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New hyper-Kähler structures on tangent bundles

Xuerong Qi, Linfen Cao, Xingxiao Li

Abstract. Let (M, g, J) be an almost Hermitian manifold, then the tangent bundle TM carries a class of naturally defined almost hyper-Hermitian structures (G, J_1, J_2, J_3). In this paper we give conditions under which these almost hyper-Hermitian structures (G, J_1, J_2, J_3) are locally conformal hyper-Kähler. As an application, a family of new hyper-Kähler structures is obtained on the tangent bundle of a complex space form. Furthermore, by restricting these almost hyper-Hermitian structures on the unit tangent sphere bundle T_1M, we obtain a class of almost contact metric 3-structures. By virtue of these almost contact metric 3-structures, we find a family of Sasakian 3-structures on the unit tangent sphere bundle of a complex space form of positive holomorphic sectional curvature.

1 Introduction

A Riemannian metric g on a smooth manifold M gives rise to several natural Riemannian metrics and almost complex structures on the tangent bundle TM and the cotangent bundle T^*M of M. Maybe the best known examples are the Sasaki metric g_s and the canonical almost complex structure J_s (see [11], [18]). The Sasaki metric g_s and the canonical almost complex structure J_s determine an almost Hermitian structure on TM. Although the natural almost Hermitian structure (g_s, J_s) is almost Kähler, it is very rigid in the following sense. For example, the Sasaki metric g_s has never constant scalar curvature unless the metric g on the base manifold M is flat (see [7], [10]). On the other hand, the canonical almost complex structure J_s is integrable if and only if the base manifold (M, g) is flat (see [6], [19]). The rigidity of the natural almost Hermitian structure (g_s, J_s) has incited many authors to tackle the problem of the construction and the study of other almost Hermitian structures on TM or T^*M ([1], [9], [12], [14], [17], [22]). Especially, Oproiu and Papaghiuc [15] has constructed a class of Kähler-Einstein structures on the cotangent bundle of a real space form. Recently, the authors [8]

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have proved that the tangent bundle of any Riemannian manifold admits a class of locally conformal almost Kähler structures. By virtue of these structures, they have also shown that there exists a class of Sasakian structures on the unit tangent sphere bundle of a real space form of positive sectional curvature.

For an almost Hermitian manifold (M, g, J), almost hyper-Hermitian structures can be found on TM and T^*M ([3], [4], [20]). In particular, Calabi [4] constructed a hyper-Kähler structure on the cotangent bundle of a complex projective space $\mathbb{C}P^n$. Tahara, Vanhecke and Watanabe [20] gave a class of almost hyper-Hermitian structures (G, J_1, J_2, J_3) on TM, which is determined by some parameters. Furthermore, suitably choosing these parameters, they obtained a family of hyper-Kähler structures on the tangent bundle of a complex space form of positive holomorphic sectional curvature. Oproiu [13] studied a family of almost hyper-Hermitian structures on the tangent bundle of a Kähler manifold, and obtained the necessary and sufficient conditions for these almost hyper-Hermitian structures to be hyper-Kähler structures.

In this paper, we study the geometry of these almost hyper-Hermitian structures (G, J_1, J_2, J_3) on the tangent bundle TM of an almost Hermitian manifold (M, g, J). The arrangement of this paper is as follows: section 2 are some necessary preliminaries and known results. In section 3, we give conditions under which these almost hyper-Hermitian structures (G, J_1, J_2, J_3) are locally conformal hyper-Kähler. As an application, a class of new hyper-Kähler structures is obtained on the tangent bundle of a complex space form. Our result is more general than one of [20]. It shows that there also exists a class of hyper-Kähler structures on the tangent bundle of a complex space form of non-positive holomorphic sectional curvature. In the end of this section, we present some concrete examples of hyper-Kähler structures on the tangent bundle of a complex space form. In section 4, using these almost hyper-Hermitian structures (G, J_1, J_2, J_3), we obtain a class of almost contact metric 3-structures $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, \widehat{G})_{\alpha=1,2,3}$ on the unit tangent sphere bundle T_1M. By studying some properties of these almost contact metric 3-structures $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, \widehat{G})_{\alpha=1,2,3}$, we find a family of Sasakian 3-structures on the unit tangent sphere bundle of a complex space form of positive holomorphic sectional curvature.

To simplify matters, we first make the following conventions on the ranges of indices used frequently in this paper:

$$1 \le i, j, k, \ldots \le n, \quad 1 \le \alpha, \beta, \gamma, \ldots \le 3\,.$$

2 Preliminaries

Let (M, g) be a smooth n-dimensional Riemannian manifold and denote its tangent bundle by $\pi\colon TM \to M$. The Levi-Civita connection ∇ of g defines a direct sum decomposition

$$TTM = HTM \oplus VTM \tag{1}$$

of the tangent bundle TTM into the vertical distribution $VTM = \ker \pi_*$ and the horizontal distribution HTM. Locally, if $(U; x^1, \ldots, x^n)$ is a local coordinate system on M, then $(\pi^{-1}(U); x^1, \ldots, x^n, y^1, \ldots, y^n)$ is a local coordinate system on TM. The metric g can be locally expressed as

$$g = g_{ij}(x)dx^i dx^j\,, \quad x \in U\,.$$

Denote by Γ^k_{ij} the Christoffel symbols of g and define

$$\Gamma^k_{i0}(x,y) = \Gamma^k_{ij}(x)y^j, \quad \text{for } y = y^i \frac{\partial}{\partial x^i}(x),\ x \in U,$$

then

$$\left\{\delta_i := \frac{\partial}{\partial x^i} - \Gamma^k_{i0}\frac{\partial}{\partial y^k}\right\}, \quad \left\{\partial_i := \frac{\partial}{\partial y^i}\right\} \tag{2}$$

define respectively a local frame field for HTM and a local frame field for VTM over $\pi^{-1}(U)$. Therefore $\{\delta_1,\ldots,\delta_n,\partial_1,\ldots,\partial_n\}$ defines a local tangent frame field on TM, adapted to the direct sum decomposition (1). Put

$$\nabla y^i = dy^i + \Gamma^i_{0k}dx^k,$$

then $\{dx^1,\ldots,dx^n,\nabla y^1,\ldots,\nabla y^n\}$ is a local cotangent frame field on TM dual to the local frame field $\{\delta_1,\ldots,\delta_n,\partial_1,\ldots,\partial_n\}$.

If we denote respectively by X^H and X^V the horizontal and the vertical lift to TM of a tangent vector field X on M, then

$$\delta_i = \left(\frac{\partial}{\partial x^i}\right)^H, \quad \partial_i = \left(\frac{\partial}{\partial x^i}\right)^V. \tag{3}$$

For any $x \in U$ and $y = y^i\frac{\partial}{\partial x^i}(x) \in \pi^{-1}(x)$, or in other words, for $(x,y) \in \pi^{-1}(U)$, put

$$t = \frac{1}{2}g_{\pi(y)}(y,y) = \frac{1}{2}g_{ij}(x)y^i y^j,$$

then it easily follows that

$$\delta_i t = 0\,, \quad \partial_i t = g_{ji}y^j := g_{0i}\,. \tag{4}$$

Furthermore, we define

$$R^l_{0ij}(x,y) = y^k R^l_{kij}(x)\,, \quad \text{for } (x,y) \in \pi^{-1}(U)\,,$$

then the Lie brackets of the vector fields ∂_i, δ_i are given by

$$[\partial_i,\partial_j] = 0\,, \qquad [\delta_i,\partial_j] = \Gamma^l_{ij}\partial_l\,, \qquad [\delta_i,\delta_j] = -R^l_{0ij}\partial_l\,. \tag{5}$$

The Sasaki metric g_s is uniquely determined by the following equations:

$$g_s(X^H,Y^H) = g_s(X^V,Y^V) = g(X,Y)\circ\pi\,,$$
$$g_s(X^H,Y^V) = 0\,, \quad \forall\, X,Y \in \mathscr{X}(M)\,,$$

where $\mathscr{X}(M)$ denotes the Lie algebra of smooth tangent vector fields on M. The canonical almost complex structure J_s on TM is given by

$$J_s X^H = X^V, \quad J_s X^V = -X^H, \quad \forall X \in \mathscr{X}(M)\,.$$

It is known that (TM, g_s, J_s) is an almost Kähler manifold. Moreover, the integrability of the almost complex structure J_s implies that (M, g) is locally Euclidean (see [6], [19]).

The Cheeger-Gromoll metric g_{CG} [5] is uniquely determined by

$$\begin{cases} g_{CG}(X^H, Y^H) = g(X,Y) \circ \pi\,, \\ g_{CG}(X^V, Y^V) = \dfrac{1}{1+2t}\left[g(X,Y) + g(X,y)g(Y,y)\right], \\ g_{CG}(X^H, Y^V) = 0\,, \end{cases}$$

where $X, Y \in \mathscr{X}(M)$ and $y \in TM$. Accordingly, one can define an almost complex structure J_{CG} on TM by the following equations:

$$\begin{aligned} J_{CG}X^H &= \tau X^V - \frac{1}{1+\tau} g(X,y)y^V, \\ J_{CG}X^V &= -\frac{1}{\tau}X^H - \frac{1}{\tau(1+\tau)} g(X,y)y^H, \end{aligned}$$

where $\tau = \sqrt{1+2t}$. Then (TM, g_{CG}, J_{CG}) is an almost Hermitian manifold (see [9]).

As a generalization of the above two metrics, one can define a class of metrics G on the tangent bundle TM of an almost Hermitian manifold (M, g, J) by the following equations (see [20]):

$$\begin{cases} G(X^H, Y^H) = c_1 g(X,Y) + d_1 g(X,y)g(Y,y) + f_1 g(X,Jy)g(Y,Jy)\,, \\ G(X^V, Y^V) = c_2 g(X,Y) + d_2 g(X,y)g(Y,y) + f_2 g(X,Jy)g(Y,Jy)\,, \\ G(X^H, Y^V) = 0\,, \end{cases}$$

where $c_1, c_2, d_1, d_2, f_1, f_2$ are smooth functions of $t \in [0, \infty)$, and satisfy the following conditions:

$$\begin{gathered} c_1 > 0\,, \quad c_2 > 0\,, \quad c_1 + 2td_1 > 0\,, \\ c_2 + 2td_2 > 0\,, \quad c_1 + 2tf_1 > 0, \quad c_2 + 2tf_2 > 0\,, \qquad \forall t\,. \end{gathered}$$

Accordingly, one can also define three kinds of almost complex structures J_1, J_2, J_3 on TM by the following equations (see [20]):

$$\begin{aligned} &\begin{cases} J_1 X^H = a_1 X^V + b_1 g(X,y)y^V + e_1 g(X,Jy)(Jy)^V, \\ J_1 X^V = -a_2 X^H - b_2 g(X,y)y^H - e_2 g(X,Jy)(Jy)^H, \end{cases} \\ &\begin{cases} J_2 X^H = a_1 (JX)^V + b_1 g(JX,y)y^V + e_1 g(X,y)(Jy)^V, \\ J_2 X^V = a_2 (JX)^H + e_2 g(JX,y)y^H + b_2 g(X,y)(Jy)^H, \end{cases} \\ &\begin{cases} J_3 X^H = -(JX)^H, \\ J_3 X^V = (JX)^V + p g(JX,y)y^V + q g(X,y)(Jy)^V, \end{cases} \end{aligned}$$

where $p = a_2 b_1 + a_1 e_2 + 2t b_1 e_2$, $q = a_2 e_1 + a_1 b_2 + 2t b_2 e_1$, a_1, a_2, b_1, b_2, e_1, e_2 are smooth functions of t satisfying the following conditions:

$$a_1 a_2 = 1\,, \quad (a_1 + 2tb_1)(a_2 + 2tb_2) = 1\,, \quad (a_1 + 2te_1)(a_2 + 2te_2) = 1\,, \tag{6}$$

or

$$a_2 = \frac{1}{a_1}\,, \quad b_2 = -\frac{b_1}{a_1(a_1 + 2tb_1)}\,, \quad e_2 = -\frac{e_1}{a_1(a_1 + 2te_1)}\,. \tag{7}$$

A direct computation shows that $J_3 = J_1 \circ J_2 = -J_2 \circ J_1$, so (J_1, J_2, J_3) is an almost hyper-complex structure on TM. Moreover, (7) shows that the almost hyper--complex structure (J_1, J_2, J_3) depends on three essential parameters a_1, b_1, e_1.

Remark 1. From (6) we know that the coefficients a_1, a_2, $a_1 + 2tb_1$, $a_2 + 2tb_2$, $a_1 + 2te_1$, $a_2 + 2te_2$ cannot vanish and have the same sign. In this paper, we assume that

$$\begin{gathered} a_1 > 0\,, \quad a_2 > 0\,, \quad a_1 + 2tb_1 > 0\,, \\ a_2 + 2tb_2 > 0\,, \quad a_1 + 2te_1 > 0\,, \quad a_2 + 2te_2 > 0\,, \qquad \forall t\,. \end{gathered}$$

For the metric G and almost hyper-complex structure (J_1, J_2, J_3), we have the following results. These results follow from corresponding statements in [20].

Proposition 1. *The almost complex structures J_1, J_2, J_3 are compatible with the metric G if and only if*

$$\begin{aligned} &\frac{c_1}{a_1} = \frac{c_2}{a_2} = \lambda\,, && \frac{c_1 + 2td_1}{a_1 + 2tb_1} = \frac{c_2 + 2td_2}{a_2 + 2tb_2} = \lambda + 2t\mu\,, \\ &d_1 = f_1\,, && \frac{c_1 + 2tf_1}{a_1 + 2te_1} = \frac{c_2 + 2tf_2}{a_2 + 2te_2} = \lambda + 2t\nu\,, \end{aligned}$$

where the proportionality coefficients $\lambda > 0$, $\lambda + 2t\mu > 0$ and $\lambda + 2t\nu > 0$ for all t.

Remark 2. Proposition 1 shows that the almost hyper-Hermitian structure (G, J_1, J_2, J_3) depends on five essential parameters a_1, b_1, e_1, λ, μ or a_1, b_1, e_1, λ, ν. Other parameters are determined by

$$a_2 = \frac{1}{a_1}\,, \quad b_2 = -\frac{b_1}{a_1(a_1 + 2tb_1)}\,, \quad e_2 = -\frac{e_1}{a_1(a_1 + 2te_1)}\,, \tag{8}$$

$$c_1 = \lambda a_1\,, \quad d_1 = f_1 = \lambda b_1 + \mu(a_1 + 2tb_1) = \lambda e_1 + \nu(a_1 + 2te_1)\,, \tag{9}$$

$$c_2 = \lambda a_2\,, \quad d_2 = \lambda b_2 + \mu(a_2 + 2tb_2)\,, \quad f_2 = \lambda e_2 + \nu(a_2 + 2te_2)\,. \tag{10}$$

Proposition 2. *Let (M, g, J) be a complex space form of holomorphic sectional curvature $4c$. If the functions b_1, e_1 are given by*

$$b_1 = \frac{a_1 a_1' - c}{a_1 - 2ta_1'}\,, \quad e_1 = \frac{c}{a_1}\,, \tag{11}$$

then the almost hyper-complex structures (J_1, J_2, J_3) is a hyper-complex structure on TM.

Remark 3. If we choose

$$a_1 = a_2 = c_1 = c_2 = 1\,, \quad b_1 = b_2 = e_1 = e_2 = d_1 = d_2 = f_1 = f_2 = 0\,,$$

then (G, J_1, J_2, J_3) is the canonical almost hyper-Hermitian structure on the tangent bundle TM. In this case, $G = g_s$, $J_1 = J_s$. Therefore, the canonical almost hyper-Hermitian structure (G, J_1, J_2, J_3) is a hyper-Hermitian structure if and only if the base manifold (M, g) if flat.

3 The tangent bundle TM

In the sequel (M, g, J) will be a connected almost Hermitian manifold. We study the geometry of the almost hyper-Hermitian manifold $(TM, G, J_\alpha)_{\alpha=1,2,3}$ and find conditions under which the considered almost hyper-Hermitian structure is locally conformal hyper-Kähler or hyper-Kähler. Now we first introduce the definitions of locally conformal Kähler manifold and locally conformal hyper-Kähler manifold (see also [16], [21]).

Definition 1. The almost Hermitian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ is called a locally conformal almost Kähler manifold if there exists a closed 1-form $\widetilde{\omega}$, called the *Lee form*, satisfying $d\widetilde{\Omega} = \widetilde{\omega} \wedge \widetilde{\Omega}$, where $\widetilde{\Omega}$ is the fundamental 2-form. Moreover, the locally conformal almost Kähler manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J})$ is called a locally conformal Kähler manifold if $\widetilde{J}$ is integrable.

Definition 2. The hyper-Hermitian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{J}_\alpha)_{\alpha=1,2,3}$ is called a locally conformal hyper-Kähler manifold if there exists a closed 1-form $\widetilde{\omega}$, such that, for each α, $d\widetilde{\Omega}_\alpha = \widetilde{\omega} \wedge \widetilde{\Omega}_\alpha$, where $\widetilde{\Omega}_\alpha$ is the fundamental 2-form of $(\widetilde{g}, \widetilde{J}_\alpha)$.

Denote by Ω_α the fundamental 2-form of (G, J_α), given by $\Omega_\alpha(\cdot,\cdot) = G(\cdot, J_\alpha\cdot)$, then under the adapted frame field $\{\delta_1, \dots, \delta_n, \partial_1, \dots, \partial_n\}$ defined by (2) or (3), we have

$$\Omega_1 = (\lambda g_{ij} + \mu g_{i0} g_{j0} + \nu g_{i\bar{0}} g_{j\bar{0}}) \nabla y^i \wedge dx^j, \tag{12}$$

$$\Omega_2 = (\lambda g_{i\bar{j}} + \mu g_{i0} g_{\bar{j}0} + \nu g_{i\bar{0}} g_{j0}) \nabla y^i \wedge dx^j, \tag{13}$$

$$\begin{aligned} \Omega_3 = \{ & c_2 g_{i\bar{j}} + [\lambda e_2 + \mu(a_2 + 2te_2)] g_{i0} g_{\bar{j}0} \\ & + [\lambda b_2 + \nu(a_2 + 2tb_2)] g_{i\bar{0}} g_{j0} \} \nabla y^i \wedge \nabla y^j \\ & - (c_1 g_{i\bar{j}} + d_1 g_{i0} g_{\bar{j}0} + f_1 g_{i\bar{0}} g_{j0}) dx^i \wedge dx^j, \end{aligned} \tag{14}$$

where $g_{i\bar{j}} = g\big(\frac{\partial}{\partial x^i}, J\frac{\partial}{\partial x^j}\big)$, $g_{i\bar{0}} = g\big(\frac{\partial}{\partial x^i}, Jy\big)$.

By using (4) and the property that $\nabla g = 0$, we obtain

$$d\lambda = \lambda' g_{i0} \nabla y^i, \qquad d\mu = \mu' g_{i0} \nabla y^i, \qquad d\nu = \nu' g_{i0} \nabla y^i,$$

$$dg_{i0} = g_{ik} \nabla y^k + \Gamma^l_{ik} g_{l0} dx^k, \qquad dg_{\bar{i}0} = -dg_{i\bar{0}} = g_{\bar{i}k} \nabla y^k + \Gamma^l_{\bar{i}k} g_{l0} dx^k,$$

$$d\nabla y^i = \Gamma^i_{lk} \nabla y^l \wedge dx^k + \frac{1}{2} R^i_{0kl} dx^k \wedge dx^l,$$

where $\Gamma^l_{\bar{i}k} g_{l0} = g\big(\nabla_{\frac{\partial}{\partial x^k}} J\frac{\partial}{\partial x^i}, y\big)$.

Now we consider the first class of almost Hermitian structures (G, J_1). By the local expression (12) of Ω_1, we have

$$
\begin{aligned}
d\Omega_1 &= d(\lambda g_{ij} + \mu g_{i0} g_{j0} + \nu g_{i\bar{0}} g_{j\bar{0}}) \wedge \nabla y^i \wedge dx^j \\
&\quad + (\lambda g_{ij} + \mu g_{i0} g_{j0} + \nu g_{i\bar{0}} g_{j\bar{0}}) d\nabla y^i \wedge dx^j \\
&= \{\lambda' g_{ij} g_{0k} \nabla y^k + \lambda(\Gamma^l_{ik} g_{lj} + \Gamma^l_{jk} g_{li}) dx^k + \mu' g_{0i} g_{0j} g_{0k} \nabla y^k \\
&\quad + \mu g_{0j}(g_{ik} \nabla y^k + \Gamma^l_{ik} g_{0l} dx^k) + \mu g_{0i}(g_{jk} \nabla y^k + \Gamma^l_{jk} g_{0l} dx^k) + \nu' g_{i\bar{0}} g_{j\bar{0}} g_{0k} \nabla y^k \\
&\quad + \nu g_{j\bar{0}}(g_{i\bar{k}} \nabla y^k - \Gamma^l_{\bar{i}k} g_{l0} dx^k) + \nu g_{i\bar{0}}(g_{j\bar{k}} \nabla y^k - \Gamma^l_{\bar{j}k} g_{l0} dx^k)\} \wedge \nabla y^i \wedge dx^j \\
&\quad + (\lambda g_{ij} + \mu g_{0i} g_{0j} + \nu g_{i\bar{0}} g_{j\bar{0}}) \Big(\Gamma^i_{lk} \nabla y^l \wedge dx^k + \frac{1}{2} R^i_{0kl} dx^k \wedge dx^l \Big) \wedge dx^j .
\end{aligned}
$$

Clearly, if (TM, G, J_1) is a locally conformal almost Kähler manifold, then

$$
\frac{1}{2}(\lambda g_{ij} + \mu g_{0i} g_{0j} + \nu g_{i\bar{0}} g_{j\bar{0}}) R^i_{0kl} dx^k \wedge dx^l \wedge dx^j = 0 . \tag{15}
$$

Using the first Bianchi identity, (15) is equivalent to

$$
\nu g_{j\bar{0}} R_{0\bar{0}kl} dx^k \wedge dx^l \wedge dx^j = 0 ,
$$

where $R_{0\bar{0}kl} = R(y, Jy, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l})$. Let $\nu = 0$, then we find that

$$
\Omega_1 = (\lambda g_{ij} + \mu g_{i0} g_{j0}) \nabla y^i \wedge dx^j , \tag{16}
$$

$$
d\Omega_1 = (\lambda' - \mu) g_{ij} g_{0k} \nabla y^k \wedge \nabla y^i \wedge dx^j . \tag{17}
$$

Therefore, if $\nu = 0$ and $\mu = \lambda'$, then (TM, G, J_1) is almost Kähler.

Proposition 3. *If $\nu = 0$, then the almost Hermitian manifold (TM, G, J_1) is locally conformal almost Kähler.*

Proof. Set

$$
\omega = \frac{\lambda' - \mu}{\lambda} g_{0k} \nabla y^k . \tag{18}
$$

Then ω is a globally defined 1-form. Moreover

$$
\begin{aligned}
d\omega &= \left(\frac{\lambda' - \mu}{\lambda} g_{0i} \right) d\nabla y^i + \left(\frac{\lambda' - \mu}{\lambda} \right)' g_{0i} g_{0k} \nabla y^i \wedge \nabla y^k \\
&\quad + \frac{\lambda' - \mu}{\lambda} \left(g_{ik} \nabla y^k + \Gamma^l_{ik} g_{0l} dx^k \right) \wedge \nabla y^i \\
&= \frac{\lambda' - \mu}{\lambda} g_{0i} \left(\Gamma^i_{lk} \nabla y^l \wedge dx^k + \frac{1}{2} R^i_{0kl} dx^k \wedge dx^l \right) \\
&\quad + \frac{\lambda' - \mu}{\lambda} \Gamma^l_{ik} g_{0l} dx^k \wedge \nabla y^i = 0 .
\end{aligned}
$$

On the other hand, it follows from (17) that

$$\begin{aligned} d\Omega_1 &= (\lambda' - \mu) g_{ij} g_{0k} \nabla y^k \wedge \nabla y^i \wedge dx^j \\ &= \left(\frac{\lambda' - \mu}{\lambda} g_{0k} \nabla y^k \right) \wedge (\lambda g_{ij} + \mu g_{0i} g_{0j}) \nabla y^i \wedge dx^j \\ &\quad - \left(\frac{\mu(\lambda' - \mu)}{\lambda} g_{0k} g_{0i} g_{0j} \right) \nabla y^k \wedge \nabla y^i \wedge dx^j . \end{aligned} \tag{19}$$

The second term in (19) vanishes since $g_{0i}g_{0k}$ is symmetric with respect to the indices i, k. Hence

$$d\Omega_1 = \omega \wedge \Omega_1. \tag{20}$$

The closeness of ω and (20) show that (TM, G, J_1) is a locally conformal almost Kähler manifold. □

The following corollary comes directly from Proposition 2 and Proposition 3.

Corollary 1. *Let (M, g, J) be a complex space form of holomorphic sectional curvature $4c$. If the functions b_1, e_1 are given by (11) and $\nu = 0$, then the almost Hermitian manifold (TM, G, J_1) is a locally conformal Kähler manifold. Moreover, the locally conformal Kähler manifold (TM, G, J_1) is a Kähler manifold if $\mu = \lambda'$.*

Next we consider the second class of almost Hermitian structures (G, J_2). By the local expression (13) of Ω_2, we have

$$\begin{aligned} d\Omega_2 &= d(\lambda g_{i\bar{j}} + \mu g_{i0} g_{\bar{j}0} + \nu_{i\bar{0}} g_{j0}) \wedge \nabla y^i \wedge dx^j \\ &\quad + (\lambda g_{i\bar{j}} + \mu g_{i0} g_{\bar{j}0} + \nu_{i\bar{0}} g_{j0}) d\nabla y^i \wedge dx^j \\ &= \big\{ \lambda' g_{i\bar{j}} g_{0k} \nabla y^k + \lambda (\Gamma^l_{ik} g_{l\bar{j}} + \Gamma^l_{\bar{j}k} g_{li}) dx^k + \mu' g_{0i} g_{\bar{j}0} g_{0k} \nabla y^k \\ &\quad + \mu g_{0\bar{j}} (g_{ik} \nabla y^k + \Gamma^l_{ik} g_{0l} dx^k) + \mu g_{0i} (g_{\bar{j}k} \nabla y^k + \Gamma^l_{\bar{j}k} g_{0l} dx^k) + \nu' g_{i\bar{0}} g_{j0} g_{0k} \nabla y^k \\ &\quad + \nu g_{j0} (g_{i\bar{k}} \nabla y^k - \Gamma^l_{\bar{i}k} g_{l0} dx^k) + \nu g_{i\bar{0}} (g_{jk} \nabla y^k + \Gamma^l_{jk} g_{l0} dx^k) \big\} \wedge \nabla y^i \wedge dx^j \\ &\quad + (\lambda g_{i\bar{j}} + \mu g_{0i} g_{\bar{j}0} + \nu g_{i\bar{0}} g_{j0}) \Big(\Gamma^i_{lk} \nabla y^l \wedge dx^k + \frac{1}{2} R^i_{0kl} dx^k \wedge dx^l \Big) \wedge dx^j . \end{aligned}$$

Clearly, if (TM, G, J_2) is a locally conformal almost Kähler manifold, then

$$\frac{1}{2} (\lambda g_{i\bar{j}} + \mu g_{0i} g_{\bar{j}0} + \nu g_{i\bar{0}} g_{j0}) R^i_{0kl} dx^k \wedge dx^l \wedge dx^j = 0 . \tag{21}$$

In the case that (M, g, J) is a Kähler manifold, using the first Bianchi identity, (21) is equivalent to

$$\nu g_{j0} R_{0\bar{0}kl} dx^k \wedge dx^l \wedge dx^j = 0.$$

Let $\nu = 0$, then we obtain that

$$\Omega_2 = (\lambda g_{i\bar{j}} + \mu g_{i0} g_{\bar{j}0}) \nabla y^i \wedge dx^j , \tag{22}$$

$$d\Omega_2 = (\lambda' - \mu) g_{i\bar{j}} g_{0k} \nabla y^k \wedge \nabla y^i \wedge dx^j . \tag{23}$$

Proposition 4. *Let (M, g, J) be a Kähler manifold. If $\nu = 0$, then the almost Hermitian manifold (TM, G, J_2) is locally conformal almost Kähler.*

Proof. From (18), (22) and (23), it follows that

$$\begin{aligned} d\Omega_2 &= (\lambda' - \mu) g_{i\bar{j}} g_{0k} \nabla y^k \wedge \nabla y^i \wedge dx^j \\ &= \left(\frac{\lambda' - \mu}{\lambda} g_{0k} \nabla y^k \right) \wedge (\lambda g_{i\bar{j}} + \mu g_{0i} g_{0\bar{j}}) \nabla y^i \wedge dx^j \\ &= \omega \wedge \Omega_2 \, . \end{aligned} \tag{24}$$

The closeness of ω and (24) show that (TM, G, J_2) is a locally conformal almost Kähler manifold. □

The following corollary comes directly from Proposition 2 and Proposition 4:

Corollary 2. *Let (M, g, J) be a complex space form of holomorphic sectional curvature $4c$. If the functions b_1, e_1 are given by (11) and $\nu = 0$, then the almost Hermitian manifold (TM, G, J_2) is a locally conformal Kähler manifold. Moreover, the locally conformal Kähler manifold (TM, G, J_2) is a Kähler manifold if $\mu = \lambda'$.*

For the third class of almost Hermitian structures (G, J_3), we have

$$\begin{aligned} d\Omega_3 = \big(&-c_1' g_{i\bar{j}} g_{0k} - d_1' g_{0i} g_{0\bar{j}} g_{0k} - d_1' g_{0\bar{i}} g_{0j} g_{0k} - d_1 g_{0\bar{j}} g_{ik} - d_1 g_{0\bar{i}} g_{jk} \\ &- d_1 g_{0i} g_{\bar{j}k} - d_1 g_{0j} g_{i\bar{k}} + c_2 R_{0\bar{k}ij} + \frac{\sigma}{2} g_{0k} R_{0\bar{0}ij} \big) \nabla y^k \wedge dx^i \wedge dx^j \\ &+ (-c_1 \Gamma^l_{\bar{j}k} g_{li} - 2 d_1 g_{0i} \Gamma^l_{\bar{j}k} g_{l0}) dx^k \wedge dx^i \wedge dx^j \\ &+ (c_2' g_{i\bar{j}} g_{0k} + \sigma g_{0i} g_{\bar{j}k}) \nabla y^k \wedge \nabla y^i \wedge \nabla y^j \\ &+ (c_2 \Gamma^l_{\bar{j}k} g_{li} - c_2 \Gamma^l_{jk} g_{\bar{l}i} + \sigma g_{0i} \Gamma^l_{\bar{j}k} g_{l0} - \sigma g_{0i} \Gamma^l_{jk} g_{\bar{l}0}) dx^k \wedge \nabla y^i \wedge \nabla y^j \, , \end{aligned}$$

where $\sigma = \lambda(b_2 + e_2) + \mu(a_2 + 2te_2) + \nu(a_2 + 2tb_2)$.

In the case that (M, g, J) is a complex space form of holomorphic sectional curvature $4c$, we find

$$\begin{aligned} R_{0\bar{k}ij} &= -c(g_{0i} g_{\bar{k}j} - g_{0j} g_{\bar{k}i} + g_{0\bar{i}} g_{kj} - g_{0\bar{j}} g_{ki} - 2 g_{0k} g_{i\bar{j}}) \, , \\ R_{0\bar{0}ij} &= -2c(g_{0i} g_{\bar{0}j} + g_{0\bar{i}} g_{0j} - 2t g_{i\bar{j}}) \, . \end{aligned}$$

Therefore,

$$\begin{aligned} c_2 R_{0\bar{k}ij} + \frac{\sigma}{2} g_{0k} R_{0\bar{0}ij} = &-c\sigma g_{0k} (g_{0i} g_{\bar{0}j} + g_{0\bar{i}} g_{0j} - 2t g_{i\bar{j}}) \\ &- c c_2 (g_{0i} g_{\bar{k}j} - g_{0j} g_{\bar{k}i} + g_{0\bar{i}} g_{kj} - g_{0\bar{j}} g_{ki} - 2 g_{0k} g_{i\bar{j}}) \, . \end{aligned}$$

Consequently,

$$\begin{aligned} d\Omega_3 = (c_2' - \sigma) & g_{i\bar{j}} g_{0k} \nabla y^k \wedge \nabla y^i \wedge \nabla y^j + \big\{ -c_1' g_{i\bar{j}} g_{0k} - d_1' g_{0i} g_{0\bar{j}} g_{0k} \\ &- d_1' g_{\bar{0}i} g_{0j} g_{0k} - (d_1 - c c_2)(g_{0\bar{j}} g_{ik} + g_{\bar{0}i} g_{jk} + g_{0i} g_{\bar{j}k} + g_{0j} g_{i\bar{k}}) \\ &+ 2 c c_2 g_{0k} g_{i\bar{j}} - c\sigma g_{0k} (g_{0i} g_{\bar{0}j} + g_{0\bar{i}} g_{0j} - 2t g_{i\bar{j}}) \big\} \nabla y^k \wedge dx^i \wedge dx^j . \end{aligned} \tag{25}$$

Theorem 1. *Let (M, g, J) be a complex space form of holomorphic sectional curvature $4c$. If the functions b_1, e_1, ν are given by*

$$b_1 = \frac{a_1 a_1' - c}{a_1 - 2ta_1'}\,, \qquad e_1 = \frac{c}{a_1}\,, \qquad \nu = 0\,, \tag{26}$$

then the almost Hermitian structure (G, J_3) is locally conformal Kähler. In this case, the almost hyper-Hermitian structure (G, J_1, J_2, J_3) is locally conformal hyper-Kähler.

Proof. Set $\varrho = \frac{\lambda' - \mu}{\lambda}$. Then $\omega = \varrho g_{0k} \nabla y^k$ is a globally defined 1-form and $d\omega = 0$. From (14), it follows easily that

$$\begin{aligned} \omega \wedge \Omega_3 = &\big\{\varrho c_2 g_{i\bar{j}} + \varrho[\lambda e_2 + (a_2 + 2te_2)\mu] g_{i0} g_{\bar{j}0} \\ &+ \varrho[\lambda b_2 + (a_2 + 2tb_2)\nu] g_{i\bar{0}} g_{j0}\big\} g_{0k} \nabla y^k \wedge \nabla y^i \wedge \nabla y^j \\ &- \varrho(c_1 g_{i\bar{j}} + d_1 g_{i0} g_{\bar{j}0} + f_1 g_{i\bar{0}} g_{j0}) g_{0k} \nabla y^k \wedge dx^i \wedge dx^j. \end{aligned}$$

Thus $d\Omega_3 = \omega \wedge \Omega_3$ if and only if

$$\begin{aligned} &\big\{(2cc_2 + 2tc\sigma - c_1' + \varrho c_1) g_{i\bar{j}} g_{0k} - (d_1' - c\sigma - \varrho d_1)(g_{0i} g_{0\bar{j}} + g_{\bar{0}i} g_{0j}) g_{0k} \\ &\quad -(d_1 - cc_2)(g_{0\bar{j}} g_{ik} + g_{\bar{0}i} g_{jk} + g_{0i} g_{\bar{j}k} + g_{0j} g_{i\bar{k}})\big\} \nabla y^k \wedge dx^i \wedge dx^j \\ &\qquad + (c_2' - \sigma - \varrho c_2) g_{i\bar{j}} g_{0k} \nabla y^k \wedge \nabla y^i \wedge \nabla y^j = 0\,. \end{aligned} \tag{27}$$

By (26) and (8), one finds

$$\mu = \frac{\lambda[2ca_1 - a_1'(a_1^2 + 2ct)]}{a_1(a_1^2 - 2ct)}\,, \qquad \sigma = \frac{2\lambda(c - a_1 a_1')}{a_1(a_1^2 - 2ct)}\,,$$

$$d_1 = cc_2, \quad \varrho c_2 = c_2' - \sigma, \quad \varrho d_1 = d_1' - c\sigma, \quad \varrho c_1 = c_1' - 2tc\sigma - 2cc_2.$$

Thus, (27) holds, namely,

$$d\Omega_3 = \omega \wedge \Omega_3\,.$$

Moreover, from (26) and Proposition 2, it follows that J_1, J_2, J_3 are integrable. Therefore (TM, G, J_3) is a locally conformal Kähler manifold. By Proposition 3 and Proposition 4, we directly obtain that (TM, G, J_1, J_2, J_3) is a locally conformal hyper-Kähler manifold. □

Remark 4. The parameters a_1, λ are not quite arbitrary. In fact, the following conditions must be fulfilled

$$a_1 > 0\,, \qquad a_1 + 2tb_1 = \frac{a_1^2 - 2ct}{a_1 - 2ta_1'} > 0\,, \qquad a_1 + 2te_1 = \frac{a_1^2 + 2ct}{a_1} > 0\,,$$

$$\lambda > 0\,, \qquad \lambda + 2t\mu = \lambda \frac{(a_1 - 2ta_1')(a_1^2 + 2ct)}{a_1(a_1^2 - 2ct)} > 0\,.$$

Corollary 3. *Let (M, g, J) be a complex space form of holomorphic sectional curvature $4c$. If the functions b_1, e_1, λ, ν are given by*

$$b_1 = \frac{a_1 a_1' - c}{a_1 - 2ta_1'}\,, \quad e_1 = \frac{c}{a_1}\,, \quad \lambda = e^{\int \frac{2ca_1 - a_1'(a_1^2+2ct)}{a_1(a_1^2-2ct)}dt}\,, \quad \nu = 0\,, \tag{28}$$

then the almost hyper-Hermitian structure (G, J_1, J_2, J_3) is hyper-Kähler.

Proof. Under the assumption of (28), (TM, G, J_1, J_2, J_3) is a locally conformal hyper-Kähler manifold, and the Lee form $\omega = 0$. This shows that

$$d\Omega_\alpha = \omega \wedge \Omega_\alpha = 0,$$

for each α. Therefore, the almost hyper-Hermitian structure (G, J_1, J_2, J_3) on TM is hyper-Kähler. □

Example 1. Let (M, g, J) be a complex space form of holomorphic sectional curvature $4c$. If $c \geq 0$, we can consider the following functions

$$a_1 = \sqrt{e^{-2t} + 2ct}\,, \qquad b_1 = -\frac{\sqrt{e^{-2t} + 2ct}}{1+2t}\,,$$
$$e_1 = \frac{c}{\sqrt{e^{-2t} + 2ct}}\,, \qquad \nu = 0\,.$$

Clearly, all the conditions of Remark 4 and Theorem 1 are fulfilled. Therefore, we obtain a family of locally conformal hyper-Kähler structures (G, J_1, J_2, J_3) on TM. In particular, putting

$$\lambda = e^{\int \frac{e^{-2t}+4ct+c}{e^{-2t}+2ct}dt}, \tag{29}$$

we can obtain a family of hyper-Kähler structures (G, J_1, J_2, J_3) on TM. If $c < 0$, we can consider the following functions

$$a_1 = \sqrt{e^{-2t} - 2ct}\,, \quad b_1 = -\frac{(2c + e^{-2t})\sqrt{e^{-2t} - 2ct}}{(1+2t)e^{-2t}}\,,$$
$$e_1 = \frac{c}{\sqrt{e^{-2t} - 2ct}}\,, \quad \nu = 0\,.$$

Clearly, all the conditions of Remark 4 and Theorem 1 are fulfilled. Therefore, we obtain a family of locally conformal hyper-Kähler structures (G, J_1, J_2, J_3) on TM. In particular, putting

$$\lambda = e^{\int \frac{e^{-4t}+3ce^{-2t}-4c^2t}{(e^{-2t}-2ct)(e^{-2t}-4ct)}dt}, \tag{30}$$

we can obtain a family of hyper-Kähler structures (G, J_1, J_2, J_3) on TM.

Example 2. Let (M, g, J) be a complex space form of holomorphic sectional curvature $4c$. If $c \geq 0$, we can consider the following functions

$$a_1 = A + \sqrt{A^2 + 2ct}\,, \quad b_1 = e_1 = \frac{c}{A + \sqrt{A^2+2ct}}\,, \quad \lambda = B\,, \quad \mu = \nu = 0\,,$$

where A, B are two positive constants. If $c < 0$, we can consider the following functions

$$a_1 = A + \sqrt{A^2 - 2ct}\,, \quad b_1 = \frac{-c(A + 2\sqrt{A^2 - 2ct})}{A(A + \sqrt{A^2 - 2ct})}\,, \quad e_1 = \frac{c}{A + \sqrt{A^2 - 2ct}}\,,$$

$$\lambda = \frac{B}{\sqrt{A^2 - 2ct}}\,, \qquad \mu = \nu = 0\,.$$

Clearly, all the conditions of Remark 4 and Corollary 3 are fulfilled. Therefore, we obtain a family of hyper-Kähler structures (G, J_1, J_2, J_3) on TM.

In the end of the section, we compute the Levi-Civita connection $\widetilde{\nabla}$ of (TM, G). The explicit expression of $\widetilde{\nabla}$ is obtained from the following well-known formula

$$\begin{aligned} 2G(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, \widetilde{Z}) &= \widetilde{X}(G(\widetilde{Y}, \widetilde{Z})) + \widetilde{Y}(G(\widetilde{X}, \widetilde{Z})) - \widetilde{Z}(G(\widetilde{X}, \widetilde{Y})) \\ &\quad + G([\widetilde{X}, \widetilde{Y}], \widetilde{Z}) + G([\widetilde{Z}, \widetilde{X}], \widetilde{Y}) + G(\widetilde{X}, [\widetilde{Z}, \widetilde{Y}])\,, \end{aligned} \tag{31}$$

where $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \mathscr{X}(TM)$. A direct computation using (5) and (31) gives the following proposition.

Proposition 5. *Let (M, g, J) be a Kähler manifold, then the Levi-Civita connection $\widetilde{\nabla}$ of (TM, G) is given by the following identities:*

$$\widetilde{\nabla}_{\partial_i}\partial_j = Q^l_{ij}\partial_l\,, \qquad \widetilde{\nabla}_{\delta_i}\partial_j = \Gamma^l_{ij}\partial_l + P^l_{ji}\delta_l\,,$$
$$\widetilde{\nabla}_{\partial_i}\delta_j = P^l_{ij}\delta_l\,, \qquad \widetilde{\nabla}_{\delta_i}\delta_j = \Gamma^l_{ij}\delta_l + S^l_{ij}\partial_l\,,$$

where Q^l_{ij}, P^l_{ij}, S^l_{ij} are defined by

$$\begin{aligned} Q^l_{ij} &= \frac{2d_2 - c_2'}{2(c_2 + 2td_2)} g_{ij} y^l + \frac{c_2'}{2c_2}(g_{0i}\delta^l_j + g_{0j}\delta^l_i) + \frac{c_2 d_2' - 2c_2' d_2}{2c_2(c_2 + 2td_2)} g_{0i} g_{0j} y^l \\ &\quad + \frac{c_2 f_2' - c_2' f_2 - 2f_2^2}{2c_2(c_2 + 2tf_2)}(g_{0i} g_{\bar{0}j} + g_{0j} g_{\bar{0}i}) y^{\bar{l}} + \frac{2d_2 f_2 - c_2 f_2'}{2c_2(c_2 + 2td_2)} g_{\bar{0}i} g_{\bar{0}j} y^l \\ &\quad + \frac{f_2}{c_2}(J^l_i g_{\bar{0}j} + J^l_j g_{\bar{0}i})\,, \qquad Jy = y^k J^l_k \frac{\partial}{\partial x^l} := y^{\bar{l}} \frac{\partial}{\partial x^l}\,, \end{aligned}$$

$$\begin{aligned} P^l_{ij} &= \frac{c_1'}{2c_1} g_{0i}\delta^l_j + \frac{d_1}{2c_1} g_{0j}\delta^l_i + \frac{d_1}{2(c_1 + 2td_1)} g_{ij} y^l + \frac{c_1 d_1' - c_1' d_1 - d_1^2}{2c_1(c_1 + 2td_1)} g_{0i} g_{0j} y^l \\ &\quad + \frac{c_2}{2c_1} R^l_{j0i} + \frac{c_2 d_1}{2c_1(c_1 + 2td_1)} R_{0i0j} y^l + \frac{c_1 f_1' - c_1' f_1 - f_1^2}{2c_1(c_1 + 2tf_1)} g_{0i} g_{\bar{0}j} y^{\bar{l}} \\ &\quad + \frac{d_1 f_1}{2c_1(c_1 + 2td_1)} g_{\bar{0}i} g_{\bar{0}j} y^l + \frac{f_1}{2(c_1 + 2tf_1)} g_{\bar{i}j} y^{\bar{l}} - \frac{d_1 f_1}{2c_1(c_1 + 2tf_1)} g_{\bar{0}i} g_{0j} y^{\bar{l}} \\ &\quad + \frac{f_1}{2c_1} g_{\bar{0}j} J^l_i + \frac{f_2}{2c_1} R^l_{j0\bar{0}} g_{\bar{0}i} + \frac{d_1 f_2}{2c_1(c_1 + 2td_1)} R_{0\bar{0}0j} g_{\bar{0}i} y^l \\ &\quad + \frac{c_2 f_1}{2c_1(c_1 + 2tf_1)} R_{0i\bar{0}j} y^{\bar{l}} + \frac{f_1 f_2}{2c_1(c_1 + 2tf_1)} R_{0\bar{0}\bar{0}j} g_{\bar{0}i} y^{\bar{l}}\,, \end{aligned}$$

$$
\begin{aligned}
S^l_{ij} = &-\frac{1}{2}R^l_{0ij} - \frac{c_1'}{2(c_2+2td_2)}g_{ij}y^l + \frac{2d_1d_2 - c_2d_1'}{2c_2(c_2+2td_2)}g_{0i}g_{0j}y^l \\
&- \frac{d_1}{2c_2}(g_{0i}\delta^l_j + g_{0j}\delta^l_i) + \frac{2d_2f_1 - c_2f_1'}{2c_2(c_2+2td_2)}g_{\bar{0}i}g_{\bar{0}j}y^l \\
&+ \frac{(d_1-f_1)f_2}{2c_2(c_2+2tf_2)}(g_{0i}g_{\bar{0}j} + g_{\bar{0}i}g_{0j})y^{\bar{l}} + \frac{f_1}{2c_2}(J^l_i g_{\bar{0}j} + g_{\bar{0}i}J^l_j)\,.
\end{aligned}
$$

4 The unit tangent sphere bundle T_1M

In this section, we restrict the almost hyper-Hermitian structure $(G, J_\alpha)_{\alpha=1,2,3}$ on the unit tangent sphere bundle T_1M, obtaining an almost contact metric 3-structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, \widehat{G})_{\alpha=1,2,3}$. We study the geometry of $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, \widehat{G})_{\alpha=1,2,3}$ and find conditions under which $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, \widehat{G})_{\alpha=1,2,3}$ is a Sasakian 3-structure.

Let $T_1M = \{y \in TM \colon g(y,y) = 1\}$ be the unit tangent sphere bundle on M and $\pi_1 \colon T_1M \to M$ be the canonical projection. Clearly, T_1M is a hypersurface of TM with a local expression

$$g_{ij}(x)y^iy^j = 1$$

in a local coordinates (x^i, y^i) on TM.

Let ∂_i and δ_i be as in Section 2, and define the vector field N on T_1M given by

$$N = \frac{1}{\sqrt{c_2+d_2}}y^i\partial_i\,.$$

Clearly, N is a unit normal vector field of T_1M in TM. Using this fact, one can find the tangent vector field Y_i on T_1M given by

$$Y_i = \left(\frac{\partial}{\partial x^i}\right)^T := \partial_i - g_{0i}y^l\partial_l\,, \quad i = 1, \ldots, n\,. \tag{32}$$

Then it is easy to see that $y^iY_i = 0$. So, $Y_1, \ldots, Y_n$ are not linearly independent. But we can verify that these n vectors together with $\{\delta_i\}$ span $T_y(T_1M)$ at each point $y \in T_1M$. Denote by $\widehat{G}$ the induced metric on T_1M from (TM, G). Then we have

$$
\begin{cases}
\widehat{G}(\delta_i, \delta_j) = c_1g_{ij} + d_1g_{0i}g_{0j} + f_1g_{\bar{0}i}g_{\bar{0}j}\,, \\
\widehat{G}(Y_i, Y_j) = c_2(g_{ij} - g_{0i}g_{0j}) + f_2g_{\bar{0}i}g_{\bar{0}j}\,, \\
\widehat{G}(\delta_i, Y_j) = 0\,,
\end{cases}
$$

where $c_1, c_2, d_1, d_2, f_1, f_2$ are constants because $t = \frac{1}{2}g(y,y) \equiv \frac{1}{2}$ on T_1M.

Using the almost hyper-Hermitian structure $(G, J_\alpha)_{\alpha=1,2,3}$ on TM, we can naturally construct an almost contact metric 3-structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, \widehat{G})_{\alpha=1,2,3}$ on T_1M as follows.

First we put

$$\xi_\alpha = -J_\alpha N\,.$$

Then ξ_α is a unit tangent vector field on T_1M. Next we define a one-form η_α and a $(1,1)$ tensor field φ_α on T_1M by

$$\varphi_\alpha(V) = \tan(J_\alpha V)\,, \quad \eta_\alpha(V)N = \mathrm{nor}(J_\alpha V)\,, \quad \forall\, V \in TT_1M\,,$$

where tan: $TTM \to TT_1M$, nor: $TTM \to T^{\perp}T_1M$ are the usual projection operators. Here $T^{\perp}T_1M$ denotes the normal bundle of T_1M in TM.

By the above identities, one easily verifies that the following relations hold.

$$\varphi_\alpha^2 = -\mathrm{id} + \eta_\alpha \otimes \xi_\alpha\,, \quad \varphi_\alpha(\xi_\alpha) = 0\,, \quad \eta_\alpha \circ \varphi_\alpha = 0\,, \quad \eta_\alpha(\xi_\alpha) = 1\,,$$
$$\eta_\alpha(V) = \widehat{G}(V, \xi_\alpha)\,, \quad \widehat{G}(\varphi_\alpha V, \varphi_\alpha W) = \widehat{G}(V, W) - \eta_\alpha(V)\eta_\alpha(W)\,,$$

for all vector fields V, W on T_1M. This shows that $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, \widehat{G})$ is an almost contact metric structure. Moreover,

$$\begin{gathered}\varphi_\gamma = \varphi_\alpha\varphi_\beta - \eta_\beta \otimes \xi_\alpha = -\varphi_\beta\varphi_\alpha + \eta_\alpha \otimes \xi_\beta\,, \qquad \eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}\,, \\ \xi_\gamma = \varphi_\alpha\xi_\beta = -\varphi_\beta\xi_\alpha\,, \qquad \eta_\gamma = \eta_\alpha \circ \varphi_\beta = -\eta_\beta \circ \varphi_\alpha\,,\end{gathered} \tag{33}$$

where (α, β, γ) is a cyclic permutation of $(1, 2, 3)$. Thus we have proved the following result.

Proposition 6. *$(\varphi_\alpha, \xi_\alpha, \eta_\alpha, \widehat{G})_{\alpha=1,2,3}$ is an almost contact metric 3-structure on T_1M.*

The almost contact metric 3-structure $(T_1M, \varphi_\alpha, \xi_\alpha, \eta_\alpha, \widehat{G})_{\alpha=1,2,3}$ can be locally expressed as

$$\xi_1 = \frac{1}{\sqrt{c_1 + d_1}} y^k \delta_k\,, \qquad \eta_1(\delta_i) = \sqrt{c_1 + d_1}\; g_{0i}\,, \qquad \eta_1(Y_i) = 0\,,$$
$$\varphi_1(\delta_i) = (a_1\delta_i^k + e_1 g_{\bar{0}i} y^{\bar{k}})Y_k\,, \qquad \varphi_1(Y_i) = -[a_2(\delta_i^k - g_{0i}y^k) + e_2 g_{\bar{0}i} y^{\bar{k}}]\delta_k\,,$$
$$\xi_2 = -\frac{1}{\sqrt{c_1 + d_1}} y^{\bar{k}} \delta_k\,, \qquad \eta_2(\delta_i) = \sqrt{c_1 + d_1}\; g_{0\bar{i}}\,, \qquad \eta_2(Y_i) = 0\,,$$
$$\varphi_2(\delta_i) = (a_1 J_i^k + e_1 g_{0i} y^{\bar{k}})Y_k\,, \qquad \varphi_2(Y_i) = [a_2(J_i^k - g_{0i}y^{\bar{k}}) + e_2 g_{0\bar{i}} y^k]\delta_k\,,$$
$$\xi_3 = -\frac{1}{\sqrt{c_2 + f_2}} y^{\bar{k}} Y_k\,, \qquad \eta_3(\delta_i) = 0, \qquad \eta_3(Y_i) = \sqrt{c_2 + f_2}\; g_{0\bar{i}}\,,$$
$$\varphi_3(\delta_i) = -J_i^k\delta_k\,, \qquad \varphi_3(Y_i) = (J_i^k - g_{0i}y^{\bar{k}})Y_k\,.$$

Furthermore, we find from (32) and (5) that

$$[Y_i, Y_j] = -g_{0j}Y_i + g_{0i}Y_j\,, \qquad [\delta_i, Y_j] = \Gamma_{ij}^k Y_k\,, \qquad [\delta_i, \delta_j] = -R_{0ij}^k Y_k\,.$$

Then we have

$$d\eta_1(\delta_i, \delta_j) = 0\,, \quad d\eta_1(Y_i, Y_j) = 0\,, \quad d\eta_1(\delta_i, Y_j) = -\sqrt{c_1 + d_1}(g_{ij} - g_{0i}g_{0j})\,,$$
$$\widehat{G}(\delta_i, \varphi_1 Y_j) = -\lambda(g_{ij} - g_{0i}g_{0j}) - \nu g_{\bar{0}i}g_{\bar{0}j}\,, \quad \widehat{G}(\delta_i, \varphi_1\delta_j) = \widehat{G}(Y_i, \varphi_1 Y_j) = 0\,,$$
$$d\eta_2(\delta_i, \delta_j) = -\sqrt{c_1 + d_1}\Big\{F\Big(\frac{\partial}{\partial x^i}, y, \frac{\partial}{\partial x^j}\Big) - F\Big(\frac{\partial}{\partial x^j}, y, \frac{\partial}{\partial x^i}\Big)\Big\}\,,$$
$$d\eta_2(\delta_i, Y_j) = \sqrt{c_1 + d_1}(g_{i\bar{j}} - g_{0j}g_{\bar{0}i})\,, \quad d\eta_2(Y_i, Y_j) = 0\,,$$

$$\widehat{G}(\delta_i,\varphi_2 Y_j)=\lambda(g_{i\bar{j}}-g_{\bar{0}i}g_{0j})+\nu g_{0i}g_{0\bar{j}}\,,\quad \widehat{G}(\delta_i,\varphi_2\delta_j)=\widehat{G}(Y_i,\varphi_2 Y_j)=0\,,$$

$$d\eta_3(\delta_i,\delta_j)=\sqrt{c_2+f_2}R^k_{0ij}g_{\bar{k}0}\,,\quad d\eta_3(\delta_i,Y_j)=\sqrt{c_2+f_2}F\Big(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j},y\Big)\,,$$

$$d\eta_3(Y_i,Y_j)=2\sqrt{c_2+f_2}(g_{i\bar{j}}-g_{i0}g_{\bar{j}0}+g_{j0}g_{\bar{i}0})\,,\quad \widehat{G}(\delta_i,\varphi_3 Y_j)=0\,,$$

$$\widehat{G}(\delta_i,\varphi_3\delta_j)=c_1 g_{\bar{i}j}+d_1(g_{i0}g_{\bar{0}j}-g_{i\bar{0}}g_{j0})\,,$$

$$\widehat{G}(Y_i,\varphi_3 Y_j)=c_2(g_{i\bar{j}}-g_{i0}g_{\bar{j}0}-g_{0j}g_{i\bar{0}})\,,$$

where $F(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j},y)=g\big((\nabla_{\frac{\partial}{\partial x^i}}J)\frac{\partial}{\partial x^j},y\big)$.

In order to get a contact metric 3-structure on T_1M, i.e.

$$\widehat{G}(V,\varphi_\alpha W)=\frac{1}{2}d\eta_\alpha(V,W),\quad \forall\, V,W\in\mathscr{X}(T_1M),$$

we have to modify the almost contact metric 3-structure in the following way (see e.g. [2]):

$$\varphi^{\text{new}}_\alpha=\varphi_\alpha\,,\quad \xi^{\text{new}}_\alpha=\frac{2\lambda}{\sqrt{c_1+d_1}}\xi_\alpha\,,\quad \eta^{\text{new}}_\alpha=\frac{\sqrt{c_1+d_1}}{2\lambda}\eta_\alpha\,,\quad \widehat{G}^{\text{new}}=\frac{c_1+d_1}{4\lambda^2}\widehat{G}\,.$$

In the next discussion, we still denote by $(\varphi_\alpha,\xi_\alpha,\eta_\alpha,\widehat{G})$ the new almost contact metric 3-structure, then we obtain immediately

Proposition 7. *$(\varphi_\alpha,\xi_\alpha,\eta_\alpha,\widehat{G})_{\alpha=1,2,3}$ is a contact metric 3-structure on T_1M if and only if (M,g,J) is a Kähler manifold and*

$$\nu=0\,,\qquad a_1=e_1\,,\qquad R_{0\bar{0}ij}=2a_1^2(g_{i\bar{j}}+g_{i\bar{0}}g_{j0}-g_{j\bar{0}}g_{i0})\,. \tag{34}$$

Definition 3. A contact metric 3-structure $(\varphi_\alpha,\xi_\alpha,\eta_\alpha,\widehat{G})_{\alpha=1,2,3}$ is called a Sasakian 3-structure if each ξ_α is a Killing vector field and

$$(\widehat{\nabla}_V\varphi_\alpha)W=\widehat{G}(V,W)\xi_\alpha-\eta_\alpha(W)V\,,$$

for all vector fields V,W.

By Proposition 5, we can obtain the following proposition.

Proposition 8. *Let (M,g,J) be a Kähler manifold and denote by $\widehat{\nabla}$ the Levi-Civita connection of $(T_1M,\widehat{G})$, then*

$$\widehat{\nabla}_{\delta_i}\delta_j=\Gamma^k_{ij}\delta_k-\left(\frac{d_1}{2c_2}\left(\delta^k_i g_{0j}+\delta^k_j g_{0i}\right)+\frac{1}{2}R^k_{0ij}+\frac{f_1}{2c_2}(g_{0\bar{j}}J^k_i+g_{0\bar{i}}J^k_j)\right)Y_k\,,$$

$$\begin{aligned}
\widehat{\nabla}_{\delta_i} Y_j = \Gamma^k_{ij} Y_k + \frac{1}{2}\biggl(& \frac{d_1}{c_1+d_1} g_{ij} y^k + \frac{d_1}{c_1} g_{0i}\delta^k_j + \frac{c_2}{c_1} R^k_{i0j} - \frac{d_1(2c_1+d_1)}{c_1(c_1+d_1)} g_{0i} g_{0j} y^k \\
& - \frac{c_2 d_1}{c_1(c_1+d_1)} R_{0ji0} y^k + \frac{d_1 f_1}{c_1(c_1+d_1)} g_{\bar{0}i} g_{\bar{0}j} y^k + \frac{f_1}{c_1+f_1} g_{i\bar{j}} y^{\bar{k}} \\
& - \frac{d_1 f_1}{c_1(c_1+f_1)} g_{0i} g_{\bar{0}j} y^{\bar{k}} + \frac{f_1}{c_1} g_{\bar{0}i} J^k_j + \frac{f_2}{c_1} R^k_{i0\bar{0}} g_{\bar{0}j} \\
& + \frac{d_1 f_2}{c_1(c_1+d_1)} R_{0\bar{0}0i} g_{\bar{0}j} y^k - \frac{f_1(2c_1+f_1)}{c_1(c_1+f_1)} g_{\bar{0}i} g_{0j} y^{\bar{k}} \\
& + \frac{c_2 f_1}{c_1(c_1+f_1)} R_{0j\bar{0}i} y^{\bar{k}} + \frac{f_1 f_2}{c_1(c_1+f_1)} R_{0\bar{0}\bar{0}i} g_{\bar{0}j} y^{\bar{k}} \biggr) \delta_k\,, \\
\widehat{\nabla}_{Y_i} Y_j = -g_{0j} Y_i - \frac{f_2}{c_2} & \left(g_{0\bar{j}} J^k_i + g_{0\bar{i}} J^k_j - g_{0\bar{j}} g_{0i} y^{\bar{k}} - g_{0j} g_{0\bar{i}} y^{\bar{k}} \right) Y_k\,, \\
\widehat{\nabla}_{Y_i} \delta_j = \frac{1}{2}\biggl(& \frac{d_1}{c_1+d_1} g_{ij} y^k - \frac{d_1(2c_1+d_1)}{c_1(c_1+d_1)} g_{0i} g_{0j} y^k - \frac{c_2 d_1}{c_1(c_1+d_1)} R_{0ij0} y^k \\
& + \frac{c_2}{c_1} R^k_{j0i} + \frac{d_1}{c_1} g_{0j}\delta^k_i + \frac{d_1 f_1}{c_1(c_1+d_1)} g_{\bar{0}i} g_{\bar{0}j} y^k + \frac{f_1}{c_1+f_1} g_{\bar{i}j} y^{\bar{k}} \\
& - \frac{d_1 f_1}{c_1(c_1+d_1)} g_{\bar{0}i} g_{0j} y^{\bar{k}} + \frac{f_1}{c_1} g_{\bar{0}j} J^k_i + \frac{f_2}{c_1} R^k_{j0\bar{0}} g_{\bar{0}i} + \frac{d_1 f_2}{c_1(c_1+d_1)} R_{0\bar{0}0j} g_{\bar{0}i} y^k \\
& - \frac{f_1(2c_1+f_1)}{c_1(c_1+f_1)} g_{0i} g_{\bar{0}j} y^{\bar{k}} + \frac{c_2 f_1}{c_1(c_1+f_1)} R_{0i\bar{0}j} y^{\bar{k}} + \frac{f_1 f_2}{c_1(c_1+f_1)} R_{0\bar{0}\bar{0}j} g_{\bar{0}i} y^{\bar{k}} \biggr) \delta_k.
\end{aligned}$$

Remark 5. If the base manifold (M, g, J) is a complex space form of holomorphic sectional curvature $4c > 0$ and the parameters a_1, e_1, ν satisfy

$$a_1^2 = c\,, \qquad a_1 = e_1\,, \qquad \nu = 0\,, \tag{35}$$

then (34) holds. In this case, $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, \widehat{G})_{\alpha=1,2,3}$ is a contact metric 3-structure on T_1M. From (35) and (8), it follows that

$$c_1 = d_1 = f_1 = \lambda a_1\,, \qquad f_2 = \lambda e_2\,, \qquad e_2 = -\frac{1}{2a_1}\,.$$

By Proposition 8 and Remark 5, we have the following

Proposition 9. *Let (M, g, J) be a complex space form of positive holomorphic sectional curvature $4c$. If the parameters a_1, e_1, ν satisfy (35), then the Levi-Civita connection of the metric $\widehat{G}$ is determined by the following equations.*

$$\begin{aligned}
\widehat{\nabla}_{\delta_i}\delta_j &= \Gamma^k_{ij}\delta_k + c\left(-g_{0j}\delta^k_i + g_{\bar{0}j} J^k_i - g_{i\bar{j}} y^{\bar{k}}\right) Y_k \\
\widehat{\nabla}_{\delta_i} Y_j &= \frac{1}{2}\left(g_{ij} y^k - g_{0i} g_{0j} y^k + g_{i\bar{j}} y^{\bar{k}} - g_{\bar{0}j} J^k_i + g_{0i} g_{\bar{0}j} y^{\bar{k}} - g_{\bar{0}i} g_{\bar{0}j} y^k - g_{\bar{0}i} g_{0j} y^{\bar{k}} \right) \delta_k \\
&\quad + \Gamma^k_{ij} Y_k \\
\widehat{\nabla}_{Y_i} Y_j &= -g_{0j} Y_i - \frac{1}{2}\left(g_{\bar{0}j} J^k_i + g_{\bar{0}i} J^k_j - g_{\bar{0}i} g_{0j} y^{\bar{k}} - g_{0i} g_{\bar{0}j} y^{\bar{k}} \right) Y_k \\
\widehat{\nabla}_{Y_i}\delta_j &= \frac{1}{2}\left(g_{ij} y^k - g_{0i} g_{0j} y^k + g_{\bar{i}j} y^{\bar{k}} - g_{\bar{0}i} J^k_j + g_{\bar{0}i} g_{0j} y^{\bar{k}} - g_{\bar{0}i} g_{\bar{0}j} y^k - g_{0i} g_{\bar{0}j} y^{\bar{k}} \right) \delta_k
\end{aligned}$$

Remark 6. Under the assumption of Proposition 9, a direct calculation shows that the following relations hold:

$$\widehat{G}(\widehat{\nabla}_V \xi_\alpha, W) = -\widehat{G}(\widehat{\nabla}_W \xi_\alpha, V)\,,$$
$$(\widehat{\nabla}_V \varphi_\alpha)W = \widehat{G}(V, W)\xi_\alpha - \eta_\alpha(W)V\,,$$

for each α and all vector fields V, W on T_1M. Therefore, we have proved the following

Theorem 2. *There exists a class of Sasakian 3-structures on the unit tangent sphere bundle of a complex space form of positive holomorphic sectional curvature.*

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Authors' addresses:
Xuerong Qi: Department of Mathematics, Zhengzhou University, Zhengzhou 450001, P. R. China

E-mail: xrqi@zzu.edu.cn

Linfen Cao, Xingxiao Li: Department of Mathematics, Henan Normal University, Xinxiang 453007, P. R. China

E-mail: caolf2010@yahoo.com, xxl@henannu.edu.cn

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