.2}$$

where ρ is given by (2.3).

Proof of Theorems 3.1 and 3.2. Suppose that $\sigma < -3$ or that $\sigma = -3$ and (2.1) holds. Since q is nearly regularly varying of index σ, there exist positive constants k and K ($k \leq K$) and $q_\sigma \in \mathrm{RV}(\sigma)$ such that

$$k q_\sigma(t) \leq q(t) \leq K q_\sigma(t), \quad t \in [T, \infty). \tag{3.3}$$

From the first inequality in (3.3) it is clear that $\int_a^\infty t^2 q_\sigma(t) dt < \infty$. Let us define

$$X(t) = \begin{cases} \left[\frac{1+\gamma}{2} \int_t^\infty s^2 q_\sigma(s)\, ds\right]^{\frac{1}{1+\gamma}} & \text{if } \sigma = -3 \text{ and (2.1) holds;} \\ \left[\frac{t^3 q_\sigma(t)}{(-\rho)(1-\rho)(2-\rho)}\right]^{\frac{1}{1+\gamma}} & \text{if } \sigma < -3, \text{ where } \rho = \frac{\sigma+3}{1+\gamma}. \end{cases}$$

We know from the preceding section that $X(t)$ satisfies the asymptotic relation

$$X(t) \sim \int_t^\infty \int_s^\infty \int_r^\infty q_\sigma(u) X(u)^{-\gamma} du dr ds, \quad t \to \infty.$$

This implies that there exists $T > a$ such that

$$\frac{X(t)}{2} \leq \int_t^\infty \int_s^\infty \int_r^\infty q_\sigma(u) X(u)^{-\gamma} du dr ds \leq 2X(t), \quad t \geq T. \tag{3.4}$$

Let us choose positive constants m and M ($m < M$) so that

$$m M^\gamma \leq \frac{k}{2}, \quad M m^\gamma \geq 2K, \tag{3.5}$$

hold. Notice that $\dfrac{k}{2} < 2K$, so $mM^\gamma < Mm^\gamma$ have to be fulfilled, or equivalently

$$m^{1-\gamma} < M^{1-\gamma}.$$

This is possible only if $\gamma < 1$, what is a reason for this additional assumption on Eq. (A).

Next let us define the set

$$\mathcal{X} = \{x \in C[T, \infty) : mX(t) \leq x(t) \leq MX(t),\ t \geq T\} \tag{3.6}$$

and the operator

$$\mathcal{F}x(t) = \int_t^\infty \int_s^\infty \int_r^\infty q(u)x(u)^{-\gamma} dudrds, \quad t \geq T.$$

It is clear that $\mathcal{X}$ is a closed convex subset of the locally convex space $C[T,\infty)$. Let us show that $\mathcal{F}$ fulfils conditions of the Schauder–Tychonoff fixed point theorem

(i) $\mathcal{F}(\mathcal{X}) \subset \mathcal{X}$.

Using conditions (3.3)–(3.6) and bearing in mind that $x^{-\gamma}$ is a decreasing function of x, we see that for an arbitrary $x \in \mathcal{X}$ and for all $t \in [T,\infty)$

$$\mathcal{F}x(t) \leq Km^{-\gamma} \int_t^\infty \int_s^\infty \int_r^\infty q_\sigma(u)X(u)^{-\gamma} dudrds \leq Km^{-\gamma} 2X(t) \leq MX(t)$$

and

$$\mathcal{F}x(t) \geq kM^{-\gamma} \int_t^\infty \int_s^\infty \int_r^\infty q_\sigma(u)X(u)^{-\gamma} dudrds \geq kM^{-\gamma} \frac{X(t)}{2} \geq mX(t)$$

holds. So we have $mX(t) \leq \mathcal{F}x(t) \leq MX(t)$ for $t \geq T$, which means that $\mathcal{F}x \in \mathcal{X}$. Since $x \in \mathcal{X}$ is arbitrary it follows that $\mathcal{F}(\mathcal{X}) \subset \mathcal{X}$.

(ii) $\mathcal{F}(\mathcal{X})$ is relatively compact.

From (i) we see that $mX(t) \leq \mathcal{F}x(t) \leq MX(t)$, $t \geq T$, for all $x \in \mathcal{X}$. Since neither upper bound nor lower bound for $\mathcal{F}x(t)$ depends on $x \in \mathcal{X}$, $\mathcal{F}(\mathcal{X})$ is uniformly bounded on $[T,\infty)$.

To show equicontinuity we will show the uniform boundedness of the operator $(\mathcal{F}x)'$:

$$(\mathcal{F}x)'(t) = -\int_t^\infty \int_s^\infty q(r)x(r)^{-\gamma} drds \leq 0$$

and

$$(\mathcal{F}x)'(t) = -\int_t^\infty \int_s^\infty q(r)x(r)^{-\gamma} drds \geq -m^{-\gamma} \int_t^\infty \int_s^\infty q(r)X(r)^{-\gamma} drds$$

for all $x \in \mathcal{X}$ and all $t \in [T,\infty)$. Since $\int_t^\infty \int_s^\infty q(r)X(r)^{-\gamma} drds$ is integrable on $[T,\infty)$, it is bounded there, and so also $(\mathcal{F}x)'$ is uniformly bounded, consequently $\mathcal{F}(\mathcal{X})$ is equicontinuous and we can use Arzela–Ascoli Theorem to see that $\mathcal{F}(\mathcal{X})$ is relative compact.

(iii) $\mathcal{F}$ is continuous. Let $\{x_n\}_{n=1}^\infty \subset \mathcal{X}$ be a sequence converging to $x \in \mathcal{X}$ in a topology on $C[T,\infty)$, i.e. $x_n(t) \to x(t)$ uniformly on every compact subinterval of $[T,\infty)$. To show that $\mathcal{F}x_n \to \mathcal{F}x$ in topology on $C[T,\infty)$ we observe that for $t \geq T$

$$|\mathcal{F}x_n(t) - \mathcal{F}x(t)| \leq \int_t^\infty \int_s^\infty \int_r^\infty q(u)\left|x_n(u)^{-\gamma} - x(u)^{-\gamma}\right| dudrds$$

$$\leq K\frac{1}{2}\int_t^\infty (s-t)^2 q_\sigma(s)\left|x_n(s)^{-\gamma} - x(s)^{-\gamma}\right| ds$$

$$\leq K\frac{1}{2}\int_t^\infty s^2 q_\sigma(s)\left|x_n(s)^{-\gamma} - x(s)^{-\gamma}\right| ds$$

$$\leq K\frac{1}{2}\int_T^\infty t^2 q_\sigma(t)\left|x_n(t)^{-\gamma} - x(t)^{-\gamma}\right| dt.$$

Put $f_n(t) := t^2 q_\sigma(t)\,|x_n(t)^{-\gamma} - x(t)^{-\gamma}|$. We see that
(1) $f_n(t) \to 0$ as $n \to \infty$ for each fixed $t \in [T,\infty)$ and
(2) $|f_n(t)| \leq 2m^{-\gamma}t^2 q_\sigma(t)X(t)^{-\gamma} =: G(t)$ (the dominant function).
Since

$$4m^{-\gamma}\int_t^\infty \int_s^\infty q_\sigma(r)X(r)^{-\gamma}drds$$

is integrable on $[T,\infty)$, such is also G (they are asymptotically equivalent). Thanks to (1) and (2) we can use the Lebesque dominated convergence theorem to see that also

$$K\frac{1}{2}\int_T^\infty t^2 q_\sigma(t)\left|x_n(t)^{-\gamma} - x(t)^{-\gamma}\right| dt \to 0,$$

for $n \to \infty$. Since the convergence does not depend on $t \in [T,\infty)$, it is uniform on $[T,\infty)$ and so also on every compact subinterval of $[T,\infty)$, i.e. $\mathcal{F}x_n \to \mathcal{F}x$ in topology on $C[T,\infty)$, from what follows that $\mathcal{F}$ is continuous.

Using results of (i), (ii) and (iii), by the Schauder–Tychonoff fixed point theorem we know that there is $x \in \mathcal{X}$ such that $x(t) = \mathcal{F}x(t)$, i.e.

$$x(t) = \int_t^\infty \int_s^\infty \int_r^\infty q(u)x(u)^{-\gamma}dudrds, \quad t \geq T,$$

which after differentiating three times gives that x is a solution of Eq. (A) and simultaneously $mX(t) \leq x(t) \leq MX(t),\ t \geq T$, so x is nearly slowly varying, resp. nearly regularly varying of index $\rho = (\sigma+3)/(1+\gamma)$, when $\sigma = -3$, resp. $\sigma < -3$.

4 Regularly varying solutions

Returning to the property of regular variation of q, we show that there are also strongly decaying solutions of Eq. (A) which are regularly varying. The following lemma which is a generalization of the L'Hospital rule is useful for this purpose.

Lemma 4.1 [11] *Let f, $g \in C^1[T,\infty)$ and suppose that*

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = \infty \quad \text{and} \quad g'(t) > 0, \quad t \in [T,\infty),$$

or

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0 \quad \text{and} \quad g'(t) < 0, \quad t \in [T,\infty).$$

Then

$$\liminf_{t\to\infty} \frac{f'(t)}{g'(t)} \le \liminf_{t\to\infty} \frac{f(t)}{g(t)}; \quad \limsup_{t\to\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to\infty} \frac{f'(t)}{g'(t)}.$$

Theorem 4.1 *Let $\gamma < 1$ and let q be a regularly varying function of index σ. Eq.* (A) *possesses strongly decaying slowly varying solutions if and only if $\sigma = -3$ and* (2.1) *holds, in which case any such solution x has the asymptotic behavior*

$$x(t) \sim \left[\frac{1+\gamma}{2}\int_t^\infty s^2 q(s)ds\right]^{\frac{1}{1+\gamma}}, \quad t \to \infty. \tag{4.1}$$

Theorem 4.2 *Let $\gamma < 1$ and let q be a regularly varying function of index σ. Eq.* (A) *possesses regularly varying solutions of index $\rho < 0$ if and only if $\sigma < -3$, in which case index ρ is given by $\rho = \dfrac{\sigma+3}{1+\gamma}$ and any such solution x has the asymptotic behavior*

$$x(t) \sim \left[\frac{t^3 q(t)}{(-\rho)(1-\rho)(2-\rho)}\right]^{\frac{1}{1+\gamma}}, \quad t \to \infty. \tag{4.2}$$

Proof of Theorems 4.1 and 4.2. (The "only if" part) All regularly varying solutions of index $\rho < 0$ and strongly decaying slowly varing solutions of Eq. (A) satisfy also the integral asymptotic relation (AR), and so the "only if" parts of these theorems are implied by the "only if" parts of Theorems 2.1 and 2.2.

(The "if" part) Let σ and q fulfil required conditions. From Theorem 3.1 and 3.2 we know that there exists a nearly regularly varying solution of Eq. (A) obtained as the solution of the integral equation

$$x(t) = \int_t^\infty \int_s^\infty \int_r^\infty q(u)x(u)^{-\gamma} du dr ds.$$

Let us define again X as in (2.10) and let us show that $x(t) \sim X(t)$ (from what (4.1) and (4.2) will follow).

Define the function J by

$$J(t) = \int_t^{\infty}\int_s^{\infty}\int_r^{\infty} q(u)X(u)^{-\gamma}dudrds.$$

From the proof of Theorems 2.1 and 2.2 we know that

$$J(t) \sim X(t), \quad t \to \infty, \tag{4.3}$$

and from Theorems 3.1 and 3.2 that

$$x(t) \asymp X(t), \quad t \to \infty. \tag{4.4}$$

Put

$$l := \liminf_{t\to\infty} \frac{x(t)}{J(t)}, \quad L := \limsup_{t\to\infty} \frac{x(t)}{J(t)} \tag{4.5}$$

Combining (4.3) and (4.4), we see that $x(t) \asymp J(t)$, $t \to \infty$, so constants l and L defined by (4.5) satisfy

$$0 < l \le L < \infty. \tag{4.6}$$

Since $\lim_{t\to\infty} J^{(i)}(t) = \lim_{t\to\infty} x^{(i)}(t) = 0$, $i = 0, 1, 2$, we can apply Lemma 4.1 three times to l to get

$$\begin{aligned} l &= \liminf_{t\to\infty} \frac{x(t)}{J(t)} \ge \liminf_{t\to\infty} \frac{x'(t)}{J'(t)} = \liminf_{t\to\infty} \frac{\int_t^\infty \int_s^\infty q(r)x(r)^{-\gamma}drds}{\int_t^\infty \int_s^\infty q(r)X(r)^{-\gamma}drds} \\ &\ge \liminf_{t\to\infty} \frac{\int_t^\infty q(s)x(s)^{-\gamma}ds}{\int_t^\infty q(s)X(s)^{-\gamma}ds} \ge \liminf_{t\to\infty} \frac{q(t)x(t)^{-\gamma}}{q(t)X(t)^{-\gamma}} \\ &= \liminf_{t\to\infty} \left[\frac{x(t)}{X(t)}\right]^{-\gamma} = \left[\liminf_{t\to\infty} \frac{x(t)}{J(t)}\right]^{-\gamma} = l^{-\gamma}, \end{aligned}$$

where we used (4.3) in the final step. Thus $l \ge l^{-\gamma}$ holds. Together with (4.6) and the fact that $-\gamma < 0$, we have $l \ge 1$.

Analogously we get that $L \le L^{-\gamma}$ and that $L \le 1$ holds.

Simultaneously $L \ge l$ what implies that $L = l = 1$ has to hold, i.e. there exists $\lim_{t\to\infty} \frac{x(t)}{J(t)} = 1$. This implies $x(t) \sim J(t) \sim X(t)$ as $t \to \infty$. □

Example 4.1 Let $0 < \gamma < 1$ and consider the equation

$$x''' + q_1(t)x^{-\gamma} = 0, \quad q_1(t) \sim \frac{2}{t^3}(\log t)^{-\gamma-2}, \quad t \to \infty. \tag{A1}$$

The function $q_1 \in \mathrm{RV}(-3)$ and satisfies the condition (2.1) (due to a need of the positivity of the function q on $[a, \infty)$, we consider $a > 1$)

$$\int_a^\infty t^2 q_1(t)dt \sim \left[\frac{2(\log t)^{-\gamma-1}}{-\gamma-1}\right]_a^\infty = \frac{2(\log a)^{-\gamma-1}}{\gamma+1}.$$

Consequently by Theorem 4.1 Eq. (A1) possesses strongly decaying solutions which are nontrivial slowly varying and all such solutions have the unique asymptotic behavior

$$x(t) \sim \left[\frac{1+\gamma}{2}\int_t^\infty s^2 q_1(s)\,ds\right]^{\frac{1}{1+\gamma}} \sim \frac{1}{\log t}, \quad t \to \infty.$$

If in particular

$$q_1(t) = \frac{2}{t^3(\log t)^{\gamma+2}}\left(1 + \frac{3}{\log t} + \frac{3}{(\log t)^2}\right),$$

then Eq. (A1) has an exact nontrivial slowly varying solution $x(t) = 1/\log t$ for any values of $\gamma > 0$.

Example 4.2 Let $0 < \gamma < 1$ and consider the equation

$$x''' + q_2(t)x^{-\gamma} = 0, \quad q_2(t) = \frac{15}{8}t^{\frac{-7-\gamma}{2}} \exp((1+\gamma)(\log t)^{\frac{1}{3}} \cos(\log t)^{\frac{1}{3}}). \quad \text{(A2)}$$

Notice that $q_2 \in \mathrm{RV}(\sigma)$, with $\sigma = \frac{-7-\gamma}{2} < -3$. By Theorem 4.2 Eq. (A2) possesses strongly decaying solutions which are regularly varying of index $\rho = -\frac{1}{2}$ and behave like

$$x(t) \sim t^{-\frac{1}{2}} \exp((\log t)^{\frac{1}{3}} \cos(\log t)^{\frac{1}{3}}), \quad t \to \infty.$$

Remark 4.1 As we can see recent results for some ordinary differential equations obtained in context of Karamata's functions can be extended also to strongly decaying solutions of the singular differential equation (A). The increasing solutions (of type (II)) of Eq. (A) will be the subject of our forthcoming paper. In view of Example 4.1 and the fact that complete analysis of (AR) in terms of strongly decreasing regularly varying solutions can be made for any values $\gamma > 0$, it can be expected that even for $\gamma > 1$ Eq. (A) with regularly varying q may possess strongly decaying solutions which are regularly varying with accurate asymptotic behavior at infinity.

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