Martin Steigemann; Maria Specovius-Neugebauer

On the change of energy caused by crack propagation in 3-dimensional anisotropic solids

Mathematica Bohemica, Vol. 139 (2014), No. 2, 401–416

Persistent URL: `http://dml.cz/dmlcz/143865`

139 (2014) MATHEMATICA BOHEMICA No. 2, 401–416

ON THE CHANGE OF ENERGY CAUSED BY CRACK PROPAGATION IN 3-DIMENSIONAL ANISOTROPIC SOLIDS

Martin Steigemann, Maria Specovius-Neugebauer, Kassel

(Received September 30, 2013)

Abstract. Crack propagation in anisotropic materials is a persistent problem. A general concept to predict crack growth is the energy principle: A crack can only grow, if energy is released. We study the change of potential energy caused by a propagating crack in a fully three-dimensional solid consisting of an anisotropic material. Based on methods of asymptotic analysis (method of matched asymptotic expansions) we give a formula for the decrease in potential energy if a smooth inner crack grows along a small crack extension.

Keywords: crack propagation; energy principle; stress intensity factor

MSC 2010: 74R10, 41A60, 35Q74

1. Introduction

Still today, crack propagation in structural components is a problem in many areas of modern engineering. While there are many ideas how to predict crack growth for plane problems, the situation is much more complicated in three dimensions and especially in anisotropic materials. A very general approach is the energy principle. Formulated by Griffith in 1921, a crack can only advance, if energy can be released [12]. Whereas this fracture criterion is easy to understand, the precise mathematical formulation and also the practical application in numerical simulations is still a challenge. Recent developments [1] give asymptotic formulas for the change of potential energy caused by crack propagation for plane problems and for plane cracks in three dimensional situations [2]. In this work we generalize these ideas to smooth crack surfaces of arbitrary shape completely contained in three-dimensional solids.

This work is based on investigations of the collaborative research center SFB Transregio 30, supported by the DFG.

Problems of this kind were considered by many authors since the early works of Irwin, Sih and co-workers, see e.g. [15], [27], [26], [13], [10]. Three-dimensional crack problems were intensively discussed by Leblond, Lazarus and co-workers [20], [19], see also [11] and the literature cited there for more details. We also want to refer to [3] for a deeper discussion about the energy principle.

We consider an anisotropic structure with an inner crack. The solid is represented by a bounded domain $G \subset \mathbb{R}^3$ with polygonal boundary ∂G, by $n(x)$ we denote the exterior unit normal vector in $x \in \partial G$. To describe the crack we fix a smooth simply connected submanifold $\Xi \subset \mathbb{R}^3$ with boundary Γ and to avoid technicalities we assume that Ξ is parameterized in the form

$$\Xi := \{(x_1, x_2, H(x_1, x_2)) \colon\ (x_1, x_2) \subset U\},$$

where $U \subset \mathbb{R}^2$ is a smoothly surrounded simply connected domain, and H is a smooth function defined on $\overline{U}$ at least. For a point $x \in \Xi$ we must distinguish between the approximation of x by points where $x_3 > H(x_1, x_2)$ or $x_3 < H(x_1, x_2)$, hence we understand Ξ as a union of two surfaces Ξ^+, Ξ^- with unit normal vectors

$$n^{\pm}(x_1, x_2) = \pm \frac{1}{\sqrt{1 + |\nabla H(x_1, x_2)|^2}} (\nabla H(x_1, x_2), -1)^{\top} \tag{1.1}$$

for $x \in \Xi^{\pm}$. It is convenient to think of vectors in $\mathbb{R}^3$ always as columns, $\top$ indicates the transposition. Since we deal with an internal crack we require $\overline{\Xi} = \Xi \cup \Gamma \subset G$, we put $\Omega := G \setminus \overline{\Xi}$, hence $\partial\Omega = \partial G \cup \Xi^+ \cup \Xi^- \cup \Gamma$, while the crack front is represented by the simple closed smooth curve Γ (see Figure 1).

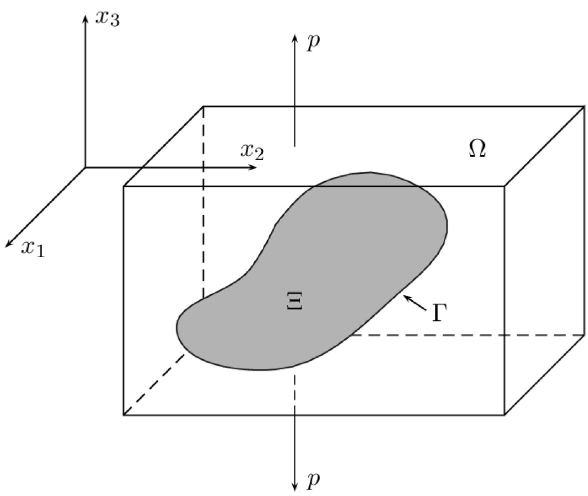

Figure 1. Elastic solid Ω with an inner crack Ξ.

We assume that the solid is under the influence of an external loading p, for simplicity no body forces are present, the crack surfaces are traction free and we deal with a linear elastic material behavior, which leads to the following Neumann problem:

$$
\begin{aligned}
-\nabla \cdot \boldsymbol{\sigma}(u;x) &= 0, \quad x \in \Omega,\\
\boldsymbol{\sigma}(u;x) \cdot n(x) &= 0, \quad x \in \Xi^{+} \cup \Xi^{-},\\
\boldsymbol{\sigma}(u;x) \cdot n(x) &= p(x), \quad x \in \partial G,
\end{aligned} \tag{1.2}
$$

where $u = (u_1, u_2, u_3)^\top$ denotes the displacement field. The dot "$\cdot$" indicates the usual matrix vector multiplication. The strain tensor $\boldsymbol{\varepsilon}(u;x) = \frac{1}{2}(\nabla u + (\nabla u)^\top)$ is related to the stress tensor $\boldsymbol{\sigma}$ by Hooke's law:

$$
\sigma_{ij}(u;x) = \sum_{k,l=1}^{3} a_{ij}^{kl} \varepsilon_{kl}(u;x), \quad i,j = 1,2,3, \tag{1.3}
$$

where the rank-4 tensor $\boldsymbol{A} = (a_{ij}^{kl})_{i,j,k,l=1,2,3}$ contains the elastic moduli and fulfills the usual symmetry and positivity conditions. We assume that p is smooth and self-balanced such that a solution $u \in H^1(\Omega)$ to problem (1.2) exists, unique up to rigid motions only.

2. Local curvilinear coordinates

In order to describe possible advance of the crack, we introduce local curvilinear coordinates in a tubular neighborhood of the crack front Γ. Global Cartesian coordinates are fixed by $x = (x_1, x_2, x_3)^\top$. Due to our assumptions on Ξ the crack front Γ is represented by a closed smooth regular curve which is parameterized by the arc length s, i.e., $\Gamma = \{\gamma(s) \colon s \in [0,l)\}$. In each point $x(s) \in \Gamma$ the tangent vector $\boldsymbol{t}(s) = (\mathrm{d}/\mathrm{ds})\gamma(s)$ and the normal plane perpendicular to $\boldsymbol{t}(s)$ exist and are well-defined.

In a vicinity of the crack front Γ, the intersection of the normal plane at arc length s with Ξ defines a curve $\eta(S,s)$ in $\mathbb{R}^3$, which is now parameterized by the arc length $S \in [0,\delta)$ for sufficiently small δ. We use the convention $\eta(0;s) = x(s) \in \Gamma$. For $S > 0$ the tangent $\boldsymbol{T}(S;s) = (\partial/\partial S)\eta(S;s)$ on the curve $\eta(\cdot;s)$ is contained in the tangential space of the surface Ξ at the point $x = \eta(S;s) \in \Xi$, moreover $|\boldsymbol{T}(S;s)| = 1$. If we set $\boldsymbol{E}_1(S;s) = \boldsymbol{T}(S;s)$ and

$$
\boldsymbol{E}_3(S;s) := n(\eta_1(S;s), \eta_2(S;s)) := n^{-}(\eta_1(S;s), \eta_2(S;s)),
$$

i.e., $\boldsymbol{E}_3$ is the unit normal vector (1.1), the so-called Darboux frame $\{\boldsymbol{E}_1(S;s), \boldsymbol{E}_2(S;s), \boldsymbol{E}_3(S;s)\}$ is completed by $\boldsymbol{E}_2(S;s) = \boldsymbol{E}_3(S;s) \times \boldsymbol{E}_1(S;s)$, see Figure 2. The Darboux frame defines an orthonormal basis of the Euclidean space attached to the point $\eta(S;s)$ of the crack surface Ξ. Because Ξ is a smooth submanifold, for any arc length parameter s the limits

$$\lim_{S\to 0} \boldsymbol{E}_j(S;s) =: \boldsymbol{E}_j(0;s), \quad j = 1,2,3,$$

exist and are well-defined. Moreover, $\boldsymbol{E}_2(0;s)$ is a unit tangent vector to the crack front, assumming that the curve γ is orientated in such a way that tangent $\boldsymbol{t}(s) = \frac{\mathrm{d}}{\mathrm{d}s}\gamma(s) = \boldsymbol{E}_2(0;s)$. Setting $\boldsymbol{n}(s) := -\boldsymbol{E}_1(0;s)$, $\boldsymbol{b}(s) :=\boldsymbol{E}_3(0;s)$, the frame $\{\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$ defines a positively oriented orthonormal basis of $\mathbb{R}^3$ at each point $x(s) \in \Gamma$, see Figure 2. Due to the Frenet-Serret formulas (see e.g. [18]) the change of the coordinate frame along the crack front can be calculated as:

$$\begin{aligned} \boldsymbol{t}'(s) &= \kappa_g(s)\boldsymbol{n}(s) + \kappa_n(s)\boldsymbol{b}(s), \\ \boldsymbol{n}'(s) &= -\kappa_g(s)\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{b}(s), \\ \boldsymbol{b}'(s) &= -\kappa_n(s)\boldsymbol{t}(s) - \tau_g(s)\boldsymbol{n}(s), \end{aligned} \tag{2.1}$$

with the normal and geodesic curvature $\kappa_n(s) := \boldsymbol{t}'(s)\cdot\boldsymbol{b}(s)$, $\kappa_g(s) := \boldsymbol{t}'(s)\cdot\boldsymbol{n}(s)$, and the geodesic torsion $\tau_g(s) := \boldsymbol{n}'(s)\cdot\boldsymbol{b}(s)$. The normal curvature $\kappa_n(s) = \cos(\sphericalangle(\boldsymbol{t}'(s), \boldsymbol{b}(s)))$ is the curvature of the crack front projected onto the plane spanned by $\boldsymbol{t}(s)$ and $\boldsymbol{b}(s)$, the geodesic curvature $\kappa_g(s)$ is the curvature of the crack front projected onto the surface's tangent plane. The geodesic torsion $\tau_g(s)$ is a measure for the change of the normal on the surface along the crack front. All these quantities are characteristic of the crack surface itself and are determined by the submanifold Ξ, see e.g. [18], [4] for more details.

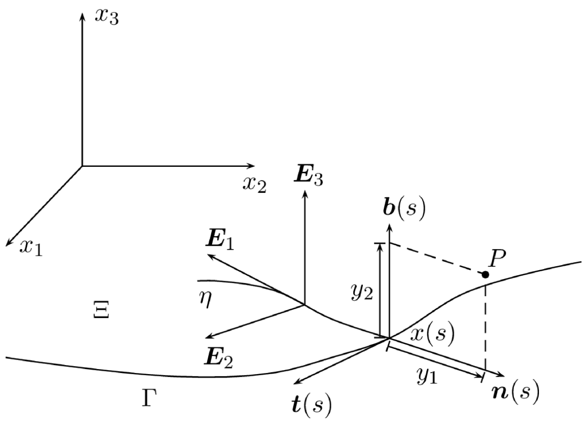

Figure 2. Point P in local coordinates near the crack front Γ.

Now we are able to define curvilinear coordinates. Since Γ is compact, we can find $\delta > 0$ such that for every x in the tubular neighborhood

$$\mathcal{T}_\delta(\Gamma) := \{x \in \mathbb{R}^3 \colon \operatorname{dist}(x, \Gamma) < \delta\} \tag{2.2}$$

there exist uniquely determined parameters y_1, y_2, s such that

$$x = x(s) + y_1 \boldsymbol{n}(s) + y_2 \boldsymbol{b}(s) =: \Theta(y_1, y_2, s), \tag{2.3}$$

i.e., the vector $Y := (y_1, y_2, s)^\top$ represents the curvilinear coordinates of a point $P \in \mathcal{T}_\delta(\Gamma)$. At each point $x = \Theta(Y) \in \mathcal{T}_\delta(\Gamma)$ the columns $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3\}$ of the matrix $\boldsymbol{g} := (\boldsymbol{g}_{ij}) = \nabla_Y \Theta$ define the so-called *covariant* basis of the tangent space ($\simeq \mathbb{R}^3$). From (2.1) and (2.3) it is clear that

$$\begin{aligned} \boldsymbol{g}_1(Y) &= \frac{\partial \Theta(Y)}{\partial y_1} = \boldsymbol{n}(s), \quad \boldsymbol{g}_2(Y) = \frac{\partial \Theta(Y)}{\partial y_2} = \boldsymbol{b}(s), \\ \boldsymbol{g}_3(Y) &= \frac{\partial \Theta(Y)}{\partial s} = (1 - \kappa_g(s) y_1 - \kappa_n(s) y_2) \boldsymbol{t}(s) + \tau_g(s)(y_1 \boldsymbol{b}(s) - y_2 \boldsymbol{n}(s)). \end{aligned} \tag{2.4}$$

We recall that the Riemannian metric tensor is $\boldsymbol{G} = (\boldsymbol{G}_{ij}) = (\nabla_Y \Theta)^\top \cdot (\nabla_Y \Theta)$. In a similar manner, the rows $\{\boldsymbol{g}^1, \boldsymbol{g}^2, \boldsymbol{g}^3\}$ of $(\nabla_Y \Theta)^{-1} = \boldsymbol{g}^{-1} = (\boldsymbol{g}^{ij})$ define the basis of $\mathbb{R}^3$, the so-called *contravariant* basis. The Jacobian of the transformation Θ reads $|\det(\nabla_Y \Theta)| = \sqrt{\det(\boldsymbol{G})} = |1 - \kappa_g(s) y_1 - \kappa_n(s) y_2|$. Thus, depending on the curvature, the transformation (2.3) is one-to-one only in a very small neighborhood around the crack front, since $\sqrt{\det(\boldsymbol{G})} = 0$ for $y_1 \kappa_g(s) + y_2 \kappa_n(s) = 1$.

The displacement field in Cartesian coordinates $u \colon \Omega \to \mathbb{R}^3$ with (smooth enough) components u_i can be rewritten near the crack front in curvilinear coordinates by the defining relation

$$u(x) = u_i(x) \boldsymbol{e}_i =: \hat{u}_i(Y) \boldsymbol{g}^i(Y) = \boldsymbol{g}^{-\top}(Y) \cdot \hat{u}(Y) \quad \text{for all } x = \Theta(Y) \in \mathcal{T}_\delta(\Gamma) \tag{2.5}$$

with sum convention and shorter notation $\boldsymbol{g}^{-\top} := (\boldsymbol{g}^{-1})^\top$. We assume that components of vector fields in global Cartesian coordinates $u(x) = (u_1(x), u_2(x), u_3(x))^\top$ are related to the standard unit basis of $\mathbb{R}^3$, $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, and identify the vector $\hat{u} = (\hat{u}_i)$ with the vector of covariant components, whereas $\hat{\boldsymbol{u}}(Y) := \boldsymbol{g}^{-\top}(Y) \cdot \hat{u}(Y) = u(x)$ is the field at point $x = \Theta(Y) \in \mathcal{T}_\delta(\Gamma)$, see e.g. [4] for more details.

3. Equilibrium equations in curvilinear coordinates

Using the representation (2.5), the rules of calculus imply

$$\partial_j v_i(x) = \hat{v}_{k\|l}(Y)\boldsymbol{g}^{ki}(Y)\boldsymbol{g}^{lj}(Y), \quad x = \Theta(Y) \in \mathcal{T}_\delta(\Gamma),$$

where $\hat{v}_{i\|j}(Y) := \partial_j \hat{v}_i(Y) - \Gamma^p_{ij}(Y)\hat{v}_p(Y)$ is the so-called covariant derivative and $\Gamma^p_{ij}(Y) := \boldsymbol{g}^p(Y)\cdot\partial_i\boldsymbol{g}_j(Y)$ are the Christoffel symbols of the second kind. For derivatives with respect to Y the choice of the basis (2.4) implies

$$\nabla_Y \hat{v} = (\partial_1\hat{v}, \partial_2\hat{v}, \partial_3\hat{v}), \quad \partial_1\hat{v} = \frac{\partial\hat{v}}{\partial y_1}, \quad \partial_2\hat{v} = \frac{\partial\hat{v}}{\partial y_2}, \quad \partial_3\hat{v} = \frac{\partial\hat{v}}{\partial s}. \tag{3.1}$$

The (covariant) components of the strain tensor in curvilinear coordinates for $i, j = 1, 2, 3$ are defined by the relation

$$\hat{\varepsilon}_{ij}(\hat{u}; Y) = \frac{1}{2}(\hat{u}_{i\|j}(Y) + \hat{u}_{j\|i}(Y)), \quad x = \Theta(Y) \in \mathcal{T}_\delta(\Gamma).$$

In a similar manner, the (contravariant) components of the stress tensor are

$$\hat{\sigma}^{ij}(Y) := \sigma_{kl}(x)\boldsymbol{g}^{ik}(Y)\boldsymbol{g}^{jl}(Y), \quad i, j = 1, 2, 3, \quad x = \Theta(Y) \in \mathcal{T}_\delta(\Gamma).$$

In curvilinear coordinates Hooke's law reads:

$$\hat{\sigma}^{ij}(Y) = \hat{a}^{kl}_{ij}(Y)\hat{\varepsilon}_{kl}(\hat{u}; Y)$$

and a short calculation shows

$$\hat{\sigma}^{ij}(Y) = a^{pq}_{kl}\hat{\varepsilon}_{mn}(\hat{u}; Y)\boldsymbol{g}^{mp}(Y)\boldsymbol{g}^{nq}(Y)\boldsymbol{g}^{ik}(Y)\boldsymbol{g}^{jl}(Y) =: \hat{a}^{mn}_{ij}(Y)\hat{\varepsilon}_{mn}(\hat{u}; Y).$$

Put $\widehat{\mathcal{T}}_\delta(\Gamma) := \{Y\colon\ \Theta(Y) \in \mathcal{T}_\delta(\Gamma)\}$, then the equations of equilibrium in curvilinear coordinates are:

$$-\hat{\sigma}^{ij}\|_j(Y) = 0 \quad \text{in } \widehat{\mathcal{T}}_\delta(\Gamma), \quad i = 1, 2, 3,$$

where $\hat{\sigma}^{ij}\|_j(Y) := \hat{\partial}_j\hat{\sigma}^{ij}(Y) + \Gamma^i_{pj}\hat{\sigma}^{pj}(Y) + \Gamma^j_{jq}\hat{\sigma}^{iq}(Y)$. Also, transforming the components of normal stresses to curvilinear coordinates, boundary conditions on the crack faces can be formulated as follows:

$$\hat{\sigma}^{ij}(Y)\hat{n}^\pm_j(Y) = 0 \quad \text{in } \Theta(Y) = x \in \Xi^\pm, \quad i = 1, 2, 3,$$

where $\hat{n}^\pm_j$ are the covariant components of the outer normal vector on $\Xi^\pm$. In curvilinear coordinates, we use the operator notation

$$-\nabla_\| \cdot \hat{\boldsymbol{\sigma}}(Y) =: \mathscr{L}(Y, \nabla_Y)\hat{u}(Y), \quad \hat{\boldsymbol{\sigma}}(Y)\cdot\hat{n}(Y) =: \mathscr{N}(Y, \nabla_Y)\hat{u}(Y). \tag{3.2}$$

4. Asymptotic behavior near the crack front

The asymptotic behavior near the tip of a crack in a plane structure was investigated in the early works from Williams [29] and Kondrat'ev [16]. These results were generalized to more complicated geometrical singularities and crack fronts in several works of Maz'ya, Plamenevsky and Nazarov [21], [24], [17] and also of Costabel & Dauge and co-workers [9], [5], [6], [7], [8]. The list of authors treating this kind of problems cannot be complete and we refer also to the works cited in the given literature.

In order to calculate the asymptotic expansion of the displacement field at the curved crack front we flatten out the crack and obtain a boundary value problem with nonconstant coefficients in a wedge with a straight edge and opening angle equal to 2π for all parameters s, see e.g. [24]. In local curvilinear coordinates y, the intersection of the normal plane with the crack surface can be represented as the graph $(y_1, \boldsymbol{h}(y_1,s))^\top$ of a mapping $\boldsymbol{h}$ for which we have the Taylor approximation

$$y_1 \mapsto \boldsymbol{h}(y_1,s) = \frac{\tau_g(s)}{2} y_1^2 + \mathcal{O}(y_1^3), \quad y_1 < 0. \tag{4.1}$$

The function $\boldsymbol{h}$ can be extended to small positive y_1 by $\boldsymbol{h}(y_1,s) := \boldsymbol{h}(-y_1,s)$, thus $\boldsymbol{h}(\cdot,s) \in C^2(-\delta,\delta)$. Now we introduce a second coordinate transformation to flatten out the crack surface Ξ near the crack front Γ, namely we put

$$Z := (z_1, z_2, s)^\top = \Pi(Y) := (y_1, y_2 - \boldsymbol{h}(y_1,s), s)^\top, \quad |y| < \delta(s). \tag{4.2}$$

We introduce the semi-infinite crack

$$\Upsilon_\infty = \{z \in \mathbb{R}^2\colon\ z_1 \leqslant 0,\ z_2 = 0\}$$

in the plane, and think of $\Upsilon_\infty = \Upsilon_\infty^+ \cup \Upsilon_\infty^-$, where $\Upsilon_\infty^\pm$ denote the upper and lower surface as usual. The transformation of the boundary value problem (1.2) restricted to $\mathcal{T}_{\delta_0}(\Gamma)$ leads to a boundary value problem

$$\left.\begin{aligned} \mathscr{L}(Z,\nabla_Z)\tilde{u}(Z) &= 0, \quad z \in \mathbb{R}^2 \setminus \Upsilon_\infty, \\ \mathscr{N}(Z,\nabla_Z)\tilde{u}(Z) &= 0, \quad z \in \Upsilon_\infty^+ \cup \Upsilon_\infty^-, \end{aligned}\right\} \quad |z| < \delta(s), \quad s \in [0,l),$$

where $\hat{u}(Y) =: (\nabla_Z \Pi^{-1}(Z))^{-\top} \cdot \tilde{u}(Z)$, $Y = \Pi^{-1}(Z)$. It is a classical result that in flat coordinates near the edge the displacement field admits an asymptotic representation of the well-known square-root characteristic [22], [5], [7]:

$$\begin{aligned} \tilde{u}(z,s) = C(s) &+ \sum_{j=1}^{3} K_j(s) r^{1/2} \Phi^{j,1}(\varphi) + \sum_{j=1}^{2} T_j(s) r \Phi^{j,2}(\varphi) + R(s) \begin{pmatrix} -z_2 \\ z_1 \\ 0 \end{pmatrix} \\ &+ \mathcal{O}(r^{3/2} \ln(r)), \quad r \to 0, \end{aligned} \tag{4.3}$$

where r, φ denote plane polar coordinates, i.e., $z = r(\cos(\varphi), \sin(\varphi))$. The coefficients $K_j(s)$, $T_j(s)$ are called stress intensity factors and T-stresses, respectively. $C(s)$ and $R(s)$ denote rigid motions in the normal plane, depending on the arc length s. The main asymptotic terms

$$U^{j,k}(z,s) = r^{k/2}\Phi^{j,k}(\varphi,s), \quad j = 1,2,3, \ k = 0,1,2$$

can be found as solutions of the homogeneous model problem

$$\begin{aligned} \mathscr{L}^0(\nabla_z, s)U^{j,k}(z,s) &= 0, \quad z \in \mathbb{R}^2 \setminus \Upsilon_\infty, \\ \mathscr{N}^0(\nabla_z, s)U^{j,k}(z,s) &= 0, \quad z \in \Upsilon_\infty^+ \cup \Upsilon_\infty^-, \end{aligned} \tag{4.4}$$

where the operators are obtained from expanding the coefficients near the crack front:

$$\begin{aligned} \mathscr{L}(Z, \nabla_Z) &= \mathscr{L}^0(\nabla_z, s) + \mathscr{L}^1(z, \nabla_z, s, \partial_s) + \ldots, \\ \mathscr{N}(Z, \nabla_Z) &= \mathscr{N}^0(\nabla_z, s) + \mathscr{N}^1(z, \nabla_z, s, \partial_s) + \ldots. \end{aligned}$$

Note that the arc length s merely appears as a parameter,

$$\mathscr{L}^0(\nabla_z, s) = \mathscr{L}(Z, \nabla_Z)\big|_{z=0,\partial_s=0}, \quad \mathscr{N}^0(\nabla_z, s) = \mathscr{N}(Z, \nabla_Z)\big|_{z=0,\partial_s=0},$$

hence $\mathscr{L}^0$, $\mathscr{N}^0$ are differential operators acting in the plane, depending only on ∂_{z_1}, ∂_{z_2} and the elastic constants a_{ij}^{kl} with $k + l \neq 6$. We refer e.g. to [24] for more details about higher order terms and the regularity of the coefficients in (4.3).

5. Calculation of stress intensity factors

For calculating stress intensity factors along the crack front, the power-law solutions with finite elastic energy near $z = 0$

$$U^{j,k}(z,s) := r^{k/2}\Phi^{j,k}(\varphi,s), \quad j = 1,2,3, \ k = 0,1,2,\ldots$$

have to be normalized in a mechanical reliable sense, see e.g. [25], [23]. There also exist singular power-law solutions to problem (4.4):

$$V^{i,l}(z,s) := r^{-l/2}\Psi^{i,l}(\varphi,s), \quad i = 1,2,3, \ l = 0,1,2,\ldots.$$

These solutions can be normalized by fulfilling an orthogonality condition in the following sense: for any smooth curve $\omega \subset \mathbb{R}^2$ around the crack tip connecting the crack faces and sufficiently smooth vector fields u and v we define the form

$$\mathcal{Q}(u, v; \omega) := \int_\omega (\mathscr{N}^0(\nabla_z, s)u(z,s) \cdot v(z,s) - u(z,s) \cdot \mathscr{N}^0(\nabla_z, s)v(z,s))\,\mathrm{d}o,$$

where do denotes the surface (line) element. The singular solutions $V^{i,l,0}$ can be normalized in such a way that

$$\mathcal{Q}(U^{j,k}, V^{i,l}; \omega) = \delta_{i,j}\delta_{k,l}, \quad i, j = 1, 2, 3, \ k, l = 0, 1, 2, \ldots \tag{5.1}$$

where $\delta_{i,j}$ is the Kronecker symbol and they are called the dual power law solutions. The integral is path-independent [24]. Following the classical approach of Maz'ya and Plamenevsky deriving integral formulae for SIFs [21], we look for solutions ζ of the homogeneous global problem

$$-\nabla \cdot \boldsymbol{\sigma}(\zeta; x) = 0, \quad x \in \Omega, \qquad \boldsymbol{\sigma}(\zeta; x) \cdot n(x) = 0, \quad x \in \partial\Omega. \tag{5.2}$$

Since the only solutions $u \in H^1(\Omega)$ of (5.2) are rigid motions, other solutions, the so-called weight functions, must have singularities at the crack front. For a smooth function F defined on the crack front there exist solutions of (5.2) with asymptotic decomposition in flat coordinates

$$\tilde{\zeta}^{j,k}(F; Z) = F(s) r^{-k/2} \Psi^{j,k,0}(\varphi, s) + \mathcal{O}(r^{(2-k)/2}), \quad k = 1, 2,$$

see e.g. [24, §11.4, §12.8] for more details. Of main importance for our considerations is the following integral representation for stress intensity factors, which can be obtained by the same arguments as in [24, §12.8], [2]:

Theorem 5.1. *For any smooth function $F(s)$ on the crack front γ, the following integral representation hold, see e.g.* [2]*:*

$$\int_{\partial G} p(x) \cdot \zeta^{j,1}(F; x)\, \mathrm{d}O = \int_\gamma F(s) K_j(s)\, \mathrm{d}s, \quad j = 1, 2, 3,$$
$$\int_{\partial G} p(x) \cdot \zeta^{j,2}(F; x)\, \mathrm{d}O = \int_\gamma F(s) T_j(s)\, \mathrm{d}s, \quad j = 1, 2.$$

6. Energy release caused by crack propagation

Let us assume that the crack has grown along a small area with new crack front $\Gamma(t)$. We define the crack extension in local curvilinear coordinates along the crack front Γ by

$$\Gamma(t) := \{x_t(s) = x(s) + th(s)(\cos(\vartheta(s))\boldsymbol{n}(s) + \sin(\vartheta(s))\boldsymbol{b}(s))\colon\ x(s) \in \Gamma\}.$$

Here, $t > 0$ is a small time-like parameter and $th(s)$ is the (small) length of the crack extension at arc length s into direction $\vartheta(s)$. The parameter t is independent of s, $h(s) \geqslant 0$ is a smooth function vanishing on parts of the crack front where no crack propagation appears. Moreover, $\vartheta(s)$ is the kink angle in the normal plane, for simplicity also assumed to be smooth in the arc length s (compare also with Figure 2). We assume further that the mapping

$$P_t\colon \Gamma = \Gamma(0) \to \Gamma(t) \quad \text{with} \quad P_t(x(s)) = x_t(s)$$

is one-to-one. Put $\Xi_t = \Xi \cup \{\Gamma(\tau)\colon\ 0 \leqslant \tau \leqslant t\}$ and $\Omega_t = G \setminus \Xi_t$. We remark that the surface Ξ_t does not inherit the smoothness of Ξ and is not necessarily a smooth submanifold of $\mathbb{R}^3$, because we allow kinking of the crack front along $\Gamma(0)$. Moreover, if $h(s) = 0$, the new crack front $\Gamma(t)$ is not necessarily a smooth curve and there can be points on $\Gamma(t)$, where local coordinates can not be defined uniquely.

Let u^t be the solution to (1.2) where Ω is replaced by Ω_t. Our approach to obtain the change of potential energy for small t is based on the method of matched asymptotic expansion. We approximate the displacement field u^t in some distance to the crack front in terms of the displacement field u^0 and certain functions. Close to the crack front $\Gamma(t)$ the influence on the displacements is much larger and here we approximate u^t by solutions of the so-called second limit problem.

In local curvilinear coordinates, we denote the crack shoot by

$$\Upsilon_{th(s)}(\vartheta(s)) = \{y\colon\ 0 < y_1 \leqslant th(s)\cos(\vartheta(s)),\ y_2 = y_1 \tan(\vartheta(s))\}.$$

Transformed to flat coordinates, the crack shoot is not necessarily a straight line segment and we use the following notation in flat coordinates, see (4.2):

$$\Upsilon_{th(s)}(s) = \{z\colon\ (z,s) = \Pi(y,s),\ y \in \Upsilon_{th(s)}(\vartheta(s))\}.$$

The whole plane without the kinked crack is denoted by

$$\Omega_\infty^{th(s)}(s) = \mathbb{R}^2 \setminus (\Upsilon_\infty \cup \Upsilon_{th(s)}(s)), \quad s \in [0,l).$$

Introducing stretched coordinates, $\xi := t^{-1}z$, and sending $t \to 0$, we obtain the so-called *second limit problem* in the plane with a semi-infinite kinked crack:

$$\begin{aligned} \mathscr{L}(\xi, \nabla_\xi, s, \partial_s) w(\xi, s) &= 0, \quad \xi \in \Omega_\infty^{h(s)}(s), \\ \mathscr{N}(\xi, \nabla_\xi, s, \partial_s) w(\xi, s) &= 0, \quad \xi \in \partial\Omega_\infty^{h(s)}(s),\ s \in [0,l). \end{aligned}$$

We use the method of matched asymptotic expansions, see [14], [25] and the literature cited there. Near to the crack shoot we approximate the displacement field u^t by an inner expansion, which reads in plane coordinates

$$\tilde{u}^t(h^{-1}z, s) = \tilde{u}^t(\xi, s) \sim w(\xi, s; t) = t^{1/2} w^1(\xi, s) + t w^2(\xi, s) + \dots .$$

In some distance to the crack front Γ, the displacement field u^t will not differ much from the displacement field u^0 of the initial configuration and here we approximate the displacement field by an outer expansion:

$$u^t(x) \sim v(x; t) = u^0(x) + v^1(x; t) + v^2(x; t) + \dots .$$

Inner and outer expansion approximate the same solution u^t only in different regions and must coincide for small $|z|$ and large $|\xi|$:

$$\{z\colon\ c_1 t^{1/2} < |z| < c_2 t^{1/2}\}.$$

Moreover, both expansions must fulfill the elasticity equations. The displacement field u^0 fulfills the boundary conditions on the outer boundary and the functions v^j must be solutions of the homogeneous problem, only nontrivial if singular at the crack front, and so combinations of the weight functions $\zeta^{j,k}$. Decomposing the operator into a series,

$$\begin{aligned}
\mathscr{L}(\xi, \nabla_\xi, \partial_s, s) w(\xi, s; t) &= t^{-3/2} \mathscr{L}^0(\nabla_\xi, s) w^1(\xi, s) + t^{-1} \mathscr{L}^0(\nabla_\xi, s) w^2(\xi, s) \\
&\quad + t^{-1/2} (\mathscr{L}^1(\xi, \nabla_\xi, \partial_s, s) w^1(\xi, s) + \mathscr{L}^0(\nabla_\xi, s) w^3(\xi, s; t)) + \dots \\
\mathscr{N}(\xi, \nabla_\xi, \partial_s; s) w(\xi, s; t) &= t^{-1/2} \mathscr{N}^0(\nabla_\xi; s) w^1(\xi, s) + \mathscr{N}^0(\nabla_\xi; s) w^2(\xi, s) \\
&\quad + t^{1/2} (\mathscr{N}^1(\xi, \nabla_\xi, \partial_s; s) w^1(\xi, s) + \mathscr{N}^0(\nabla_\xi; s) w^3(\xi, s; t)) + \dots
\end{aligned}$$

the equations for the first terms of the inner expansion read

$$\begin{aligned}
\mathscr{L}^0(\nabla_\xi, s) w^k(\xi; s) &= 0, \quad \xi \in \Omega_\infty^{h(s)}(s), \\
\mathscr{N}^0(\nabla_\xi, s) w^k(\xi; s) &= 0, \quad \xi \in \partial\Omega_\infty^{h(s)}(s),
\end{aligned} \qquad k = 1, 2.$$

We construct the functions w^1 and w^2. The asymptotic decomposition of u^0 near the crack front rewritten in ξ-coordinates reads

$$\begin{aligned}
\tilde{u}^0(z, s) &= r^{1/2} \Big(\sum_{j=1}^{3} K_j(s) \Phi^{j,1}(\varphi) \Big) + r \Big(\sum_{j=1}^{2} T_j(s) \Phi^{j,2}(\varphi) \Big) + \dots \\
&= t^{1/2} \varrho^{1/2} \Big(\sum_{j=1}^{3} K_j(s) \Phi^{j,1}(\varphi) \Big) + t\varrho \Big(\sum_{j=1}^{2} T_j(s) \Phi^{j,2}(\varphi) \Big) + \dots
\end{aligned} \tag{6.1}$$

and we look for the first two terms in the form

$$w^1(\xi,s)=\sum_{j=1}^{3}K_j(s)w^{j,1}(\xi,s)=\sum_{j=1}^{3}K_j(s)(\varrho^{1/2}\Phi^{j,1}(\varphi,s)+w^{j,1,0}(\xi,s)),$$
$$w^2(\xi,s)=\sum_{j=1}^{2}T_j(s)w^{j,2}(\xi,s)=\sum_{j=1}^{2}T_j(s)(\varrho\Phi^{j,2}(\varphi,s)+w^{j,2,0}(\xi,s)).$$

If $h(s)>0$, the functions $\varrho^{k/2}\Phi^{j,k}(\varphi,s)$, $k=1,2$ do not fulfill homogeneous boundary conditions on the kink $\Upsilon_{h(s)}(\vartheta(s))$ and the equations for $w^{j,k,0}$ read

$$\begin{aligned}\mathscr{L}^0(\nabla_\xi,s)w^{j,k,0}(\xi,s)&=0,\quad \xi\in\Omega_\infty^{h(s)}(s),\\ \mathscr{N}^0(\nabla_\xi,s)w^{j,k,0}(\xi,s)&=-\mathscr{N}^0(\nabla_\xi,s)\varrho^{k/2}\Phi^{j,k}(\varphi,s)\\ &=:g^{j,k}(\xi,s),\quad \xi\in\partial\Omega_\infty^{h(s)}(s).\end{aligned}\tag{6.2}$$

To solve these equations and get the dependency on the length $h(s)$ of the kinked crack, we fix the parameter s and assume $h(s)>0$. Now, we transform the problem to a domain with a crack of fixed length one: $\boldsymbol{\xi}:=h^{-1}\xi$. The boundary value problem (6.2) transforms to

$$\begin{aligned}\mathscr{L}^0(\nabla_{\boldsymbol{\xi}})W^{j,k}(\boldsymbol{\xi},s)&=0,\quad \boldsymbol{\xi}\in\Omega_\infty^1(s),\\ \mathscr{N}^0(\nabla_{\boldsymbol{\xi}})W^{j,k}(\boldsymbol{\xi},s)&=G^{j,k}(\boldsymbol{\xi},s),\quad \boldsymbol{\xi}\in\partial\Omega_\infty^1(s),\end{aligned}$$

where $W^{j,k}(\boldsymbol{\xi},s)=w^{j,k,0}(h\boldsymbol{\xi},s)$ and the right-hand side $G^{j,k}(\boldsymbol{\xi},s)=hg^{j,k}(h\boldsymbol{\xi},s)$. For readability, we drop the parameter s now. For investigating the solvability conditions and asymptotic behavior of such solutions, the appropriate framework are weighted Sobolev spaces. As shown e.g. in [24], [1], [28], there exist unique solutions $W^{j,k}$ with asymptotic behavior for $\boldsymbol{\varrho}\to\infty$:

$$W^{j,k}(\boldsymbol{\xi})=\sum_{i=1}^{3}(M_{i,1}^{j,k}(\Upsilon(s);h)\boldsymbol{\varrho}^{-1/2}\Psi^{i,1}(\varphi)+M_{i,2}^{j,k}(s;h)\boldsymbol{\varrho}^{-1}\Psi^{i,2}(\varphi))+\dots.$$

Here, the coefficients $M_{i,l}^{j,k}(\Upsilon(s);h)$ depend only on the geometry of the crack shoot $\Upsilon(s)$ (and elasticity properties at arc length s). Also shown in [24], the coefficients can be calculated to

$$\begin{aligned}M_{i,l}^{j,k}(\Upsilon(s);h)&=\sum_{\pm}\left(\int_{\Upsilon^\pm(\vartheta)}W^{i,l}(\boldsymbol{\xi})\mathscr{N}^0(\nabla_{\boldsymbol{\xi}})W^{j,k}(\boldsymbol{\xi})\,\mathrm{d}S\right)\\ &=-h^{k/2}\sum_{\pm}\left(\int_{\Upsilon^\pm(\vartheta)}W^{i,l}(\boldsymbol{\xi})\mathscr{N}^0(\nabla_{\boldsymbol{\xi}})\boldsymbol{\varrho}^{k/2}\Phi^{j,k}(\varphi)\,\mathrm{d}S\right)=:h^{k/2}M_{i,l}^{j,k}(\Upsilon(s)).\end{aligned}$$

We remark $M^{3,2}_{i,l} = M^{i,l}_{3,2} = 0$. The geometry of the shoot $\Upsilon(s)$ depends on the curvature of the crack surface Ξ^0 and the kink angle $\vartheta(s)$ at arc length s. The elements $M^{j,k}_{i,l}$ are symmetric in the sense $M^{j,k}_{i,l} = M^{i,l}_{j,k}$, see e.g. [1], [28].

We remark that if $h(s) = 0$, the domain is the whole plane with the semi-infinite crack and the power-law solutions $\varrho^{k/2}\Phi^{j,k}(\varphi, s)$, $k = 1, 2$, fulfill homogeneous boundary conditions on $\partial\Omega^{h(s)}_\infty(s)$. This means $g^{j,k}(\zeta, s) = 0$ and the coefficients $M^{j,k}_{i,l}$ are zero. Of course, if there is no crack extension, there is nothing to do.

The transformation back leads to the asymptotic behavior of the solutions $w^{j,k,0}$:

$$\begin{aligned} w^{j,k,0}(\xi, s) = \sum_{i=1}^{3} (h(s)^{(k+1)/2} M^{j,k}_{i,1}(\vartheta(s))\varrho^{-1/2}\Psi^{i,1}(\varphi, s) \\ + h(s)^{(k+2)/2} M^{j,k}_{i,2}(\vartheta(s))\varrho^{-1}\Psi^{i,2}(\varphi, s)) + \mathcal{O}(h(s)^{(k+3)/2}\varrho^{-3/2}), \quad \varrho \to \infty. \end{aligned}$$

Rewriting the decomposition at infinity in local coordinates, we find

$$\begin{aligned} t^{1/2}w^1(\xi, s) + tw^2(\xi, s) = {} & \sum_{j=1}^{3} K_j(s)U^{j,1}(z, s) + \sum_{j=1}^{2} T_j(s)U^{j,2}(z, s) \\ & + \sum_{i=1}^{3}\sum_{j=1}^{3} [th(s)(K_j(s)M^{j,1}_{i,1}(\vartheta(s))V^{i,1}(z, s)) \\ & + t^{3/2}h(s)^{3/2}(K_j(s)M^{j,1}_{i,2}(\vartheta(s))V^{i,2}(z, s))] \\ & + \sum_{i=1}^{3}\sum_{j=1}^{2} [t^{3/2}h(s)^{3/2}(T_j(s)M^{j,2}_{i,1}(\vartheta(s))V^{i,1}(z, s)) \\ & + t^2h(s)^2(T_j(s)M^{j,2}_{i,2}(\vartheta(s))V^{i,2}(z, s))] + \dots . \end{aligned} \tag{6.3}$$

As mentioned previously, the functions $v^1(\cdot; t)$ and $v^2(\cdot; t)$ in the outer expansion must contain singular functions at the crack front and have to fulfill the homogeneous elasticity problem (because the displacement field u^0 fulfills boundary conditions). Comparing the asymptotic decomposition (6.1) with (6.3), we see that

$$v^1(x; t) = t\left(\sum_{j=1}^{3} \zeta^{j,1}(a^{j,1}; x)\right), \quad v^2(x; t) = t^{3/2}\left(\sum_{j=1}^{2} \zeta^{j,2}(a^{j,2}; x)\right)$$

and the coefficients have to admit an expansion in t and depend on s:

$$a^{i,1} = h(s)\left(\sum_{j=1}^{3} K_j(s)M^{j,1}_{i,1}(\vartheta(s))\right) + t^{1/2}h(s)^{3/2}\left(\sum_{j=1}^{2} T_j(s)M^{j,2}_{i,1}(\vartheta(s))\right) + \dots,$$

$$a^{i,2} = h(s)^{3/2}\left(\sum_{j=1}^{3} K_j(s)M^{j,1}_{i,2}(\vartheta(s))\right) + t^{1/2}h(s)^2\left(\sum_{j=1}^{2} T_j(s)M^{j,2}_{i,2}(\vartheta(s))\right) + \dots .$$

With these asymptotic expansions and technical results at hand, the change of energy can be calculated asymptotically. If we assume that the outer boundary ∂G is not deformed (much) by the applied load p, the potential energy at time t can be calculated using Green's formula as

$$\boldsymbol{U}(t) = -\frac{1}{2}\int_{\partial G} p(x)\cdot u^t(x)\,\mathrm{d}O.$$

Here, u^t denotes the displacement field of Ω_t with the crack Ξ_t. Then the change of potential energy caused by the propagated crack reads

$$\Delta\boldsymbol{U} := \boldsymbol{U}(t) - \boldsymbol{U}(0) = -\frac{1}{2}\int_{\partial G} p(x)\cdot\left(u^t(x) - u^0(x)\right)\mathrm{d}O.$$

Using the outer expansion and formula (5.3), we calculate

$$\begin{aligned}
\Delta\boldsymbol{U} &= -\frac{1}{2}\int_{\partial G} p(x)\cdot\left(u^0(x) + tv^1(x) + t^{3/2}v^2(x) - u^0(x)\right)\mathrm{d}O + \mathcal{O}(t^2)\\
&= -\frac{1}{2}\Bigg[t\int_{\partial G}\left(\sum_{j=1}^{3} p(x)\cdot\zeta^{j,1}(a^{j,1};x)\right)\mathrm{d}O\\
&\quad + t^{3/2}\int_{\partial G}\left(\sum_{j=1}^{2} p(x)\cdot\zeta^{j,2}(a^{j,2};x)\right)\mathrm{d}O\Bigg] + \mathcal{O}(t^2)\\
&= -\frac{1}{2}\Bigg[t\int_{\gamma}\left(\sum_{j=1}^{3} a^{j,1}(s)K_j(s)\right)\mathrm{d}s + t^{3/2}\int_{\gamma}\left(\sum_{j=1}^{2} a^{j,2}(s)T_j(s)\right)\mathrm{d}s\Bigg] + \mathcal{O}(t^2)\\
&= -\frac{1}{2}\Bigg[t\int_{\gamma} h(s)\Bigg(\sum_{i=1}^{3}\sum_{j=1}^{3} K_i(s)M^{i,1}_{j,1}(\vartheta(s))K_j(s)\Bigg)\mathrm{d}s\\
&\quad + 2t^{3/2}\int_{\gamma} h(s)^{3/2}\Bigg(\sum_{i=1}^{3}\sum_{j=1}^{2} K_i(s)M^{i,1}_{j,2}(\vartheta(s))T_j(s)\Bigg)\mathrm{d}s\Bigg] + \mathcal{O}(t^2).
\end{aligned}$$

This is a generalization of the asymptotic formula for the change of potential energy given in [1] for plane crack problems to three dimensions and also an extension of the results for crack surfaces contained in a plane obtained in [2] to arbitrarily shaped crack surfaces.

[1] *I. I. Argatov, S. A. Nazarov*: Energy release caused by the kinking of a crack in a plane anisotropic solid. Translated from the Russian. J. Appl. Math. Mech. *66* (2002), 491–503.

[2] *M. Bach, S. A. Nazarov, W. L. Wendland*: Stable propagation of a mode-1 planar crack in an isotropic elastic space. Comparison of the Irwin and the Griffth approaches. Problemi attuali dell'analisi e della fisica matematica (P. E. Ricci, ed.). Dipartimento di Matematica, Univ. di Roma, 2000, pp. 167–189.

[3] *B. Bourdin, G. A. Francfort, J.-J. Marigo*: Numerical experiments in revisited brittle fracture. J. Mech. Phys. Solids *48* (2000), 797–826.

[4] *P. G. Ciarlet*: An introduction to differential geometry with applications to elasticity. J. Elasticity *78-79* (2005), 1–215.

[5] *M. Costabel, M. Dauge*: General edge asymptotics of solutions of second-order elliptic boundary value problems I. Proc. R. Soc. Edinb., Sect. A *123* (1993), 109–155.

[6] *M. Costabel, M. Dauge*: General edge asymptotics of solutions of second-order elliptic boundary value problems II. Proc. R. Soc. Edinb., Sect. A *123* (1993), 157–184.

[7] *M. Costabel, M. Dauge*: Crack singularities for general elliptic systems. Math. Nachr. *235* (2002), 29–49.

[8] *M. Costabel, M. Dauge, Z. Yosibash*: A quasi-dual function method for extracting edge stress intensity functions. SIAM J. Math. Anal. *35* (2004), 1177–1202.

[9] *M. Dauge*: Elliptic Boundary Value Problems on Corner Domains. Lecture Notes in Mathematics 1341, Springer, Berlin, 1988.

[10] *H. G. de Lorenzi*: On the energy release rate and the J-integral for 3-D crack configurations. Int. J. Fract. *19* (1982), 183–193.

[11] *E. Favier, V. Lazarus, J.-B. Leblond*: Coplanar propagation paths of 3D cracks in infinite bodies loaded in shear. Int. J. Solids Struct. *43* (2006), 2091–2109.

[12] *A. A. Griffith*: The phenomena of rupture and flow in solids. Philos. Trans. Roy. Soc. London *221* (1921), 163–198.

[13] *R. J. Hartranft, G. C. Sih*: Stress singularity for a crack with an arbitrarily curved front. Engineering Fracture Mechanics *9* (1977), 705–718.

[14] *A. M. Il'in*: Matching of Asymptotic Expansions of Solutions of Boundary Value Problems. Translated from the Russian. Translations of Mathematical Monographs 102, American Mathematical Society, Providence, 1992.

[15] *G. Irwin*: Fracture. Handbuch der Physik. Bd. 6: Elastizität und Plastizität (S. Flügge, ed.). Springer, Berlin, 1958, pp. 551–590.

[16] *V. A. Kondrat'ev*: Boundary value problems for elliptic equations in domains with conical or angular points. Trans. Mosc. Math. Soc. *16* (1967), 227–313.

[17] *V. A. Kozlov, V. G. Maz'ya, J. Rossmann*: Elliptic Boundary Value Problems in Domains with Point Singularities. Mathematical Surveys and Monographs 52, American Mathematical Society, Providence, 1997.

[18] *W. Kühnel*: Differential Geometry. Curves–Surfaces–Manifolds. Translated from the German. Student Mathematical Library 16, American Mathematical Society, Providence, 2002.

[19] *V. Lazarus*: Brittle fracture and fatigue propagation paths of 3D plane cracks under uniform remote tensile loading. Int. J. Fract. *122* (2003), 23–46.

[20] *J.-B. Leblond, O. Torlai*: The stress field near the front of an arbitrarily shaped crack in a three-dimensional elastic body. J. Elasticity *29* (1992), 97–131.

[21] *V. G. Maz'ya, B. A. Plamenevsky*: The coefficients in the asymptotic of the solutions of elliptic boundary-value problems in domains with conical points. Math. Nachr. *76* (1977), 29–60. (In Russian.)

[22] *V. G. Maz'ya, J. Rossmann*: Über die Asymptotik der Lösungen elliptischer Randwertaufgaben in der Umgebung von Kanten. Math. Nachr. *138* (1988), 27–53. (In German.)
[23] *S. A. Nazarov*: Stress intensity factors and crack deviation conditions in a brittle anisotropic solid. J. Appl. Mech. Techn. Phys. *46* (2005), 386–394.
[24] *S. A. Nazarov, B. A. Plamenevsky*: Elliptic Problems in Domains with Piecewise Smooth Boundaries. De Gruyter Expositions in Mathematics 13, Walter de Gruyter, Berlin, 1994.
[25] *S. A. Nazarov, O. R. Polyakova*: Rupture criteria, asymptotic conditions at crack tips, and selfadjoint extensions of the Lamé operator. Tr. Mosk. Mat. Obs. *57* (1996), 16–74. (In Russian.)
[26] *D. M. Parks*: A stiffness derivative finite element technique for determination of crack tip stress intensity factors. Int. J. Fract. *10* (1974), 487–502.
[27] *G. C. Sih, P. C. Paris, G. R. Irwin*: On cracks in rectilinearly anisotropic bodies. Int. J. Fract. *1* (1965), 189–203.
[28] *M. Steigemann*: Verallgemeinerte Eigenfunktionen und lokale Integralcharakteristiken bei quasi-statischer Rissausbreitung in anisotropen Materialien. Berichte aus der Mathematik, Shaker, Aachen, 2009. (In German.)
[29] *M. L. Williams*: Stress singularities resulting from various boundary conditions in angular corners of plates in extension. J. Appl. Mech. *19* (1952), 526–528.

Authors' address: *Martin Steigemann*, *Maria Specovius-Neugebauer*, Department of Mathematics and Natural Sciences, Institute of Mathematics, University of Kassel, Heinrich-Plett-Str. 40, 34132 Kassel, Germany, e-mail: `martin.steigemann@mathematik.uni-kassel.de`, `specovi@mathematik.uni-kassel.de`.