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ON THE STRONG CONVERGENCE FOR WEIGHTED SUMS OF ASYMPTOTICALLY ALMOST NEGATIVELY ASSOCIATED RANDOM VARIABLES

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Applying the moment inequality of asymptotically almost negatively associated (AANA, in short) random variables which was obtained by Yuan and An (2009), some strong convergence results for weighted sums of AANA random variables are obtained without assumptions of identical distribution, which generalize and improve the corresponding ones of Zhou et al. (2011), Sung (2011, 2012) to the case of AANA random variables, respectively.

Keywords: AANA random variables, strong convergence, weighted sums

Classification: 60F15

1. INTRODUCTION

Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed random variables and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real constants. The strong convergence properties for weighted sums $\sum_{i=1}^{n} a_{ni}X_i$ have been studied by many authors (see, for example, Cuzick [3]; Bai and Cheng [1]; Wu [19]; Cai [2]; Wang et al. [11, 12, 16]; Wu [20, 21], etc).

Sung [8] obtained the following strong convergence result for weighted sums of negatively associated (NA, in short) random variables.

Theorem 1.1. Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed NA random variables, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real constants satisfying

$$A_\alpha = \limsup_{n\to\infty} A_{\alpha,n} < \infty, \qquad A_{\alpha,n} = \frac{1}{n}\sum_{i=1}^{n} |a_{ni}|^\alpha, \tag{1.1}$$

for some $0 < \alpha \leq 2$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Furthermore, suppose that $EX = 0$ when $1 < \alpha \leq 2$. If

$$E|X|^\alpha < \infty, \quad \text{for} \quad \alpha > \gamma,$$
$$E|X|^\alpha \log(1 + |X|) < \infty, \quad \text{for} \quad \alpha = \gamma, \tag{1.2}$$

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$$E|X|^\gamma < \infty, \quad \text{for} \quad \alpha < \gamma,$$

then

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1\le j\le n} \left| \sum_{i=1}^{j} a_{ni} X_i \right| > \varepsilon b_n \right) < \infty \quad \text{for} \quad \forall\, \varepsilon > 0. \tag{1.3}$$

Recently, Zhou et al. [25] generalized the above Theorem 1.1 to the case ρ^* − mixing random variables when $\alpha \neq \gamma$ by using different methods from those of Sung [8].

Theorem 1.2. Let $\{X, X_n, n \ge 1\}$ be a sequence of identically distributed ρ^*- mixing random variables, and let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of real constants satisfying

$$A_\beta = \limsup_{n\to\infty} A_{\beta,n} < \infty, \qquad A_{\beta,n} = \frac{1}{n} \sum_{i=1}^{n} |a_{ni}|^\beta, \tag{1.4}$$

where $\beta = \max(\alpha, \gamma)$ for some $0 < \alpha \le 2$ and $\gamma > 0$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$. If $EX = 0$ for $1 < \alpha \le 2$ and (1.2) for $\alpha \ne \gamma$, then (1.3) holds.

Sung [9] solved the case $\alpha = \gamma$ of Theorem 1.2 and obtained the following complete convergence result as follows.

Theorem 1.3. Let $\{X, X_n, n \ge 1\}$ be a sequence of identically distributed ρ^*- mixing random variables, and let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of real constants satisfying (1.1) for some $0 < \alpha \le 2$. Let $b_n = n^{1/\alpha}(\log n)^{1/\alpha}$. If $EX = 0$ for $1 < \alpha \le 2$ and $E|X|^\alpha \log(1 + |X|) < \infty$, then (1.3) holds.

In the following, we will recall the definitions of NA random variables and AANA random variables.

Definition 1.4. A finite collection of random variables $\{X_i, 1 \le i \le n\}$ is said to be NA if for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, \ldots, n\}$,

$$Cov(f_1(X_i, i \in A_1),\ f_2(X_j, j \in A_2)) \le 0, \tag{1.5}$$

whenever f_1 and f_2 are nondecreasing functions such that this covariance exists. An infinite collection of random variables $\{X_i, i \ge 1\}$ is NA if every finite subcollection is NA.

Definition 1.5. A sequence of random variables $\{X_n, n \ge 1\}$ is called AANA if there exists a nonnegative sequence $\mu(n) \to 0$ as $n \to \infty$ such that

$$\begin{aligned} &Cov(f_1(X_n), f_2(X_{n+1}, X_{n+2}, \ldots, X_{n+k})) \\ &\qquad \le \mu(n)(Var(f_1(X_n))Var(f_2(X_{n+1}, X_{n+2}, \ldots, X_{n+k})))^{1/2}, \end{aligned} \tag{1.6}$$

for all $n, k \ge 1$ and for all coordinatewise nondecreasing continuous functions f_1 and f_2 whenever the variances exist.

The family of AANA sequence contains NA (in particular, independent) sequences with $q(n) = 0$ for $n \geq 1$ and some more sequences of random variables which are not much deviated from being NA. An example of an AANA sequence which is not NA was constructed by Chandra and Ghosal [4]. So, AANA is much weaker than NA.

Since the concept of AANA sequence was introduced by Chandra and Ghosal [4], many applications have been established. For example, Chandra and Ghosal [4], Wang et al. [17], Ko et al. [6], Yuan and An [22, 23], Yuan and Wu [24], Wang et al [13, 14, 15], Yang et al. [18], Hu et al. [5], Shen and Wu [7], Tang [10], and so forth. Hence, it is very significant to study limit properties of this wider AANA random variables in probability theory and practical applications.

The main purpose of this paper is to further study strong convergence of AANA random variables. We shall generalize and improve Theorem 1.1−1.3 to the case of AANA random variables and obtain complete convergence for weighted sums of AANA random variables without assumptions of identical distribution. The results presented in this paper are obtained by using the truncated method and the classical moment inequality of AANA random variables.

We will use the following concept in this paper.

Definition 1.6. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_n| > x) \leq CP(|X| > x), \tag{1.7}$$

for all $x \geq 0$ and $n \geq 1$.

2. MAIN RESULTS

Throughout this paper, let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with the mixing coefficients $\{\mu(n), n \geq 1\}$, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real constants. Let $I(A)$ be the indicator function of the set A . The symbol C denotes a positive constant which may be different in various places, and $a_n = O(b_n)$ stands for $a_n \leq C(b_n)$.

Our main results are as follows, the proofs will be detailed in the next section.

Theorem 2.1. Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables which is stochastically dominated by a random variable X, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real constants satisfying (1.4) where $\beta = \max(\alpha, \gamma)$ for some $0 < \alpha \leq 2$ and $\gamma > 0$. Assume that $EX_n = 0$ for $1 < \alpha \leq 2$ and $E|X|^\beta < \infty$. Then,

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon b_n\right) < \infty \quad \text{for} \quad \forall\, \varepsilon > 0, \tag{2.1}$$

where $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$.

Theorem 2.2. Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables which is stochastically dominated by a random variable X and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real constants satisfying (1.1) for some $0 < \alpha \leq 2$. Assume that $EX_n = 0$ for $1 < \alpha \leq 2$ and $E|X|^\alpha \log(1 + |X|) < \infty$. Then (2.1) holds, where $b_n = n^{1/\alpha}(\log n)^{1/\alpha}$.

Corollary 2.3. Under the conditions of Theorem 2.1 or Theorem 2.2,

$$\lim_{n\to\infty}\frac{1}{b_n}\sum_{i=1}^{n}a_{ni}X_i=0 \qquad \text{a.s.} \tag{2.2}$$

Remark 2.4. In Theorem 2.1 and Theorem 2.2, we consider the case $\alpha\neq\gamma$ and $\alpha=\gamma$ for $0<\alpha\leq 2$ respectively, and obtain some complete convergence results for weighted sums of AANA random variables without assumptions of identical distribution. We use different methods from those of Sung [8]. The obtained results not only extend the corresponding results of Zhou et al. [25], Sung [8, 9] to AANA case, but also improve them.

3. PROOFS

To prove the main results of this paper, we need the following lemmas.

Lemma 3.1. (Yuan and An [22]) Let $\{X_n, n\geq 1\}$ be a sequence of AANA random variables with mixing coefficients $\{\mu(n), n\geq 1\}$, let $\{f_n, n\geq 1\}$ be a sequence of all nondecreasing (or all nonincreasing) continuous functions, then $\{f_n(X_n), n\geq 1\}$ is still a sequence of AANA random variables with mixing coefficients $\{\mu(n), n\geq 1\}$.

Lemma 3.2. (Yuan and An [22]) Let $\{X_n, n\geq 1\}$ be a sequence of AANA random variables with mixing coefficients $\{\mu(n), n\geq 1\}$, $EX_n=0$. If $\sum_{i=1}^{\infty}\mu^{1/(M-1)}(i)<\infty$ for some $M\in(3\cdot 2^{k-1}, 4\cdot 2^{k-1}]$, where integer number $k\geq 1$, then there exists a positive constant $C=C(M)$ depending only on M such that for all $n\geq 1$,

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}X_i\right|^M\right)\leq C\left(\sum_{i=1}^{n}E\left|X_i\right|^M+\left(\sum_{i=1}^{n}EX_i^2\right)^{M/2}\right). \tag{3.1}$$

Lemma 3.3. (Sung [8]) Let X be a random variable and $\{a_{ni}, 1\leq i\leq n, n\geq 1\}$ be an array of real constants satisfying (1.1) for some $\alpha>0$. Let $b_n=n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma>0$. Then,

$$\sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}P\left(|a_{ni}X|>b_n\right)\leq\begin{cases}CE|X|^{\alpha}, & \text{for}\quad \alpha>\gamma,\\ CE|X|^{\alpha}\log(1+|X|), & \text{for}\quad \alpha=\gamma,\\ CE|X|^{\gamma}, & \text{for}\quad \alpha<\gamma.\end{cases} \tag{3.2}$$

Lemma 3.4. (Sung [9]) Let X be a random variable and $\{a_{ni}, 1\leq i\leq n, n\geq 1\}$ be an array of real constants satisfying $a_{ni}=0$ or $a_{ni}>1$ and (1.1) for some $\alpha>0$. Let $b_n=n^{1/\alpha}(\log n)^{1/\alpha}$. If $q>\alpha$, then,

$$\sum_{n=1}^{\infty}n^{-1}b_n^{-q}\sum_{i=1}^{n}E|a_{ni}X|^qI(|a_{ni}X|\leq b_n)\leq CE|X|^{\alpha}\log(1+|X|). \tag{3.3}$$

Lemma 3.5. Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $b > 0$, the following two statements hold:

$$E|X_n|^\alpha I\left(|X_n| \leq b\right) \leq C_1 \left(E|X|^\alpha I(|X| \leq b) + b^\alpha P(|X| > b)\right), \tag{3.4}$$

$$E|X_n|^\alpha I(|X_n| > b) \leq C_2 E|X|^\alpha I(|X| > b), \tag{3.5}$$

where C_1 and C_2 are positive constants.

Proof. (Theorem 2.1) Without loss of generality, we assume that $a_{ni} \geq 0$ (otherwise, we use a_{ni}^+ and a_{ni}^- instead of a_{ni}, and note that $a_{ni} = a_{ni}^+ - a_{ni}^-$). Define that

$$Y_i = -b_n I(a_{ni}X_i < -b_n) + a_{ni}X_i I(|a_{ni}X_i| \leq b_n) + b_n I(a_{ni}X_i > b_n);$$

$$T_j = \sum_{i=1}^{j} (Y_i - EY_i) \text{ for all } i \geq 1.$$

It is obvious to check that for $\forall \varepsilon > 0$,

$$\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon b_n\right) \subset \left(\max_{1\leq j\leq n}|a_{nj}X_j| > b_n\right)\bigcup\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} Y_i\right| > \varepsilon b_n\right),$$

which implies that

$$\begin{aligned} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon b_n\right) &\leq P\left(\max_{1\leq j\leq n}|a_{nj}X_j| > b_n\right) + P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} Y_i\right| > \varepsilon b_n\right) \\ &\leq \sum_{j=1}^{n} P\left(|a_{nj}X_j| > b_n\right) \\ &\quad + P\left(\max_{1\leq j\leq n}|T_j| > \varepsilon b_n - \max_{1\leq j\leq n}\left|\sum_{i=1}^{j} EY_i\right|\right). \end{aligned} \tag{3.6}$$

Firstly, we will show that

$$b_n^{-1}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} EY_i\right| \to 0 \quad \text{as} \quad n\to\infty. \tag{3.7}$$

For $\alpha > \gamma$, it follows from (1.4) that $\max_{1\leq i\leq n}|a_{ni}|^\alpha \leq C$ for $\forall n \geq 1$.
For $\alpha < \gamma$, by $\sum_{i=1}^{n}|a_{ni}|^\beta = \sum_{i=1}^{n}|a_{ni}|^\gamma \leq Cn$ and Lyapunov's inequality,

$$\frac{1}{n}\sum_{i=1}^{n}|a_{ni}|^\alpha \leq \left(\frac{1}{n}\sum_{i=1}^{n}|a_{ni}|^\alpha\right)^{\alpha/\gamma} \leq C, \tag{3.8}$$

which implies that $\frac{1}{n} \times n \times \max_{1\le i\le n} |a_{ni}|^\alpha = \max_{1\le i\le n} |a_{ni}|^\alpha \le C$ for $\forall n \ge 1$. When $0 < \alpha \le 1$, we have by Lemma 3.5, Definition 1.6 and $E|X|^\beta < \infty$ that

$$\begin{aligned}
&b_n^{-1} \max_{1\le j\le n} \left|\sum_{i=1}^{j} EY_i\right| \le b_n^{-1} \sum_{i=1}^{n} |EY_i| \\
&\le b_n^{-1} \sum_{i=1}^{n} E|a_{ni}X_i| I(|a_{ni}X_i| \le b_n) + \sum_{i=1}^{n} P(|a_{ni}X_i| > b_n) \\
&\le Cb_n^{-1} \sum_{i=1}^{n} (E|a_{ni}X| I(|a_{ni}X| \le b_n) + b_n P(|a_{ni}X| > b_n)) + C\sum_{i=1}^{n} P(|a_{ni}X| > b_n) \\
&\le Cb_n^{-1} \sum_{i=1}^{n} (E|a_{ni}X| I(|a_{ni}X| \le b_n)) + C\sum_{i=1}^{n} P(|a_{ni}X| > b_n) \\
&\le C\sum_{i=1}^{n} \left(E\left(\frac{|a_{ni}X|}{b_n}\right)^\alpha I\left(|a_{ni}X| \le b_n\right)\right) + C\sum_{i=1}^{n} P(|a_{ni}X| > b_n) \\
&\le C(\log n)^{-\alpha/\gamma} E|X|^\alpha + Cn\frac{E|a_{ni}X|^\alpha}{b_n^\alpha} \\
&\le C(\log n)^{-\alpha/\gamma} E|X|^\alpha \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned} \tag{3.9}$$

When $1 < \alpha \le 2$, we have by Lemma 3.5, $EX_n = 0$, C_r inequality, Definition 1.6 and $E|X|^\beta < \infty$ again that

$$\begin{aligned}
b_n^{-1} \max_{1\le j\le n} \left|\sum_{i=1}^{j} EY_i\right| &\le b_n^{-1} \sum_{i=1}^{n} |EY_i| \\
&\le b_n^{-1} \sum_{i=1}^{n} |Ea_{ni}X_i I(|a_{ni}X_i| \le b_n)| + \sum_{i=1}^{n} P(|a_{ni}X_i| > b_n) \\
&\le Cb_n^{-1} \sum_{i=1}^{n} (E|a_{ni}X| I(|a_{ni}X| \ge b_n) + b_n P(|a_{ni}X| > b_n)) \\
&\le C\sum_{i=1}^{n} (E\left(\frac{|a_{ni}X|}{b_n}\right)^\alpha + CnP(|a_{ni}X| > b_n) \\
&\le C(\log n)^{-\alpha/\gamma} E|X|^\alpha + C(\log n)^{-\alpha/\gamma} |a_{ni}|^\alpha E|X|^\alpha \to 0 \\
&\qquad \text{as } n \to \infty.
\end{aligned} \tag{3.10}$$

By (3.9) and (3.10), we can get (3.7) immediately. Hence, for n large enough,

$$P\left(\max_{1\le j\le n} \left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon b_n\right) \le \sum_{j=1}^{n} P\left(|a_{nj}X_j| > b_n\right) + P\left(\max_{1\le j\le n} |T_j| > \frac{\varepsilon b_n}{2}\right). \tag{3.11}$$

To prove (2.1), we need only to show that

$$I \triangleq \sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^{n} P\left(|a_{nj}X_j| > b_n\right) < \infty; \tag{3.12}$$

$$J \triangleq \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n} |T_j| > \frac{\varepsilon b_n}{2}\right) < \infty. \tag{3.13}$$

By Lemma 3.5, Lemma 3.3 and $E|X|^\beta < \infty$, we can get that

$$\begin{aligned} I &\triangleq \sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^{n} P\left(|a_{nj}X_j| > b_n\right) \\ &\le C\sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^{n} P\left(|a_{nj}X| > b_n\right) \\ &\le CE|X|^\beta < \infty. \end{aligned} \tag{3.14}$$

For fixed $n \ge 1$, it is easily seen that $\{Y_i, 1 \le i \le n\}$ are still a sequence of AANA random variables by Lemma 3.1. For $M > 2$, it follows from Lemma 3.2 and Markov inequality that

$$\begin{aligned} J &\triangleq \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n} |T_j| > \frac{\varepsilon b_n}{2}\right) \\ &\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} E\left(\max_{1\le j\le n} |T_j|^M\right) \\ &\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \left[\sum_{i=1}^{n} E\left|Y_i - EY_i\right|^M + \left(\sum_{i=1}^{n} E\left|Y_i - EY_i\right|^2\right)^{M/2}\right] \\ &\triangleq J_1 + J_2. \end{aligned} \tag{3.15}$$

Next, we will show $J_1 < \infty$ and $J_2 < \infty$ in the following two cases, respectively. The detailed proofs are as follows.
For $\alpha < \gamma$, take $M > \max(2, \gamma)$. It follows from Lemma 3.5, Lemma 3.3, Markov inequality and $E|X|^\beta < \infty$ that

$$\begin{aligned} J_1 &= C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \sum_{i=1}^{n} E\left|Y_i - EY_i\right|^M \\ &\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \sum_{i=1}^{n} E\left|Y_i\right|^M \\ &\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \sum_{i=1}^{n} \left(E|a_{ni}X_i|^M I(|a_{ni}X_i| \le b_n) + b_n^M P(|a_{ni}X_i| > b_n)\right) \\ &\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \sum_{i=1}^{n} |a_{ni}|^M E|X|^M I(|a_{ni}X| \le b_n) \\ &\quad + C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \sum_{i=1}^{n} b_n^M P(|a_{ni}X| > b_n) \\ &\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-\gamma} \sum_{i=1}^{n} |a_{ni}|^\gamma E|X|^\gamma + C\sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P(|a_{ni}X| > b_n) \\ &\le C\sum_{n=1}^{\infty} n^{-\gamma/\alpha} (\log n)^{-1} + CE|X|^\beta < \infty. \end{aligned} \tag{3.16}$$

For $\alpha > \gamma$, note that $E|X|^\beta = E|X|^\alpha < \infty$. We can get that

$$\begin{aligned}
J_1 &\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \sum_{i=1}^{n} E\left|Y_i\right|^M \\
&\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \sum_{i=1}^{n} \left(|a_{ni}|^M E|X_i|^M I(|a_{ni}X_i| \le b_n) + b_n^M P(|a_{ni}X_i| > b_n)\right) \\
&\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i=1}^{n} |a_{ni}|^\alpha E|X|^\alpha + C\sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P(|a_{ni}X| > b_n) \\
&\le C\sum_{n=1}^{\infty} n^{-1} (\log n)^{-\alpha/\gamma} + CE|X|^\alpha < \infty.
\end{aligned} \tag{3.17}$$

For $\alpha < \gamma \le 2$ or $\gamma < \alpha \le 2$, taking $M \ge \max(2, \dfrac{2\gamma}{\alpha})$, we can get that

$$\begin{aligned}
J_2 &\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \left(\sum_{i=1}^{n} E\left|Y_i\right|^2\right)^{M/2} \\
&\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \left(\sum_{i=1}^{n} E|a_{ni}X_i|^2 I(|a_{ni}X_i| \le b_n)\right)^{M/2} \\
&\quad + C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \left(\sum_{i=1}^{n} |b_n|^2 P\left(|a_{ni}X_i| > b_n\right)\right)^{M/2} \\
&\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \left(\sum_{i=1}^{n} \left(E|a_{ni}X|^2 I(|a_{ni}X| \le b_n) + b_n^2 P\left(|a_{ni}X| > b_n\right)\right)\right)^{M/2} \\
&\quad + C\sum_{n=1}^{\infty} n^{-1} \left(\sum_{i=1}^{n} P\left(|a_{ni}X| > b_n\right)\right)^{M/2} \\
&\le C\sum_{n=1}^{\infty} n^{-1} \left(\sum_{i=1}^{n} E\frac{|a_{ni}X|^2}{b_n^2} I(|a_{ni}X| \le b_n)\right)^{M/2} \\
&\quad + C\sum_{n=1}^{\infty} n^{-1} \left(\sum_{i=1}^{n} P\left(|a_{ni}X| > b_n\right)\right)^{M/2} \\
&\le C\sum_{n=1}^{\infty} n^{-1} \left(\sum_{i=1}^{n} E\frac{|a_{ni}X|^\alpha}{b_n^\alpha} I(|a_{ni}X| \le b_n)\right)^{M/2} + C\sum_{n=1}^{\infty} n^{-1} \left(\sum_{i=1}^{n} \frac{E|a_{ni}X|^\alpha}{b_n^\alpha}\right)^{M/2} \\
&\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-\alpha M/2} \left(\sum_{i=1}^{n} |a_{ni}|^\alpha E|X|^\alpha\right)^{M/2} \\
&\le C\sum_{n=1}^{\infty} n^{-1} (\log n)^{\frac{-\alpha M}{2\gamma}} < \infty.
\end{aligned} \tag{3.18}$$

For $\alpha \le 2 < \gamma$ or $\alpha < 2 \le \gamma$, it follows $\sum\limits_{i=1}^{n} |a_{ni}|^{\beta} = \sum\limits_{i=1}^{n} |a_{ni}|^{\gamma} \le Cn$ and Lyapunov's inequality that

$$\frac{1}{n}\sum_{i=1}^{n}|a_{ni}|^{\alpha} \le \left(\frac{1}{n}\sum_{i=1}^{n}|a_{ni}|^{\gamma}\right)^{\alpha/\gamma} \le C,$$

which implies that $\sum_{i=1}^{n}|a_{ni}|^{\alpha} \le Cn$. Hence, by $\max_{1\le i\le n}|a_{ni}|^{\alpha} \le Cn$ and Hölder inequality, we can have that

$$\sum_{i=1}^{n}|a_{ni}|^{k} = \sum_{i=1}^{n}|a_{ni}|^{\alpha}|a_{ni}|^{k-\alpha} \le Cnn^{\frac{k-\alpha}{\alpha}} \le Cn^{\frac{k}{\alpha}} \quad \text{for} \quad \forall k \ge \alpha. \tag{3.19}$$

For $M > \gamma$, it follows from C_r inequality, Markov inequality and (3.19) (for $k = 2$) that

$$\begin{aligned} J_2 \quad &\le C\sum_{n=1}^{\infty} n^{-1}b_n^{-M}\left(\sum_{i=1}^{n} E\left|Y_i\right|^2\right)^{M/2} \\ &\le C\sum_{n=1}^{\infty} n^{-1}b_n^{-M}\left(\sum_{i=1}^{n} E|a_{ni}X_i|^2 I(|a_{ni}X_i| \le b_n)\right)^{M/2} \\ &\quad +C\sum_{n=1}^{\infty} n^{-1}b_n^{-M}\left(\sum_{i=1}^{n} |b_n|^2 P\left(|a_{ni}X_i| > b_n\right)\right)^{M/2} \\ &\le C\sum_{n=1}^{\infty} n^{-1}b_n^{-M}\left(\sum_{i=1}^{n}|a_{ni}|^2\right)^{M/2} + C\sum_{n=1}^{\infty} n^{-1}b_n^{-M}\left(\sum_{i=1}^{n}|a_{ni}|^2\right)^{M/2} \\ &\le C\sum_{n=1}^{\infty} n^{-1}b_n^{-M}n^{M/\alpha} \\ &\le C\sum_{n=1}^{\infty} n^{-1}(\log n)^{-M/\gamma} < \infty. \end{aligned} \tag{3.20}$$

Therefore, the desired result (2.1) follows from the above statements immediately. The proof of Theorem 2.1 is complete. □

P r o o f. (Theorem 2.2) Without loss of generality, suppose that $\sum_{i=1}^{n}|a_{ni}|^{\alpha} \le Cn$ and $a_{ni} \ge 0$ for all $1 \le i \le n, n \ge 1$.For fixed $n \ge 1$, we use the same notation and truncated methods of the proof in Theorem 2.1.

Firstly, we will show that (3.7) holds true. For $0 < \alpha \le 1$, it follows from (3.4) of

Lemma 3.5, C_r inequality, Markov inequality and $E|X|^\alpha \log(1+|X|) < \infty$ that

$$\begin{aligned}
b_n^{-1} \max_{1\le j\le n} \left|\sum_{i=1}^{j} EY_i\right| &\le Cb_n^{-1} \sum_{i=1}^{n} |EY_i| \\
&\le Cb_n^{-1} \sum_{i=1}^{n} (E|a_{ni}X_i|I(|a_{ni}X_i| \le b_n) + b_n P(|a_{ni}X_i| > b_n)) \\
&\le Cb_n^{-1} \sum_{i=1}^{n} (E|a_{ni}X|I(|a_{ni}X| \le b_n) + b_n P(|a_{ni}X| > b_n)) \\
&\quad + C\sum_{i=1}^{n} P(|a_{ni}X| > b_n) \\
&\le Cb_n^{-\alpha} \sum_{i=1}^{n} (E|a_{ni}X|^\alpha I(|a_{ni}X| \le b_n)) + Cb_n^{-\alpha} \sum_{i=1}^{n} E|a_{ni}X|^\alpha \\
&\le Cb_n^{-\alpha} \sum_{i=1}^{n} |a_{ni}|^\alpha E|X|^\alpha \\
&\le C(\log n)^{-1} E|X|^\alpha \to 0 \quad \text{as} \quad n\to\infty.
\end{aligned} \tag{3.21}$$

For $1 < \alpha \le 2$, it follows from $EX_n = 0$, (3.5) of Lemma 3.5, C_r inequality, Markov inequality and $E|X|^\alpha \log(1+|X|) < \infty$ again that

$$\begin{aligned}
b_n^{-1} \max_{1\le j\le n} \left|\sum_{i=1}^{j} EY_i\right| &\le b_n^{-1} \max_{1\le j\le n} \left|\sum_{i=1}^{j} (Ea_{ni}X_i I(|a_{ni}X_i| \le b_n) + b_n P(|a_{ni}X_i| > b_n))\right| \\
&\le Cb_n^{-1} \sum_{i=1}^{n} E|a_{ni}X|I(|a_{ni}X| > b_n) + C\sum_{i=1}^{n} P(|a_{ni}X| > b_n) \\
&\le Cb_n^{-\alpha} \sum_{i=1}^{n} E|a_{ni}X|^\alpha I(|a_{ni}X| > b_n) + Cb_n^{-\alpha} \sum_{i=1}^{n} E|a_{ni}X|^\alpha \\
&\le Cb_n^{-\alpha} \sum_{i=1}^{n} E|a_{ni}X|^\alpha \\
&\le C(\log n)^{-1} E|X|^\alpha \to 0 \quad \text{as} \quad n\to\infty.
\end{aligned} \tag{3.22}$$

By (3.21) and (3.22), we can get (3.7) immediately. Hence, to prove (2.1), we need only to show that $I < \infty$ and $J < \infty$.
It follows from Lemma 3.3 and $E|X|^\alpha \log(1+|X|) < \infty$ that

$$\begin{aligned}
I &\triangleq \sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^{n} P\left(|a_{nj}X_j| > b_n\right) \\
&\le C\sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^{n} P\left(|a_{nj}X| > b_n\right) \\
&\le E|X|^\alpha \log(1+|X|) < \infty.
\end{aligned} \tag{3.23}$$

From the proof of Theorem 2.1, we need only to show that $J_1 < \infty$ and $J_2 < \infty$. It follows from (3.4) of Lemma 3.5, C_r inequality and Markov inequality that

$$\begin{aligned} J_1 &\triangleq C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \sum_{i=1}^{n} E\left|Y_i - EY_i\right|^M \\ &\leq C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \sum_{i=1}^{n} \left(|a_{ni}|^M E|X_i|^M I(|a_{ni}X_i| \leq b_n) + b_n^M P(|a_{ni}X_i| > b_n)\right) \\ &\leq C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \left(\sum_{i=1}^{n} |a_{ni}|^M E|X|^M I(|a_{ni}X| \leq b_n) + \sum_{i=1}^{n} b_n^M P(|a_{ni}X| > b_n)\right) \\ &\leq C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \sum_{i=1}^{n} |a_{ni}|^M E|X|^M I(|a_{ni}X| \leq b_n) + C\sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P(|a_{ni}X| > b_n) \\ &\triangleq J_{11} + J_{12}. \end{aligned} \tag{3.24}$$

By Lemma 3.4 and $E|X|^\alpha \log(1+|X|) < \infty$, it follows that

$$J_{12} = C\sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P(|a_{ni}X| > b_n) \leq E|X|^\alpha \log(1+|X|) < \infty. \tag{3.25}$$

Next, we will show that $J_{11} < \infty$. Divide $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ into three subsets $\{a_{ni} : |a_{ni}| \leq 1/(\log n)^m\}$, $\{a_{ni} : 1/(\log n)^m < |a_{ni}| \leq 1\}$ and $\{a_{ni} : |a_{ni}| > 1\}$, where $m = \dfrac{1}{(M-\alpha)}$. Then,

$$\begin{aligned} J_{11} &= C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \sum_{i=1}^{n} |a_{ni}|^M E|X|^M I(|a_{ni}X| \leq b_n) \\ &= C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \sum_{i:|a_{ni}| \leq 1/(\log n)^m} |a_{ni}|^M E|X|^M I(|a_{ni}X| \leq b_n) \\ &+ C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \sum_{i:1/(\log n)^m < |a_{ni}| \leq 1} |a_{ni}|^M E|X|^M I(|a_{ni}X| \leq b_n) \\ &+ C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \sum_{i:|a_{ni}| > 1} |a_{ni}|^M E|X|^M I(|a_{ni}X| \leq b_n) \\ &\triangleq J_{11}^{(1)} + J_{11}^{(2)} + J_{11}^{(3)}. \end{aligned} \tag{3.26}$$

By Lemma 3.4 and $E|X|^\alpha \log(1+|X|) < \infty$ again, we can get that

$$\begin{aligned} J_{11}^{(3)} &= C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \sum_{i:|a_{ni}| > 1} |a_{ni}|^M E|X|^M I(|a_{ni}X| \leq b_n) \\ &\leq E|X|^\alpha \log(1+|X|) < \infty \quad \text{for} \quad M > 2 \geq \alpha > 0. \end{aligned}$$

It follows from $\sum_{i:|a_{ni}|\le 1/(\log n)^m} |a_{ni}|^\alpha \le Cn(\log n)^{-m\alpha}$ that

$$\begin{aligned} J_{11}^{(1)} &= C\sum_{n=1}^{\infty} n^{-1}b_n^{-M} \sum_{i:|a_{ni}|\le 1/(\log n)^m} |a_{ni}|^M E|X|^M I(|a_{ni}X|\le b_n) \\ &\le C\sum_{n=1}^{\infty} n^{-1}b_n^{-\alpha} \sum_{i:|a_{ni}|\le 1/(\log n)^m} |a_{ni}|^\alpha E|X|^\alpha I(|a_{ni}X|\le b_n) \\ &\le CE|X|^\alpha \sum_{n=1}^{\infty} n^{-1}b_n^{-\alpha} \sum_{i:|a_{ni}|\le 1/(\log n)^m} |a_{ni}|^\alpha \\ &\le CE|X|^\alpha \sum_{n=1}^{\infty} n^{-1}n^{-1}(\log n)^{-1}n(\log n)^{-m\alpha} < \infty. \end{aligned} \tag{3.27}$$

It follows from $\sum_{i:1/(\log n)^m<|a_{ni}|\le 1} |a_{ni}|^M \le Cn$ and $m = \frac{1}{(M-\alpha)}$ for $M > 2$, $0 < \alpha \le$ ≤ 2 that

$$\begin{aligned} J_{11}^{(2)} &= C\sum_{n=1}^{\infty} n^{-1}b_n^{-M} \sum_{i:1/(\log n)^m<|a_{ni}|\le 1} |a_{ni}|^M E|X|^M I(|a_{ni}X|\le b_n) \\ &\le C\sum_{n=1}^{\infty} b_n^{-M} E|X|^M I\left(|X|\le b_n(\log n)^m\right) \\ &= C\sum_{n=1}^{\infty} b_n^{-M} \sum_{k=1}^{n} E|X|^M I((k-1)^{1/\alpha}(\log(k-1))^{m+1/\alpha} < |X| \le k^{1/\alpha}(\log k)^{m+1/\alpha}) \\ &= C\sum_{k=1}^{\infty} E|X|^M I((k-1)^{1/\alpha}(\log(k-1))^{m+1/\alpha} < |X| \le k^{1/\alpha}(\log k)^{m+1/\alpha}) \\ &\quad \times \sum_{n=k}^{\infty} n^{-M/\alpha}(\log n)^{-M/\alpha} \\ &\le C\sum_{k=1}^{\infty} E|X|^M I((k-1)^{1/\alpha}(\log(k-1))^{m+1/\alpha} < |X| \le k^{1/\alpha}(\log k)^{m+1/\alpha}) \\ &\quad \times k^{1-M/\alpha}(\log i)^{-M/\alpha} \\ &\le C\sum_{k=1}^{\infty} E|X|^\alpha I((k-1)^{1/\alpha}(\log(k-1))^{m+1/\alpha} < |X| \le k^{1/\alpha}(\log k)^{m+1/\alpha}) \\ &= CE|X|^\alpha < \infty. \end{aligned} \tag{3.28}$$

Finally, we will prove that $J_2 < \infty$. It follows from C_r inequality, Markov inequality, Lemma 3.3 and $E|X|^\alpha \log(1+|X|) < \infty$ that

$$\begin{aligned}
J_2 \quad &\triangleq C\sum_{n=1}^{\infty} n^{-1} b_n^{-M}\left(\sum_{i=1}^{n} E\left|Y_i - EY_i\right|^2\right)^{M/2} \\
&\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-M}\left(\sum_{i=1}^{n} E\left|Y_i\right|^2\right)^{M/2} \\
&\le C\sum_{n=1}^{\infty} n^{-1}\left(\sum_{i=1}^{n} P(|a_{ni}X_i| > b_n)\right)^{M/2} \\
&\quad + C\sum_{n=1}^{\infty} n^{-1} b_n^{-M}\left(\sum_{i=1}^{n} E|a_{ni}X_i|^2 I(|a_{ni}X_i| \le b_n)\right)^{M/2} \\
&\le C\sum_{n=1}^{\infty} n^{-1}\left(b_n^{-\alpha}\sum_{i=1}^{n} E|a_{ni}X|^\alpha\right)^{M/2} \\
&\quad + C\sum_{n=1}^{\infty} n^{-1}\left(\sum_{i=1}^{n} b_n^{-\alpha} E|a_{ni}X|^\alpha I(|a_{ni}X| \le b_n)\right)^{M/2} \\
&\le C\sum_{n=1}^{\infty} n^{-1}\left(b_n^{-\alpha}\sum_{i=1}^{n} E|a_{ni}X|^\alpha\right)^{M/2} \\
&\le C\sum_{n=1}^{\infty} n^{-1}(\log n)^{-M/2}(E|X|^\alpha)^{M/2} < \infty,
\end{aligned} \tag{3.29}$$

which together with $J_1 < \infty$ yields $J < \infty$. The proof of Theorem 2.2 is complete. □

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