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ABSTRACT THEORY OF VARIATIONAL INEQUALITIES WITH LAGRANGE MULTIPLIERS AND APPLICATION TO NONLINEAR PDES

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Abstract. Recently, we established some generalizations of the theory of Lagrange multipliers arising from nonlinear programming in Banach spaces, which enable us to treat not only elliptic problems but also parabolic problems in the same generalized framework. The main objective of the present paper is to discuss a typical time-dependent double obstacle problem as a new application of the above mentioned generalization. Actually, we describe it as a usual parabolic variational inequality and then characterize it as a parabolic inclusion by using the Lagrange multiplier and the nonlinear maximal monotone operator associated with the time differential under time-dependent double obstacles.

Keywords: Lagrange multiplier; parabolic variational inequality

MSC 2010: 34G25, 35K86, 47J20

1. Introduction

Let $N \in \mathbb{N}$, $0 < T < \infty$, let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\Gamma := \partial\Omega$, $Q := (0,T) \times \Omega$ and $\Sigma := (0,T) \times \Gamma$. We put $H := L^2(\Omega)$, $V := W_0^{1,2}(\Omega)$ with the usual norms and denote by V^* the dual space of V. These are Hilbert spaces with standard inner products and $V \hookrightarrow\hookrightarrow H \hookrightarrow\hookrightarrow V^*$ holds with dense and compact imbeddings.

In this paper, for a given constant k_0 and functions ψ_0, ψ_1 in $C([0,T]; W^{1,2}(\Omega))$, we consider a time-dependent double obstacle problem with volume constraint of the following form (P): Find a function $u\colon\ Q \to \mathbb{R}$ satisfying

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{in } Q(u) := \{(t,x) \in Q;\ \psi_0(t,x) < u(t,x) < \psi_1(t,x)\},$$
$$\frac{\partial u}{\partial t} - \Delta u \geqslant f \quad \text{in } Q_0(u) := \{(t,x) \in Q;\ u(t,x) = \psi_0(t,x)\},$$

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \Delta u \leqslant f \quad \text{in } Q_1(u) := \{(t,x) \in Q;\ u(t,x) = \psi_1(t,x)\},\\
&\frac{\partial u}{\partial \boldsymbol{n}} = \frac{\partial \psi_i}{\partial \boldsymbol{n}} \quad \text{on } Q \cap \partial Q_i(u) \text{ for } i = 0,1,\\
&u = 0 \quad \text{on } \Sigma,\\
&u(0,\cdot) = u_0(\cdot) \quad \text{on } \Omega,
\end{aligned}$$

along with the bi-lateral constraint

$$\psi_0(t,\cdot) \leqslant u(t,\cdot) \leqslant \psi_1(t,\cdot) \quad \text{a.e. on } \Omega \text{ for all } t \in [0,T], \tag{1.1}$$

and the volume constraint

$$\int_0^T \int_\Omega u(t,x)\,\mathrm{d}x\,\mathrm{d}t \leqslant k_0, \tag{1.2}$$

where $\boldsymbol{n}$ is the unit (spatial) normal vector on $Q \cap \partial Q_i(u)$ outward from $\{x \in \Omega;\ \psi_0(t,x) < u(t,x) < \psi_1(t,x)\}$ to $\{x \in \Omega;\ u(t,x) = \psi_i(t,x)\}$, $i = 0,1$.

2. Main theorem

In this section, we give the weak formulation of (P) as a parabolic variational inequality and show that it has a solution. Moreover, the solution is characterized by that of an abstract inclusion with Lagrange multiplier.

2.1. Definition of the solution and existence theorem. We define a time-dependent convex and closed subset $K(t)$ of V for all $t \in [0,T]$ as follows:

$$K(t) := \{z \in V;\ \psi_0(t,\cdot) \leqslant z \leqslant \psi_1(t,\cdot) \quad \text{a.e. in } \Omega\}.$$

Moreover, we define a linear and continuous functional $\Psi\colon L^2(0,T;V) \to \mathbb{R}$ by $\Psi(\xi) := \int_0^T \int_\Omega \xi(t,x)\,\mathrm{d}x\,\mathrm{d}t$ for all $\xi \in L^2(0,T;V)$. Now, for some given $k_0 \in \mathbb{R}$, we put

$$\begin{aligned}
&\mathcal{C}_{k_0} := \{\xi \in L^2(0,T;V);\ \Psi(\xi) \leqslant k_0\},\\
&\mathcal{K} := \{\xi \in L^2(0,T;V);\ \xi \in K(t) \quad \text{for a.a. } t \in [0,T]\},\\
&\mathcal{K}_0 := \{\xi \in \mathcal{K};\ \xi' \in L^2(0,T;V^*)\};
\end{aligned}$$

note that these sets are convex and closed in $L^2(0,T;V)$. Denote by $\mathcal{F}$ the duality mapping from $L^2(0,T;V)$ onto $L^2(0,T;V^*)$.

Definition 2.1. A function $u \in C([0,T];H) \cap L^2(0,T;V)$ is called a weak solution of (P) if u satisfies $u \in \mathcal{K} \cap \mathcal{C}_{k_0}$ and

$$\begin{aligned}(2.1)\quad &\int_0^T \langle \eta'(t), u(t) - \eta(t)\rangle_{V^*,V}\,\mathrm{d}t + \int_0^T a\big(u(t), u(t) - \eta(t)\big)\,\mathrm{d}t \\ &\leqslant \frac{1}{2}|u_0 - \eta(0)|_H^2 + \int_0^T \langle f(t), u(t) - \eta(t)\rangle_{V^*,V}\,\mathrm{d}t \quad \text{for all } \eta \in \mathcal{K}_0 \cap \mathcal{C}_{k_0},\end{aligned}$$

where $a(\cdot,\cdot)\colon V \times V \to \mathbb{R}$ is the bilinear form of the Laplacian, namely $a(w,z) := \int_\Omega \nabla w(x) \cdot \nabla z(x)\,\mathrm{d}x$ for any $w, z \in V$.

Our first theorem is concerned with the existence of a weak solution of (P).

Theorem 2.1. *Assume that* $\Psi(z_0) < k_0$ *for some* $z_0 \in \mathcal{K}_0$, $f \in L^2(0,T;H)$ *and* $u_0 \in K(0)$. *Moreover, assume that* $\psi_0, \psi_1 \in W^{1,2}(0,T;W^{1,2}(\Omega)) \cap L^\infty(Q)$, $\psi_1 - \psi_0 \geqslant c_0$ *a.e. in* Q *for some constant* $c_0 > 0$ *and*

$$(2.2)\qquad \psi_0(t,x) \leqslant 0 \leqslant \psi_1(t,x) \quad \text{for a.e. } x \in \Gamma,$$

$$(2.3)\qquad \psi_0(t,x)(\psi_1(s,x) - \psi_0(s,x)) = \psi_0(s,x)(\psi_1(t,x) - \psi_0(t,x)) \quad \text{for a.e. } x \in \Gamma,$$

for all $s, t \in [0,T]$. *Then there exists a weak solution of* (P).

2.2. Nonlinear operator $\mathcal{L}_{u_0}$. We recall a nonlinear maximal monotone operator (cf. [9]), $\mathcal{L}_{u_0}\colon L^2(0,T;V) \to 2^{L^2(0,T;V^*)}$, which is the time differential $\mathrm{d}/\mathrm{d}t$ under constraint (1.1).

Definition 2.2. Let $u_0 \in \overline{K(0)}^H$. Then $g \in \mathcal{L}_{u_0}u$ in $L^2(0,T;V^*)$ if and only if $u \in \mathcal{K}$, $g \in L^2(0,T;V^*)$ and

$$\int_0^T \langle \eta'(t) - g(t), u(t) - \eta(t)\rangle_{V^*,V}\,\mathrm{d}t - \frac{1}{2}|u_0 - \eta(0)|_H^2 \leqslant 0 \quad \text{for all } \eta \in \mathcal{K}_0.$$

We define $A\colon L^2(0,T;V) \to L^2(0,T;V^*)$ by $\langle\langle Au, \eta\rangle\rangle := \int_0^T a(u(t),\eta(t))\,\mathrm{d}t$ for all $u, \eta \in L^2(0,T;V)$, where $\langle\langle\cdot,\cdot\rangle\rangle$ is the duality pairing between $L^2(0,T;V)$ and $L^2(0,T;V^*)$. Moreover, we denote by ∂_* the subdifferential from $L^2(0,T;V)$ into $L^2(0,T;V^*)$. With the above operator $\mathcal{L}_{u_0}$, the weak solution obtained by Theorem 2.1 is characterized as follows.

Theorem 2.2. *Under the assumptions of Theorem* 2.1, *there exist* $u \in \mathcal{K} \cap \mathcal{C}_{k_0}$ *and* $\lambda \in \mathbb{R}$ *such that*

$$\mathcal{L}_{u_0} u + Au + \lambda \partial_* \Psi(u) \ni f \quad \text{in } L^2(0,T;V^*), \tag{2.4}$$

$$\lambda \geqslant 0, \quad \Psi(u) - k_0 \leqslant 0, \quad \lambda(\Psi(u) - k_0) = 0. \tag{2.5}$$

We first prove Theorem 2.2 by applying the abstract theory on Lagrange multipliers for variational inequalities stated in the next subsection and then prove that (2.1) is equivalent to (2.4)–(2.5), which shows that Theorem 2.1 holds.

2.3. Related known results. The time-independent case of ψ_0, ψ_1 double obstacle problems were treated for instance in [1], [4], [11] and these results were extended to the case of time-dependent constraint in [14]. Also, some related results are found (e.g., [2], [3], [8]) in connection with optimization problems. Now, we recall the general theory [7] as mentioned below.

Let $\mathcal{V}$ be a real reflexive and strictly convex Banach space and $\mathcal{V}^*$ the dual space of $\mathcal{V}$, let $\mathcal{A}\colon D(\mathcal{A}) \subset \mathcal{V} \to 2^{\mathcal{V}^*}$ be a maximal monotone operator, and $\Psi\colon D(\Psi) = \mathcal{V} \to [0,\infty)$ a continuous, convex and bounded functional; hence $\Psi(B)$ is a bounded subset of $\mathbb{R}$ for each bounded subset B of $\mathcal{V}$. Now let k_0 be a number and put $\mathcal{C}_{k_0} := \{v \in \mathcal{V};\ \Psi(v) \leqslant k_0\} \neq \emptyset$. Moreover, assume the following constraint qualification of the Slater type and coercivity:

(A1) There exists $z_0 \in D(\mathcal{A})$ such that $\Psi(z_0) < k_0$;

(A2) $\mathcal{A}$ is coercive in the following sense:

$$\inf_{v^* \in \mathcal{A}v} \frac{\langle\!\langle v^*, v - z_0 \rangle\!\rangle + \Psi(v)^2}{\|v\|_{\mathcal{V}}} \to \infty \quad \text{as } \|v\|_{\mathcal{V}} \to \infty,$$

where z_0 is the same as in (A1).

Then we obtained the following result in [7]:

Proposition 2.1. *Assume* (A1) *and* (A2) *hold. Then, for each* $f \in \mathcal{V}^*$, *there exist* $u \in D(\mathcal{A}) \cap \mathcal{C}_{k_0}$ *and* $\lambda \in \mathbb{R}$ *such that*

$$\mathcal{A}u + \lambda \partial_* \Psi(u) \ni f \quad \text{in } \mathcal{V}^*, \tag{2.6}$$

$$\lambda \geqslant 0, \quad \Psi(u) - k_0 \leqslant 0, \quad \lambda(\Psi(u) - k_0) = 0. \tag{2.7}$$

Note here that if $\mathcal{A}$ is maximal cyclically monotone, namely $\mathcal{A} = \partial_* \varphi$ with some proper lower semicontinuous convex functional $\varphi\colon \mathcal{V} \to (-\infty, \infty]$, then Proposition 2.1 is well-known as a minimizing problem $\min\{\varphi(u);\ u \in \mathcal{C}_{k_0}\}$ under the

Slater constraint qualification (A1) (see [3], Chapter 3). For a similar result for parabolic variational inequalities, we focus our interest on the general case of $\mathcal{A}$ (see [12]).

3. Proof of main theorem

In this section, we first give a detailed characterization of the operator $\mathcal{L}_{u_0}$. Second, we show that $\mathcal{L}_{u_0} + A$ is maximal monotone, and then we prove the main theorems.

3.1. Characterization of $\mathcal{L}_{u_0}$. The operator $\mathcal{L}_{u_0}$ is characterized by the auxiliary operator $\widetilde{\mathcal{L}}_{u_0}\colon D(\widetilde{\mathcal{L}}_{u_0}) \subset L^2(0,T;V) \to 2^{L^2(0,T;V^*)}$:

Definition 3.1. Element $g \in \widetilde{\mathcal{L}}_{u_0}u$ in $L^2(0,T;V^*)$ if and only if $u \in \mathcal{K} \cap C([0,T];H)$ with $u(0) = u_0$ in H, $g \in L^2(0,T;V^*)$ and there exist $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{K} \cap W^{1,2}(0,T;H)$ and $\{g_n\}_{n\in\mathbb{N}} \subset L^2(0,T;H)$ such that $\langle\langle u_n' - g_n, u_n - \eta\rangle\rangle \leqslant 0$ for all $\eta \in \mathcal{K}$ with $u_n \to u$ in $C([0,T];H)$ and weakly in $L^2(0,T;V)$, $g_n \to g$ weakly in $L^2(0,T;V^*)$ as $n \to \infty$, and $\limsup\limits_{n\to\infty}\langle\langle g_n, u_n\rangle\rangle \leqslant \langle\langle g, u\rangle\rangle$.

Lemma 3.1 (cf. [9], Part 2, Lemma 1.1)**.** *For all $u_0 \in \overline{K(0)}^H$, we have that $G(\widetilde{\mathcal{L}}_{u_0}) \subset G(\mathcal{L}_{u_0})$ and $\langle\langle g-h, u-w\rangle\rangle \geqslant 0$ for all $[u,g] \in G(\widetilde{\mathcal{L}}_{u_0})$ and $[w,h] \in G(\mathcal{L}_{u_0})$. Moreover, $\widetilde{\mathcal{L}}_{u_0}$ is monotone, where $G(\widetilde{\mathcal{L}}_{u_0})$ and $G(\mathcal{L}_{u_0})$ are, respectively, the graphs of $\widetilde{\mathcal{L}}_{u_0}$ and $\mathcal{L}_{u_0}$.*

P r o o f. It is easy to see $G(\widetilde{\mathcal{L}}_{u_0}) \subset G(\mathcal{L}_{u_0})$. Next, by definition, for all $[u,g] \in G(\widetilde{\mathcal{L}}_{u_0})$ there exist $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{K} \cap W^{1,2}(0,T;H)$ and $\{g_n\}_{n\in\mathbb{N}} \subset L^2(0,T;H)$ such that
$$
\begin{aligned}
0 \geqslant \langle\langle u_n' - g_n, u_n - \eta\rangle\rangle &= \langle\langle \eta' - g_n, u_n - \eta\rangle\rangle \\
&\quad + \frac{1}{2}|u_n(T) - \eta(T)|_H^2 - \frac{1}{2}|u_n(0) - \eta(0)|_H^2 \quad \text{for all } \eta \in \mathcal{K}_0,
\end{aligned}
$$
with convergences mentioned in Definition 3.1. Therefore, taking $\liminf\limits_{n\to\infty}$ in the above, we obtain $[u,g] \in G(\mathcal{L}_{u_0})$. Next, let $[w,h] \in G(\mathcal{L}_{u_0})$. Then, since $u_n \in \mathcal{K}_0$, we have $\langle\langle u_n' - h, w - u_n\rangle\rangle \leqslant (1/2)|u_0 - u_n(0)|_H^2$ for all $n \in \mathbb{N}$. On the other hand, from the definition of $[u,g] \in G(\widetilde{\mathcal{L}}_{u_0})$ it follows that $\langle\langle u_n' - g_n, u_n - w\rangle\rangle \leqslant 0$ for all $n \in \mathbb{N}$. Therefore,
$$
\langle\langle g_n - h, u_n - w\rangle\rangle \geqslant -\langle\langle u_n' - h, w - u_n\rangle\rangle \geqslant -\frac{1}{2}|u_0 - u_n(0)|_H^2.
$$
Taking $\limsup\limits_{n\to\infty}$ in the above, we get the conclusion. $\square$

Lemma 3.2 (cf. [9], Part 2, Lemma 1.2). *For all* $u_0 \in \overline{K(0)}^H$, $f \in L^2(0,T;V^*)$ *and* $\mu \in (0,1]$ *there exists a unique* $u \in D(\widetilde{\mathcal{L}}_{u_0})$ *with* $u(t) \in K(t)$ *for all* $t \in [0,T]$ *and*

$$f \in \widetilde{\mathcal{L}}_{u_0} u + \mu \mathcal{F} u \quad \text{in } L^2(0,T;V^*), \tag{3.1}$$

namely, $\widetilde{\mathcal{L}}_{u_0}$ *is maximal monotone and* $\widetilde{\mathcal{L}}_{u_0} = \mathcal{L}_{u_0}$.

Proof. The uniqueness comes from the strict monotonicity of $\widetilde{\mathcal{L}}_{u_0} + \mu\mathcal{F}$. Next, let us define convex functionals by

$$\varphi(z) := \begin{cases} \frac{\mu}{2}|z|_V^2, & \text{if } z \in V, \\ \infty, & \text{if } z \in H \setminus V, \end{cases} \qquad I_{K(t)}(z) := \begin{cases} 0, & \text{if } z \in K(t), \\ \infty, & \text{if } z \in H \setminus K(t), \end{cases}$$

and $\varphi^t(z) := \varphi(z) + I_{K(t)}(z)$. Then φ^t is proper, lower semi-continuous on H. Thanks to the well-known result on the evolution inclusion governed by time-dependent subdifferential [10], [13] and its application [6], [14] under the assumptions on ψ_0, ψ_1 with (2.2)–(2.3) (see [14, Lemma 5.1]), there exists a unique u_n in $W^{1,2}(0,T;H)$ such that $t \mapsto \varphi(u_n(t))$ is also bounded on $[0,T]$, $u_n(t) \in K(t)$ for all $t \in [0,T]$ and

$$u_n'(t) + u_n^*(t) = f_n(t), \quad u_n^*(t) \in \partial(\varphi + I_{K(t)})(u_n(t)) \quad \text{in } H \text{ for a.e. } t \in (0,T),$$

with $u_n(0) = u_{0,n}$ in H, where $\{u_{0,n}\}_{n\in\mathbb{N}} \subset K(0)$ and $\{f_n\}_{n\in\mathbb{N}} \subset L^2(0,T;H)$ are approximate sequences so that $u_{0,n} \to u_0$ in H and $f_n \to f$ in $L^2(0,T;V^*)$ as $n \to \infty$. From the usual energy estimate we easily see that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;V)$. Here, for simplicity, we use the same notation ∂_* for the subdifferential from V into V^* as for that from $L^2(0,T;V)$ into $L^2(0,T;V^*)$. Then we obtain the characterization

$$\begin{aligned} u_n^*(t) \in \partial(\varphi + I_{K(t)})(u_n(t)) &\subset \partial_*(\varphi + I_{K(t)})(u_n(t)) \\ &= \mu F u_n(t) + \partial_* I_{K(t)}(u_n(t)) \quad \text{in } V^* \text{ for a.e. } t \in (0,T), \end{aligned}$$

because of $\partial_*\varphi = \mu F$, where $F\colon V \to V^*$ is the duality mapping. Therefore,

$$\begin{aligned} &(u_n^*(t) - u_m^*(t), u_n(t) - u_m(t))_H \\ &\quad = \mu\langle F u_n(t) - F u_m(t), u_n(t) - u_m(t)\rangle_{V^*,V} \\ &\qquad + \langle u_n^{**}(t) - u_m^{**}(t), u_n(t) - u_m(t)\rangle_{V^*,V} \\ &\quad = \mu|u_n(t) - u_m(t)|_V^2 + \langle u_n^{**}(t) - u_m^{**}(t), u_n(t) - u_m(t)\rangle_{V^*,V}, \end{aligned}$$

where $u_n^{**}(t) \in \partial_* I_{K(t)}(u_n(t))$ in V^* for all $n \in \mathbb{N}$. Consequently, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_n(t) - u_m(t)|_H^2 + \mu|u_n(\tau) - u_m(t)|_V^2 + \langle u_n^{**}(t) - u_m^{**}(t), u_n(t) - u_m(t)\rangle_{V^*,V} \\ = \langle f_n(t) - f_m(t), u_n(t) - u_m(t)\rangle_{V^*,V} \quad \text{for all } t \in [0,T],\end{aligned}$$

whence it follows that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];H) \cap L^2(0,T;V)$ and

$$\lim_{n,m\to\infty} \int_0^t \langle u_n^{**}(t) - u_m^{**}(t), u_n(t) - u_m(t)\rangle_{V^*,V}\,\mathrm{d}t = 0.$$

These facts imply that there exists $u \in C([0,T];H)\cap L^\infty(0,T;V)$ such that $u(0) = u_0$ in H, $u(t) \in K(t)$ for all $t \in [0,T]$ and

$$\begin{aligned}u_n \to u \quad &\text{in } C([0,T];H) \cap L^2(0,T;V), \quad \text{weakly-}* \text{ in } L^\infty(0,T;V),\\ f_n - \mu\mathcal{F}u_n \to f - \mu\mathcal{F}u \quad &\text{in } L^2(0,T;V^*) \quad \text{as } n \to \infty.\end{aligned}$$

Now, we note that $\langle\!\langle u_n' - (f_n - \mu\mathcal{F}u_n), u_n - \eta\rangle\!\rangle \leqslant 0$ for all $\eta \in \mathcal{K}$ and $\limsup_{n\to\infty}\langle\!\langle f_n - \mu\mathcal{F}u_n, u_n\rangle\!\rangle = \langle\!\langle f - \mu\mathcal{F}u, u\rangle\!\rangle$. This means that $f - \mu\mathcal{F}u \in \widetilde{\mathcal{L}}_{u_0}u$, namely (3.1). Thus, $\widetilde{\mathcal{L}}_{u_0}$ is maximal monotone (see [2], [5]). As a consequence of Lemma 3.1, $G(\widetilde{\mathcal{L}}_{u_0}) = G(\mathcal{L}_{u_0})$. Actually, if not, there would exist $[w,h] \in G(\mathcal{L}_{u_0})$ such that $[w,h] \notin G(\widetilde{\mathcal{L}}_{u_0})$. Then we can extend $\widetilde{\mathcal{L}}_{u_0}$ by

$$\overline{\mathcal{L}_{u_0}}u := \begin{cases} \widetilde{\mathcal{L}}_{u_0}u & \text{if } u \neq w,\\ h & \text{if } u = w,\end{cases}$$

and moreover Lemma 3.1 implies that $\overline{\mathcal{L}_{u_0}}$ is monotone and $\overline{\mathcal{L}_{u_0}}$ is a proper extension of $\widetilde{\mathcal{L}}_{u_0}$. This is a contradiction to the maximality of $\widetilde{\mathcal{L}}_{u_0}$. □

P r o o f of Theorem 2.2. We now apply Proposition 2.1 to show Theorem 2.2. Actually, put $\mathcal{V} := L^2(0,T;V)$, $\mathcal{A} := \mathcal{L}_{u_0} + A\colon D(\mathcal{A}) \subset L^2(0,T;V) \to 2^{L^2(0,T;V^*)}$. Since $D(A) = L^2(0,T;V)$, $\mathcal{A}$ is maximal monotone (see, [2], [5]) and (A1) and (A2) hold on account of $z_0 \in D(\mathcal{L}_{u_0})$. Hence we get Theorem 2.2. □

P r o o f of Theorem 2.1. We see that $u \in D(\mathcal{A}) \cap \mathcal{C}_{k_0}$, $\lambda \in \mathbb{R}$ with (2.6)–(2.7) is equivalent to $u \in D(\mathcal{A}) \cap \mathcal{C}_{k_0}$ and the variational inequality

$$\langle\!\langle \alpha^*, u - \eta\rangle\!\rangle \leqslant \langle\!\langle f, u - \eta\rangle\!\rangle \quad \text{for all } \eta \in \mathcal{C}_{k_0}, \tag{3.2}$$

with $\alpha^* \in \mathcal{A}u$ in $L^2(0,T;V^*)$. In fact, for $f \in L^2(0,T;V^*)$, assume that $u \in D(\mathcal{A})\cap\mathcal{C}_{k_0}$ and $\lambda \in \mathbb{R}$ satisfy (2.6)–(2.7). If $\lambda = 0$, then (3.2) trivially holds. If $\lambda > 0$, then it follows from (2.7) that $\Psi(u) = k_0$, and from (2.6) that $\langle\!\langle (f-\alpha^*)/\lambda, \eta - u\rangle\!\rangle \leqslant \Psi(\eta) - \Psi(u)$ for all $\eta \in \mathcal{V}$. We have $\Psi(\eta) \leqslant k_0$ for all $\eta \in \mathcal{C}_{k_0}$, thus

$$\langle\!\langle \alpha^* - f, u - \eta\rangle\!\rangle \leqslant \lambda\Psi(\eta) - \lambda k_0 \leqslant 0 \quad \text{for all } \eta \in \mathcal{C}_{k_0}.$$

Conversely, assume that $u \in D(\mathcal{A}) \cap \mathcal{C}_{k_0}$ satisfies (3.2). Put $\widehat{\mathcal{A}} := \mathcal{A} + \mathcal{F}$ and $\hat{f} := f + \mathcal{F}u$. Now, applying Proposition 2.1 to these $\widehat{\mathcal{A}}$ and $\hat{f}$, we see that there exist an element $\hat{u} \in D(\mathcal{A}) \cap \mathcal{C}_{k_0}$ and a number $\lambda \in \mathbb{R}$ such that

$$\mathcal{A}\hat{u} + \mathcal{F}\hat{u} + \lambda\partial_*\Psi(\hat{u}) \ni \mathcal{F}u + f \quad \text{in } \mathcal{V}^*,$$
$$\lambda \geqslant 0, \quad \Psi(\hat{u}) - k_0 \leqslant 0, \quad \lambda(\Psi(\hat{u}) - k_0) = 0.$$

Moreover, $\hat{u} \in D(\mathcal{A}) \cap \mathcal{C}_{k_0}$ is a solution of

$$\langle\!\langle \hat{\alpha}^* + \mathcal{F}\hat{u}, \hat{u} - \eta\rangle\!\rangle \leqslant \langle\!\langle \mathcal{F}u + f, \hat{u} - \eta\rangle\!\rangle \quad \text{for all } \eta \in \mathcal{C}_{k_0}, \tag{3.3}$$

with $\hat{\alpha}^* \in \mathcal{A}\hat{u}$ in $L^2(0,T;V^*)$. Clearly, the solution of the variational inequality (3.3) is unique, because $\mathcal{A} + \mathcal{F}$ is strictly monotone. Now, note that u is a solution of (3.3); in fact, substituting u for $\hat{u}$ in (3.3), we see from (3.2) that (3.3) holds. This implies that $\hat{u} = u$ in $\mathcal{V}$. Going back to (3.2), we conclude that u satisfies (2.6). Finally, from (3.2) and Definition 2.2 we conclude that u satisfies (2.1). $\square$

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